Research Article

Analysis of a Nonautonomous Delayed Predator-Prey System with a Stage Structure for the Predator in a Polluted Environment

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A two-species nonautonomous Lotka-Volterra type model with diffusional migration among the immature predator population, constant delay among the matured predators, and toxicant effect on the immature predators in a nonprotective patch is proposed. The scale of the protective zone among the immature predator population can be regulated through diffusive coefficients $D_i(t)$, $i = 1, 2$. It is proved that this system is uniformly persistent (permanence) under appropriate conditions. Sufficient conditions are derived to confirm that if this system admits a positive periodic solution, then it is globally asymptotically stable.

1. Introduction

In recent years, many countries have already realized that the pollution of the environment is a very serious and urgent problem. It is well known that with the rapid development of modern industries and agricultures, a large quantity of toxicants and contaminants enter into ecosystems one after another. One of the most important and meaningful questions in mathematical ecology is the permanence (uniformly persistent) and extinction of a population in a polluted environment. Organisms are often exposed to a polluted environment and take up toxicants. The change in environment is caused by pollution, affecting the long-term survival of species, humans, and biodiversity of the habitat. The question of the effects of pollutants and toxicants on ecological communities is of tremendous important from both environmental and conservational points of view.

Anthropogenically produced toxic substances that enter the aquatic food chain represent a serious health threats to human beings and to the inhabitants of such polluted

ecosystems. The ability of an aquatic ecosystem to restore itself to the prepollution state is directly related to the compartmental retention times and exchange rates among compartments and the transport rates out of the system. The effect of a toxic substance can be well understood by measuring its presence throughout an aquatic system and by determining the rates of its transport betweencompartments and storage within compartments. An understanding of the compartmental and total system stability of a toxic substance can be achieved with the use of conceptual models [1–4].

Acid rain results from certain kinds of air pollution that mix with precipitation, such as rain or fog, then falls to earth as an acidic solution and its major components are oxides of sulfur and nitrogen that are mainly the by-products of coal-burning power plants, copper melting, factory, and automobile emissions. These oxides are chemically changed in the atmosphere and return to the earth as rain, snow, fog, or dust. In the United states, the most recognized form of acid rain results from sulfur dioxide emissions, which are converted into sulfuric acid in the atmosphere. When this is mixed with precipitation and falls to earth, the effect is precisely like pouring a diluted acid solution onto everything it touches. In lakes also, this acidification process can change ecological structures. In this way toxic substances are invaded into the ecological communities [5, 6].

Other examples are oil pollution in the seas [7], degradation of forests [8, 9], and dumping of toxic waste in rivers and lakes [10, 11].

In order to protect environment and regulate the release of toxic substances wisely, we must assess the harm of the toxicant to the species exposed to it. By using mathematical models, Hallam and Clark [12], Hallam et al. [13, 14], Hallam and De Luna [15], De Luna and Hallam [16], Zhien et al. [17], Huaping and Zhien [18], Freedman and Shukla [19], Wang and Ma [20], Dubey [21], Xiao and Chen [22], Ghosh et al. [23], Buonomo and Lacitignola [24], Li et al. [25], Samanta and Maiti [26], J. Wang and K. Wang [27], He and Wang [28, 29], Das et al. [30], and many others studied the effects of toxic substances on various ecosystems.

The effect of "stage-structure" to the dynamics of predator-prey system is a natural phenomenon and represents the division of a population into immature and mature individuals. As is common, the dynamics-eating habitats, susceptibility to toxicants, and so forth. are often quite different in these two subpopulations. Hence investigation of the effects of such a subdivision on the interaction of species is very much important from the ecological point of view. Stage-structured models have already received much attention by many authors [31, 32]. Many models in mathematical ecology can be formulated as system of differential equations with time delays. The effect of the past history on the stability of system is also an important problem in population ecology. Recently uniformly persistent and stability of a population dynamical system involving time delays have been considered by many authors [33]. Dispersal is a ubiquitous phenomenon in the natural world. Its importance in understanding the ecological and evolutionary dynamics of populations is mirrored by a large number of mathematical models [34]. In order to prevent the destruction of biological resources and to protect the environment, all kinds of measures have been proposed. Establishing a protective patch in mathematical ecology is applied widely. The practical effects of the protective patch on the polluted population is worth investigation.

As species do not exist alone in nature, it is of more biological significance to study the uniformly persistent-extinction threshold of each population in systems of two or more interacting species subject to organismal and environmental toxicants. In most of the classical toxicant-population models it is assumed that each individual has the same dose-response parameter to the organismal toxicant concentration regardless of the difference between the mature and immature populations. However, in the natural world, there are many species

whose individual members have a life history that takes them through two stages, immature and mature. Moreover, there is a big difference between the immature and mature species in many aspects. For example, for less concentration of C_2H_4 in an environment, the immature cymbidium is so susceptible that it begins to wither, whereas, the mature cymbidium often has not been affected [22, 35]. Therefore, it is important to consider the effect of stage structure on population growth in a polluted environment. In this paper, we have considered pollution in a nonautonomous predator-prey system together with diffusional migration among the immature predator population between protective and nonprotective patches where there is a constant delay among the matured predators due to the gestation of the predator based on the assumption that mature adult predators can only contribute to the reproduction of the predator biomass and that the change rate of predators depends on the number of the preys and of the predators presented at some previous time. The scale of the protective zone among the immature predator population can be regulated through diffusive coefficients $D_i(t)$, $i = 1, 2$. Here we incorporate stage-structure for the predator and assume that the toxicants have no effect on the mature species. This seems to be reasonable. The main feature of the present paper is to study the asymptotical behaviour of the toxicant-population model with stage structure, diffusional migration, and time delay so as to obtain some conditions under which the population is uniformly persistent. In addition, by constructing an appropriate Lyapunov function, we have derived sufficient conditions to confirm that if the system admits a positive periodic solution, then it is globally asymptotically stable.

2. The Basic Mathematical Model

Our study is based on the hypothesis of a complete spatial homogeneous environment. We incorporate stage-structure for the predator into a nonautonomous Lotka-Volterra type predator-prey model. We consider the following delayed model to describe the dynamics of prey and predator population in a polluted environment. The state variables of the model are $x(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$, the densities of the prey, immature predator in nonprotective patch (with toxicants), immature predator in protective patch (without toxicants), and mature predator species at time t, respectively, $C_0(t)$ is the concentration of toxicant in the immature predator organism in nonprotective patch (with toxicants) at time t, and $C_e(t)$ is the concentration of toxicant in the environment at time t. Here we assume that the toxicants have no effect on the mature species and immature predator in protective patch.

Let us consider the two-species nonautonomous Lotka-Volterra type model with diffusional migration among the immature predator population, constant delay among the matured predators, and toxicant effect on immature predators in nonprotective patch:

$$\frac{dx(t)}{dt} = x(t)\left[r_1(t) - a_{11}(t)x(t) - a_{12}(t)y_3(t)\right],$$

$$\frac{dy_1(t)}{dt} = b_1(t)y_3(t) - \theta_1(t)y_1(t) - \delta_1(t)y_1(t) - d(t)C_0(t)y_1(t) - \beta_1(t)y_1^2(t) + D_1(t)\left(y_2(t) - y_1(t)\right),$$

$$\frac{dy_2(t)}{dt} = b_2(t)y_3(t) - \theta_2(t)y_2(t) - \delta_2(t)y_2(t) - \beta_2(t)y_2^2(t) + D_2(t)\left(y_1(t) - y_2(t)\right),$$

$$\frac{dy_3(t)}{dt} = y_3(t)\left[-r_2(t) - a_{22}(t)y_3(t)\right] + a_{21}(t)y_3(t-\tau)x(t-\tau) + \theta_1(t)y_1(t) + \theta_2(t)y_2(t),$$

$$\frac{dC_0(t)}{dt} = k(t)C_e(t) + l(t) - g(t)C_0(t) - m(t)C_0(t),$$

$$\frac{dC_e(t)}{dt} = -h(t)C_e(t) + u(t).$$

$$(2.1)$$

The model is derived under following assumptions.

(A1) The prey population: the growth of the species is of a Lotka-Volterra nature. At any time $t > 0$, the birth rate is proportional to the existing prey population with proportional function $r_1(t)$ and $a_{11}(t)$ is the intraspecific competition rate function of the prey. $r_1(t)$ and $a_{11}(t)$ are continuous, bounded, and strictly positive functions in the interval $[0, \infty)$.

(A2) The predator population: the predators cannot hunt prey when the predators are infant (immature). The mature predators feed only on prey; $a_{12}(t)$ is the capturing rate function of the predator, $a_{21}(t)/a_{12}(t)$ is the conversion rate function of nutrients into the reproduction of the mature predator, and $\tau \geq 0$ is a constant delay due to the gestation of the mature predator based on the assumption that mature adult predators can only contribute to the reproduction of the predator biomass and that the change rate of predators depends on the number of the preys and of the predators presented at some previous time. To protect the immature predator population, the region (Ψ) of immature predator population is divided into two patches Ψ_1 and Ψ_2. Pollution is permitted in Ψ_1 and is inhibited in Ψ_2. We call Ψ_1 and Ψ_2 as the nonprotective and protective patches, respectively. Since the difference of densities between patches Ψ_1 and Ψ_2 exists, the diffusive migration can occur between these two patches, and $D_1(t)$, $D_2(t)$ are diffusion coefficient functions of nonprotective immature and protective immature predator species, respectively. At any time $t > 0$, the rates of transitions from nonprotective immature and protective immature individuals to mature individuals are proportional to the existing nonprotective immature and protective immature populations, respectively, with proportional functions $\theta_1(t)$ and $\theta_2(t)$, respectively, and the death rates of mature, nonprotective immature, and protective immature population are proportional to the existing mature, nonprotective immature, and protective immature population, respectively, with proportional functions $r_2(t)$, $\delta_1(t)$, and $\delta_2(t)$, respectively. The birth rates of nonprotective immature and protective immature population are proportional to the existing mature population with proportional functions $b_1(t)$ and $b_2(t)$, respectively. $a_{22}(t)$, $\beta_1(t)$, and $\beta_2(t)$ are the intraspecific competition rate functions of the mature, nonprotective immature, and protective immature population, respectively. $a_{12}(t)$, $a_{21}(t)$, $a_{22}(t)$, $D_i(t)$, $\theta_i(t)$, $r_2(t)$, $\delta_i(t)$, $b_i(t)$, and $\beta_i(t)$, $(i = 1, 2)$ are continuous, bounded, and strictly positive functions in the interval $[0, \infty)$.

(A3) Toxicants: $d(t)$ is the dose response parameter of nonprotective immature predator species $y_1(t)$ to the organismal toxicant concentration. The first two terms on the right of the fifth equation in system (2.1) denote the organismal net uptake of toxicant from the environment ($k(t)C_e(t)$) and the food chain ($l(t)$), respectively,

and the third and fourth terms represent the organismal net loss of toxicant due to metabolic processing and other causes. $k(t)$ denotes environmental toxicant uptake rate per unit mass organism, $l(t)$ denotes uptake rate of toxicant in food per unit mass organism, and $-g(t)C_o(t)$ and $-m(t)C_o(t)$ represent the egestion and depuration rates of the toxicant in the organism, respectively. The first term $(-h(t)C_e(t))$ on the right of the sixth equation in system (2.1) represents the toxicant loss from the environment itself by biological transformation, chemical hydrolysis, volatilization, microbial degradation, photosynthetic degradation, and so on and the exogenous rate of input of toxicant into the environment is represented by $u(t)$. The capacity of the environment is so large that the change of toxicants in the environment that comes from uptake and egestion by the organisms can be neglected [15, 36, 37]. The coefficients $d(t), k(t), l(t), g(t), m(t), h(t)$, and $u(t)$ are assumed to be nonnegative, continuous, and bounded functions in the interval $[0, \infty)$.

The initial conditions for system (2.1) take the form

$$x(\theta) = \phi_1(\theta), \qquad y_i(\theta) = \psi_i(\theta), \qquad C_0(\theta) = \phi_2(\theta), \qquad C_e(\theta) = \phi_3(\theta), \quad (i = 1,2,3),$$

$$\phi_i(\theta) \geq 0, \qquad \psi_i(\theta) \geq 0 \quad (i = 1,2,3), \ \theta \in [-\tau, 0], \tag{2.2}$$

$$x(0) > 0, \qquad y_i(0) > 0, \qquad 0 \leq C_0(0) \leq 1, \qquad 0 \leq C_e(0) \leq 1 \quad (i = 1,2,3),$$

where $\Phi = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \psi_1(\theta), \psi_2(\theta), \psi_3(\theta)) \in C([-\tau, 0], R_+^6)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R_+^6, where $R_+^6 = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_i \geq 0, \ i = 1,2,3,4,5,6\}$.

It is well known by the fundamental theory of functional differential equations [38] that system (2.1) has a unique solution $(x(t), y_1(t), y_2(t), y_3(t), C_0(t), C_e(t))$ satisfying initial conditions (2.2).

Let $f^l = \inf_{t \geq 0} f(t)$, $f^u = \sup_{t \geq 0} f(t)$, for a continuous and bounded function $f(t)$ defined on $[0, +\infty)$.

In this paper, we always assume that the following holds:

$$\min\left\{d^l, k^l, l^l, g^l, m^l, h^l, u^l\right\} \geq 0, \qquad \min_{i,j=1,2}\left\{r_i^l, a_{ij}^l, b_i^l, \theta_i^l, \delta_i^l, \beta_i^l, D_i^l\right\} > 0,$$

$$\max\left\{d^u, k^u, l^u, g^u, m^u, h^u, u^u\right\} < \infty, \qquad \max_{i,j=1,2}\left\{r_i^u, a_{ij}^u, b_i^u, \theta_i^u, \delta_i^u, \beta_i^u, D_i^u\right\} < \infty. \tag{2.3}$$

Each of $C_0(t)$ and $C_e(t)$ is a concentration, and thus, these variables cannot be greater than 1. So we should give some conditions, such that

$$0 \leq C_0(t) \leq 1, \qquad 0 \leq C_e(t) \leq 1, \quad \forall t \geq 0. \tag{2.4}$$

Theorem 2.1. *Every solution of system* (2.1) *with initial conditions* (2.2) *exists and is unique in the interval* $[0, \infty)$ *and* $x(t) > 0, \ y_1(t) > 0, \ y_2(t) > 0, \ y_3(t) > 0, \ C_0(t) \geq 0, \ C_e(t) \geq 0,$ *for all* $t \geq 0.$

Proof. Since the right-hand side of system (2.1) is completely continuous and locally Lipschitzian on C, the solution $(x(t), y_1(t), y_2(t), y_3(t), C_0(t), C_e(t))$ of (2.1) with initial

conditions (2.2) exists and is unique on $[0, \alpha)$, where $0 < \alpha \leq +\infty$ [38, chapter 2]. From system (2.1) with initial conditions (2.2), we have

$$x(t) = x(0) \exp \int_0^t [r_1(s) - a_{11}(s)x(s) - a_{12}(s)y_3(s)] ds > 0, \quad \forall t \geq 0. \tag{2.5}$$

Next, we show that $y_1(t) > 0$ for all $t \in [0, \infty)$. Otherwise, there exists a $t_1 \in (0, \infty)$ such that $y_1(t_1) = 0$, $\dot{y}_1(t_1) \leq 0$, and $y_1(t) > 0$ for all $t \in [0, t_1)$. Hence there must have $y_2(t) > 0$ for all $t \in [0, t_1)$. If this statement is not true, then there exists a $t_2 \in (0, t_1)$ such that $y_2(t_2) = 0$ and $y_2(t) > 0$ on $[0, t_2)$. We claim that $y_3(t) > 0$ for all $t \in [0, t_2)$. If this statement is not true, then there exists a $t_3 \in (0, t_2)$ such that $y_3(t_3) = 0$ and $y_3(t) \geq 0$ for all $t \in [-\tau, t_3]$. Furthermore,

$$\frac{dy_3(t)}{dt} \geq -r_2(t)y_3(t) - a_{22}(t)y_3^2(t), \quad \forall t \in [0, t_3], \tag{2.6}$$

then, $y_3(t) \geq y_3(0) \exp \int_0^t [-r_2(s) - a_{22}(s)y_3(s)] ds > 0$, for all

$$t \in [0, t_3] \Longrightarrow y_3(t_3) \geq y_3(0) \exp \int_0^{t_3} [-r_2(s) - a_{22}(s)y_3(s)] ds > 0, \tag{2.7}$$

which is a contradiction. Hence, $y_3(t) > 0$ for all $t \in [0, t_2)$. Integrating the third equation of system (2.1) from 0 to t_2, we have

$$y_2(t_2) = y_2(0) \exp \left\{ -\int_0^{t_2} (D_2(s) + \theta_2(s) + \delta_2(s) + \beta_2(s)y_2(s)) ds \right\}$$
$$+ \int_0^{t_2} (b_2(u)y_3(u) + D_2(u)y_1(u)) \exp \left\{ \int_{t_2}^u (D_2(s) + \theta_2(s) + \delta_2(s) + \beta_2(s)y_2(s)) ds \right\} du > 0, \tag{2.8}$$

which is a contradiction with $y_2(t_2) = 0$. So $y_2(t) > 0$, for all $t \in [0, t_1)$ and hence $y_3(t) > 0$, for all $t \in [0, t_1)$. Integrating the second equation of system (2.1) from 0 to t_1, we have

$$y_1(t_1) = y_1(0) \exp \left\{ -\int_0^{t_1} (D_1(s) + \theta_1(s) + \delta_1(s) + d(s)C_0(s) + \beta_1(s)y_1(s)) ds \right\}$$
$$+ \int_0^{t_1} (b_1(u)y_3(u) + D_1(u)y_2(u)) \exp \left\{ \int_{t_1}^u (D_1(s) + \theta_1(s) + \delta_1(s) \tag{2.9} \right.$$
$$\left. + d(s)C_0(s) + \beta_1(s)y_1(s)) ds \right\} du > 0,$$

which is a contradiction with $y_1(t_1) = 0$. So $y_1(t) > 0$, for all $t \geq 0$ and hence $y_2(t) > 0$, $y_3(t) > 0$, for all $t \geq 0$.

Also,

$$C_0(t) = e^{-\int_0^t \{m(s)+g(s)\}ds}\left[\int_0^t \{k(s)C_e(s) + l(s)\}e^{\int_0^s \{m(s)+g(s)\}ds}ds + C_0(0)\right] \geq 0, \quad \forall t \geq 0,$$

$$C_e(t) = e^{-\int_0^t h(s)ds}\left[\int_0^t u(s)e^{\int_0^s h(s)ds}ds + C_e(0)\right] \geq 0 \quad \forall t \geq 0.$$

(2.10)

This completes the proof. □

Theorem 2.2. *From system* (2.1) *with initial conditions* (2.2), *if* $k^u + l^u \leq g^l + m^l$, $u^u \leq h^l$, *then* $0 \leq C_0(t) \leq 1$, $0 \leq C_e(t) \leq 1$, *for all* $t \geq 0$.

Proof. According to Theorem 2.1, we have $C_0(t) \geq 0$, $C_e(t) \geq 0$, for all $t \geq 0$. Now we have to prove that $C_0(t) \leq 1$, $C_e(t) \leq 1$, for all $t \geq 0$.

If the conclusion is false, then the maximum interval is $[0, T]$ such that $0 \leq C_0(t) \leq 1$, $0 \leq C_e(t) \leq 1$, for all $t \in [0, T]$ and at least one of the following cases will arise:

(i) $C_0(T) = 1, C_e(T) < 1$;

(ii) $C_0(T) < 1, C_e(T) = 1$;

(iii) $C_0(T) = C_e(T) = 1$.

We will prove that none of these cases is true.

(i) $C_0(T) = 1$, $C_e(T) < 1$; using the condition $k^u + l^u \leq g^l + m^l$, we have

$$\left.\frac{dC_0(t)}{dt}\right|_{t=T} = k(T)C_e(T) + l(T) - g(T)C_0(T) - m(T)C_0(T)$$

$$< k(T) + l(T) - g(T) - m(T)$$

$$\leq k^u + l^u - g^l - m^l \leq 0,$$

(2.11)

thus there exist $t_1 > 0$, s.t. $0 \leq C_0(t) \leq 1$, $0 \leq C_e(t) \leq 1$, for all $t \in [T, T + t_1]$. This contradicts the definition of the interval $[0, T]$. So there is no T such that $0 \leq C_0(t) \leq 1$, $0 \leq C_e(t) \leq 1$, for all $t \in [0, T]$ with $C_0(T) = 1, C_e(T) < 1$.

With the same reasoning as in case (i), for cases (ii) and (iii), as far as t which keeps $0 \leq C_0(t) \leq 1$ and $0 \leq C_e(t) \leq 1$ is concerned, the interval $[0, T]$ can be extended rightwards. This contradicts the property of T. Therefore, $0 \leq C_0(t) \leq 1$, $0 \leq C_e(t) \leq 1$, for all $t \geq 0$. This completes the proof. □

Putting the expression of $C_e(t)$ in $C_0(t)$, we can express $C_0(t)$ in terms of some bounded continuous functions; therefore the system (2.1) and (2.2) may be simplified as follows:

$$\frac{dx(t)}{dt} = x(t)\left[r_1(t) - a_{11}(t)x(t) - a_{12}(t)y_3(t)\right],$$

$$\frac{dy_1(t)}{dt} = b_1(t)y_3(t) - \theta_1(t)y_1(t) - \delta_1(t)y_1(t) - d(t)C_0(t)y_1(t) - \beta_1(t)y_1^2(t) + D_1(t)\left(y_2(t) - y_1(t)\right),$$

$$\frac{dy_2(t)}{dt} = b_2(t)y_3(t) - \theta_2(t)y_2(t) - \delta_2(t)y_2(t) - \beta_2(t)y_2^2(t) + D_2(t)\left(y_1(t) - y_2(t)\right),$$

$$\frac{dy_3(t)}{dt} = y_3(t)\left[-r_2(t) - a_{22}(t)y_3(t)\right] + a_{21}(t)y_3(t-\tau)x(t-\tau) + \theta_1(t)y_1(t) + \theta_2(t)y_2(t).$$

$$(2.12)$$

The initial conditions are

$$x(\theta) = \phi_1(\theta), \qquad y_i(\theta) = \psi_i(\theta) \quad (i = 1, 2, 3),$$

$$\phi_1(\theta) \geq 0, \qquad \psi_i(\theta) \geq 0 \quad (i = 1, 2, 3), \ \theta \in [-\tau, 0], \qquad (2.13)$$

$$x(0) > 0, \qquad y_i(0) > 0 \quad (i = 1, 2, 3),$$

where $\Phi = (\phi_1(\theta), \psi_1(\theta), \psi_2(\theta), \psi_3(\theta)) \in C([-\tau, 0], R_+^4)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R_+^4, where $R_+^4 = \{(x_1, x_2, x_3, x_4) : x_i \geq 0, i = 1, 2, 3, 4\}$.

3. Uniformly Persistent of System (2.12)

Here we wish to discuss the uniformly persistent of the system (2.12) with initial conditions (2.13), which demonstrates how this system will be uniformly persistent, is that, the long-term survival (i.e., will not vanish in time) of all components of the system (2.12) with initial conditions (2.13), under some conditions.

Definition 3.1. The system (2.12) is said to be uniformly persistent, that is, the long-term survival (will not vanish in time) of all components of the system (2.12), if there are positive constants v_i and w_i $(i = 1, 2, 3, 4)$ such that

$$v_1 \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) \leq w_1,$$

$$v_2 \leq \liminf_{t \to \infty} y_1(t) \leq \limsup_{t \to \infty} y_1(t) \leq w_2,$$

$$v_3 \leq \liminf_{t \to \infty} y_2(t) \leq \limsup_{t \to \infty} y_2(t) \leq w_3, \qquad (3.1)$$

$$v_4 \leq \liminf_{t \to \infty} y_3(t) \leq \limsup_{t \to \infty} y_3(t) \leq w_4$$

hold for any solution $(x(t), y_1(t), y_2(t), y_3(t))$ of (2.12) with initial conditions (2.13). Here v_i and w_i $(i = 1, 2, 3, 4)$ are independent of (2.13).

Theorem 3.2 (see [39]). *Consider the following equation:*

$$\dot{x}(t) = ax(t - \tau) - bx(t) - cx^2(t), \tag{3.2}$$

where $a, b, c, \tau > 0$; $x(t) > 0$, for $-\tau \leq t \leq 0$. One has the following:

(I) *if $a > b$, then $\lim_{t \to \infty} x(t) = (a - b)/c$;*

(II) *if $a < b$, then $\lim_{t \to \infty} x(t) = 0$.*

Theorem 3.3. *Let $X(t) = (x(t), y_1(t), y_2(t), y_3(t))^T$ denote any solution of system (2.12) and (2.13). Suppose that system (2.12) satisfies $b_1^u > \theta_1^l + \delta_1^l + d^l C_0^l$, $b_2^u > \theta_2^l + \delta_2^l$, and $a_{21}^u r_1^u > a_{11}^l (r_2^l - \theta_1^u - \theta_2^u) > 0$. Then there exist a $T_3 > 0$ such that*

$$x(t) \leq M_1, \qquad y_i(t) \leq M_4 \quad (i = 1, 2, 3), \quad \forall t \geq T_3, \text{ where } M_1 > \frac{r_1^u}{a_{11}^l},$$

$$M_4 > M^* = \max \left\{ \frac{b_1^u - \theta_1^l - \delta_1^l - d^l C_0^l}{\beta_1^l}, \frac{b_2^u - \theta_2^l - \delta_2^l}{\beta_2^l}, \frac{a_{21}^u r_1^u - a_{11}^l \left(r_2^l - \theta_1^u - \theta_2^u \right)}{a_{11}^l a_{22}^l} \right\}. \tag{3.3}$$

Proof. Let $M_1 > r_1^u / a_{11}^l$. We have $\dot{x}(t) \leq x(t)[r_1(t) - a_{11}(t)x(t)] \leq x(t)[r_1^u - a_{11}^l x(t)]$. Therefore, if $x(0) \leq M_1$, then $x(t) \leq M_1$, for all $t \geq 0$. If $x(0) > M_1$, and let $-\alpha_1 = M_1(r_1^u - a_{11}^l M_1), \alpha_1 > 0$, then there exist an $\epsilon_1 > 0$, s.t. if $t \in [0, \epsilon_1), x(t) > M_1$, and we have $\dot{x}(t) < -\alpha_1 < 0$. Therefore, there exist a $T_1 > 0$ s.t. $x(t) \leq M_1$, for all $t \geq T_1$, where $M_1 > r_1^u / a_{11}^l$. We define $V(t) = \max\{y_1(t), y_2(t), y_3(t)\}$. Calculating the upper-right derivative of $V(t)$ along the solution of system (2.12) and (2.13), we have the followings:

(1) if $y_1(t) \geq y_2(t) \geq y_3(t)$, or $y_1(t) \geq y_3(t) \geq y_2(t)$, then

$$D^+ V(t) = \dot{y}_1(t)$$

$$\leq b_1^u y_1(t) - \theta_1^l y_1(t) - \delta_1^l y_1(t) - d^l C_0^l y_1(t) - \beta_1^l y_1^2(t) \tag{3.4}$$

$$= \left(b_1^u - \theta_1^l - \delta_1^l - d^l C_0^l \right) y_1(t) - \beta_1^l y_1^2(t), \quad \forall t \geq 0;$$

(2) if $y_2(t) \geq y_1(t) \geq y_3(t)$, or $y_2(t) \geq y_3(t) \geq y_1(t)$, then

$$D^+ V(t) = \dot{y}_2(t)$$

$$\leq b_2^u y_2(t) - \theta_2^l y_2(t) - \delta_2^l y_2(t) - \beta_2^l y_2^2(t) \tag{3.5}$$

$$= \left(b_2^u - \theta_2^l - \delta_2^l \right) y_2(t) - \beta_2^l y_2^2(t), \quad \forall t \geq 0;$$

(3) if $y_3(t) \geq y_1(t) \geq y_2(t)$, or, $y_3(t) \geq y_2(t) \geq y_1(t)$, then

$$
\begin{aligned}
D^+V(t) &= \dot{y}_3(t) \\
&\leq -r_2^l y_3(t) - a_{22}^l y_3^2(t) + a_{21}^u M_1 y_3(t-\tau) + (\theta_1^u + \theta_2^u) y_3(t) \\
&= -\left(r_2^l - \theta_1^u - \theta_2^u\right) y_3(t) - a_{22}^l y_3^2(t) + a_{21}^u M_1 y_3(t-\tau), \quad \forall t \geq T_1 + \tau.
\end{aligned}
\tag{3.6}
$$

Let

$$
M_2 > M_1^* = \max \left\{ \frac{b_1^u - \theta_1^l - \delta_1^l - d^l C_0^l}{\beta_1^l}, \frac{b_2^u - \theta_2^l - \delta_2^l}{\beta_2^l} \right\}.
\tag{3.7}
$$

From (3.4) and (3.5), we can obtain the followings:

(i) if $\max\{y_1(0), y_2(0), y_3(0)\} \leq M_2$, then $\max\{y_1(t), y_2(t), y_t(t)\} \leq M_2$, for all $t \geq 0$;

(ii) if $\max\{y_1(0), y_2(0), y_3(0)\} > M_2$, and let $-\alpha_2 = \max[M_2\{(b_1^u - \theta_1^l - \delta_1^l - d^l C_0^l) - \beta_1^l M_2\}, M_2\{(b_2^u - \theta_2^l - \delta_2^l) - \beta_2^l M_2\}]$, $\alpha_2 > 0$, we consider the following two cases:

(a) $V(0) = y_1(0) > M_2$,
(b) $V(0) = y_2(0) > M_2$.

If (a) holds, then there exist an $\epsilon_2 > 0$, s.t. if $t \in [0, \epsilon_2), V(t) = y_1(t) > M_2$, and we have $D^+V(t) = \dot{y}_1(t) < -\alpha_2 < 0$.

If (b) holds, then there exist an $\epsilon_3 > 0$, s.t. if $t \in [0, \epsilon_3), V(t) = y_2(t) > M_2$, and we have $D^+V(t) = \dot{y}_2(t) < -\alpha_2 < 0$.

If (3.6) holds, then by Theorem 3.2, there exist $T_2 \geq T_1 + \tau > 0$, s.t.

$$
y_3(t) < M_3, \quad \forall t \geq T_2 \geq T_1 + \tau > 0, \text{where } M_3 > \frac{a_{21}^u r_1^u - a_{11}^l \left(r_2^l - \theta_1^u - \theta_2^u\right)}{a_{11}^l a_{22}^l}
\tag{3.8}
$$

$$
\left(\text{since } M_1 \text{ can be chosen close enough to } \frac{r_1^u}{a_{11}^l} \right).
$$

From the above discussions, we conclude that there exist $T_3 \geq T_2 \geq T_1 + \tau > 0$, s.t.

$$
x(t) \leq M_1, \qquad y_i(t) \leq M_4 \quad (i = 1, 2, 3), \quad \forall t \geq T_3, \quad \text{where } M_1 > \frac{r_1^u}{a_{11}^l},
$$

$$
M_4 > M^* = \max \left\{ \frac{b_1^u - \theta_1^l - \delta_1^l - d^l C_0^l}{\beta_1^l}, \frac{b_2^u - \theta_2^l - \delta_2^l}{\beta_2^l}, \frac{a_{21}^u r_1^u - a_{11}^l \left(r_2^l - \theta_1^u - \theta_2^u\right)}{a_{11}^l a_{22}^l} \right\}.
\tag{3.9}
$$

This completes the proof.   □

Theorem 3.4. *Suppose that system* (2.12) *and* (2.13) *satisfies the following conditions:*

$$b_1^l > \theta_1^u + \delta_1^u + d^u C_0^u, \qquad b_2^l > \theta_2^u + \delta_2^u, \qquad r_2^l > \theta_1^u + \theta_2^u, \qquad a_{21}^l r_1^l > a_{11}^u \left(r_2^u - \theta_1^l - \theta_2^l \right),$$

$$\max \left\{ \frac{b_1^u - \theta_1^l - \delta_1^l - d^l C_0^l}{\beta_1^l}, \frac{b_2^u - \theta_2^l - \delta_2^l}{\beta_2^l}, \frac{a_{21}^u r_1^u - a_{11}^l \left(r_2^l - \theta_1^u - \theta_2^u \right)}{a_{11}^l a_{22}^l} \right\} \tag{3.10}$$

$$< M_4 < \min \left\{ \frac{r_1^l}{a_{12}^u}, \frac{a_{21}^l r_1^l - a_{11}^u \left(r_2^u - \theta_1^l - \theta_2^l \right)}{a_{21}^l a_{12}^u} \right\}.$$

Then system (2.12) *and* (2.13) *is uniformly persistent.*

Proof. Let $X(t) = (x(t), y_1(t), y_2(t), y_2(t))^T$ be a solution of (2.12) and (2.13). $\therefore \dot{x}(t) \geq x(t)[r_1^l - a_{11}^u x(t) - a_{12}^u M_4]$, for all $t \geq T_3$ (using Theorem 3.3 and T_3 is defined there). Now, $M_4 < (a_{21}^l r_1^l - a_{11}^u (r_2^u - \theta_1^l - \theta_2^l))/a_{21}^l a_{12}^u$, and $r_2^l > \theta_1^u + \theta_2^u$ together imply $0 < (r_2^u - \theta_1^l - \theta_2^l)/a_{21}^l < (r_1^l - a_{12}^u M_4)/a_{11}^u$. Let us choose m_1 in such a way that

$$0 < \frac{r_2^u - \theta_1^l - \theta_2^l}{a_{21}^l} < m_1 < \frac{r_1^l - a_{12}^u M_4}{a_{11}^u} \Longrightarrow a_{21}^l m_1 > r_2^u - \theta_1^l - \theta_2^l. \tag{3.11}$$

If $x(T_3) \geq m_1$, then $x(t) \geq m_1$, for all $t \geq T_3$. If $x(T_3) < m_1$ and let $\mu_1 = x(T_3)(r_1^l - a_{11}^u m_1 - a_{12}^u M_4) > 0$, then there exist an $\epsilon_1 > 0$, s.t. $x(t) < m_1$, and $\dot{x}_1(t) > \mu_1 > 0$, for all $t \in [T_3, T_3 + \epsilon_1)$. Therefore, there exist a $T_4 > T_3 > 0$, s.t. $x(t) \geq m_1$, for all $t \geq T_4$.

Define $V(t) = \min\{y_1(t), y_2(t), y_3(t)\}$. Then calculating the lower-right derivative of $V(t)$ along the solution of system (2.12) and (2.13), we obtain the following.

(1) if $y_1(t) \leq y_2(t) \leq y_3(t)$, or, $y_1(t) \leq y_3(t) \leq y_2(t)$, then

$$\begin{aligned} D_+ V(t) &= \dot{y}_1(t) \\ &\geq b_1^l y_1(t) - \theta_1^u y_1(t) - \delta_1^u y_1(t) - d^u C_0^u y_1(t) - \beta_1^u y_1^2(t) \\ &= \left(b_1^l - \theta_1^u - \delta_1^u - d^u C_0^u \right) y_1(t) - \beta_1^u y_1^2(t), \quad \forall t \geq 0; \end{aligned} \tag{3.12}$$

(2) if $y_2(t) \leq y_1(t) \leq y_3(t)$, or, $y_2(t) \leq y_3(t) \leq y_1(t)$, then

$$\begin{aligned} D_+ V(t) &= \dot{y}_2(t) \\ &\geq b_2^l y_2(t) - \theta_2^u y_2(t) - \delta_2^u y_2(t) - \beta_2^u y_2^2(t) \\ &= \left(b_2^l - \theta_2^u - \delta_2^u \right) y_2(t) - \beta_2^u y_2^2(t), \quad \forall t \geq 0; \end{aligned} \tag{3.13}$$

(3) if $y_3(t) \le y_1(t) \le y_2(t)$, or, $y_3(t) \le y_2(t) \le y_1(t)$, then

$$
\begin{aligned}
D_+V(t) = \dot{y}_3(t) \\
\ge -r_2^u y_3(t) - a_{22}^u y_3^2(t) + a_{21}^l m_1 y_3(t-\tau) + \left(\theta_1^l + \theta_2^l\right)y_3(t) \\
= -\left(r_2^u - \theta_1^l - \theta_2^l\right)y_3(t) - a_{22}^u y_3^2(t) + a_{21}^l m_1 y_3(t-\tau), \quad \forall t \ge T_4 + \tau.
\end{aligned}
\tag{3.14}
$$

Let us choose m_2 in such a way that $0 < m_2 < \min\{(b_1^l - \theta_1^u - \delta_1^u - d^u C_0^u)/\beta_1^u,\ (b_2^l - \theta_2^u - \delta_2^u)/\beta_2^u\}$. From (3.12) and (3.13), we can obtain the followings:

(i) if $V(0) = \min\{y_1(0), y_2(0), y_3(0)\} \ge m_2$, then $\min\{y_1(t), y_2(t), y_t(t)\} \ge m_2$, for all $t \ge 0$;

(ii) if $V(0) = \min\{y_1(0), y_2(0), y_3(0)\} < m_2$, and let $\mu_2 = \min[y_1(0)\{b_1^l - \theta_1^u - \delta_1^u - d^u C_0^u - \beta_1^u m_2\}, y_2(0)\{b_2^l - \theta_2^u - \delta_2^u - \beta_2^u m_2\}] > 0$, we consider the following two cases:

(a) $V(0) = y_1(0) < m_2$,

(b) $V(0) = y_2(0) < m_2$.

If (a) holds, then there exist an $\epsilon_2 > 0$, s.t. if $t \in [0, \epsilon_2)$, $V(t) = y_1(t) < m_2$, and we have $D_+V(t) = \dot{y}_1(t) > \mu_2 > 0$.

If (b) holds, then there exist an $\epsilon_3 > 0$, s.t. if $t \in [0, \epsilon_3)$, $V(t) = y_2(t) < m_2$, and we have $D_+V(t) = \dot{y}_2(t) > \mu_2 > 0$.

If (3.14) holds, then by (3.11) and Theorem 3.2, there exist $T_5 \ge T_4 + \tau > 0$, s.t.

$$
y_3(t) > m_3, \quad \forall t \ge T_5 \ge T_4 + \tau > 0, \text{ where, } m_3 < \frac{a_{21}^l m_1 - \left(r_2^u - \theta_1^l - \theta_2^l\right)}{a_{22}^u}.
\tag{3.15}
$$

From above discussions, we conclude that there exist $T_6 \ge T_5 \ge T_4 + \tau > T_3 > 0$, s.t. $x(t) \ge m_1$ and $y_i(t) \ge m_4$ $(i = 1, 2, 3)$, for all $t \ge T_6$ where

$$
0 < \frac{r_2^u - \theta_1^l - \theta_2^l}{a_{21}^l} < m_1 < \frac{r_1^l - a_{12}^u M_4}{a_{11}^u},
$$

$$
0 < m_4 < \min\left\{\frac{b_1^l - \theta_1^u - \delta_1^u - d^u C_0^u}{\beta_1^u}, \frac{b_2^l - \theta_2^u - \delta_2^u}{\beta_2^u}, \frac{a_{21}^l m_1 - \left(r_2^u - \theta_1^l - \theta_2^l\right)}{a_{22}^u}\right\}.
\tag{3.16}
$$

From the above discussions, we conclude that there exist $T_6 \ge T_5 \ge T_4 + \tau > T_3 > 0$ s.t. every solution of system (2.12) and (2.13) eventually enters and remains in the region $\Omega = \{(x, y_1, y_2, y_3) \mid m \le x,\ y_i \le M,\ i = 1, 2, 3\}$, for all $t \ge T_6$, where $m = \min\{m_1, m_4\}$ and $M = \max\{M_1, M_4\}$, and M_1 is defined in Theorem 3.3. This completes the proof. $\square$

The results of Theorem 3.4 reveal the fact that lower values $\delta_1(t)$ (instantaneous per capita death rate function of nonprotective immature predator population), $\delta_2(t)$

(instantaneous per capita death rate function of protective immature predator population), $C_0(t)$ (concentration of toxicant in the immature predator organism in nonprotective patch at time t), $C_e(t)$ (concentration of toxicant in the environment at time t), and $d(t)$ (dose response parameter of nonprotective immature predator species to the organismal toxicant concentration at time t) lead to make the underlying system uniformly persistent.

4. Global Asymptotic Stability of Periodic Solution

From our everyday experience we know that the biological and environmental parameters are subject to fluctuation in time, and the effects of a periodically varying environment have an important selective forces on systems in a fluctuating environment. To investigate this kind of phenomenon, in the model, the coefficients should be periodic functions of time.

Suppose that system (2.12) is a periodic system with initial conditions (2.13). Here we assume that the periodic system (2.12) satisfies all conditions of Theorem 3.4. We derive sufficient conditions for all solutions of system (2.12) to converge to a periodic solution.

Theorem 4.1 (see [40]). *Suppose that the continuous operator A maps the closed and bounded convex set $Q \in R^n$ onto itself, then the operator A has at least one fixed point in set Q.*

Theorem 4.2. *If periodic system* (2.12) *satisfies conditions of Theorem 3.4, then there is at least one strictly positive periodic solution of system* (2.12)

Proof. From Theorems 3.3, 4.1 and 2 in Teng and Chen [41], it is easy to see that there is at least one strictly positive periodic solution of system (2.12). □

Theorem 4.3. *If system* (2.12) *has a periodic solution, then it is globally asymptotically stable provided* $\lim \inf_{t \to \infty} A_i(t) > 0$, $(i = 1, 2)$, $\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u > 0$, $\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u > 0$, *where* $A_1(t) = a_{11}^l - Ma_{21}(t + \tau)$, $A_2(t) = r_2^l + 2ma_{22}^l - a_{12}^u - b_1^u - b_2^u - Ma_{21}(t + \tau)$, m, M *are defined in Theorem 3.4.*

Proof. Let $(u(t), v_1(t), v_2(t), v_3(t))^T$ is a periodic solution of system (2.12) and (2.13). Suppose that $(x(t), y_1(t), y_2(t), y_3(t))^T$ is any positive solution of (2.12) and (2.13). Let

$$L_1(t) = |\ln x(t) - \ln u(t)| + \sum_{i=1}^{3} |y_i(t) - v_i(t)|. \tag{4.1}$$

Calculating the upper-right derivative of $L_1(t)$ along the positive solution of system (2.12) and (2.13), it follows that

$$D^+ L_1(t) = \left(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{u}(t)}{u(t)} \right) \operatorname{sgn}(x(t) - u(t)) + \sum_{i=1}^{3} (\dot{y}_i(t) - \dot{v}_i(t)) \operatorname{sgn}(y_i(t) - v_i(t))$$

$$= \operatorname{sgn}(x(t) - u(t)) \{ -a_{11}(t)(x(t) - u(t)) - a_{12}(t)(y_3(t) - v_3(t)) \}$$

$$+ \operatorname{sgn}(y_1(t) - v_1(t)) \{ b_1(t)(y_3(t) - v_3(t)) - (\theta_1(t) + \delta_1(t) + d(t)C_0(t))(y_1(t) - v_1(t))$$

$$- \beta_1(t) \left(y_1^2(t) - v_1^2(t) \right) + D_1(t)((y_2(t) - v_2(t)) - (y_1(t) - v_1(t))) \}$$

$$+ \text{sgn}(y_2(t) - v_2(t))\Big\{ b_2(t)(y_3(t) - v_3(t)) - (\theta_2(t) + \delta_2(t))(y_2(t) - v_2(t))$$

$$-\beta_2(t)\Big(y_2^2(t) - v_2^2(t)\Big) + D_2(t)((y_1(t) - v_1(t)) - (y_2(t) - v_2(t)))\Big\}$$

$$+ \text{sgn}(y_3(t) - v_3(t))\Big\{ -r_2(t)(y_3(t) - v_3(t)) - a_{22}(t)\Big(y_3^2(t) - v_3^2(t)\Big) + a_{21}(t)y_3(t - \tau)$$

$$\times (x(t - \tau) - u(t - \tau)) + a_{21}(t)u(t - \tau) \; (y_3(t - \tau) - v_3(t - \tau))$$

$$+\theta_1(t)(y_1(t) - v_1(t)) + \theta_2(t)(y_2(t) - v_2(t))\Big\}$$

$$\leq -a_{11}^l |x(t) - u(t)| - \Big(\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u\Big)|y_1(t) - v_1(t)|$$

$$- \Big(\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u\Big)|y_2(t) - v_2(t)| - \Big(r_2^l + 2ma_{22}^l - a_{12}^u - b_1^u - b_2^u\Big)|y_3(t) - v_3(t)|$$

$$+ a_{21}(t)y_3(t - \tau)|x(t - \tau) - u(t - \tau)| + a_{21}(t)u(t - \tau)|y_3(t - \tau) - v_3(t - \tau)|.$$

$$(4.2)$$

Let

$$L_2(t) = \int_{t-\tau}^{t} a_{21}(s + \tau)y_3(s)|x(s) - u(s)|ds + \int_{t-\tau}^{t} a_{21}(s + \tau)u(s)|y_3(s) - v_3(s)|ds. \qquad (4.3)$$

Then,

$$D^+L_2(t) = a_{21}(t + \tau)y_3(t)|x(t) - u(t)| - a_{21}(t)y_3(t - \tau)|x(t - \tau) - u(t - \tau)|$$

$$+ a_{21}(t + \tau)u(t)|y_3(t) - v_3(t)| - a_{21}(t)u(t - \tau)|y_3(t - \tau) - v_3(t - \tau)|. \qquad (4.4)$$

Let

$$L(t) = L_1(t) + L_2(t). \qquad (4.5)$$

Then,

$$D^+L(t) \leq -\Big(a_{11}^l - Ma_{21}(t + \tau)\Big)|x(t) - u(t)| - \Big(\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u\Big)|y_1(t) - v_1(t)|$$

$$- \Big(\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u\Big)|y_2(t) - v_2(t)|$$

$$- \Big(r_2^l + 2ma_{22}^l - a_{12}^u - b_1^u - b_2^u - Ma_{21}(t + \tau)\Big)|y_3(t) - v_3(t)|$$

$$= -A_1(t)|x(t) - u(t)| - \Big(\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u\Big)|y_1(t) - v_1(t)|$$

$$- \Big(\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u\Big)|y_2(t) - v_2(t)| - A_2(t)|y_3(t) - v_3(t)|,$$

$$(4.6)$$

where $A_1(t) = a_{11}^l - Ma_{21}(t+\tau)$, $A_2(t) = r_2^l + 2ma_{22}^l - a_{12}^u - b_1^u - b_2^u - Ma_{21}(t+\tau)$. By assumption, $\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u > 0$, $\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u > 0$, and there exist $\alpha_1 > 0, \alpha_2 > 0$, $T^* \geq T_6 > 0$, (T_6 is defined in Theorem 3.4), such that $A_1(t) \geq \alpha_1$, $A_2(t) \geq \alpha_2$, for all $t \geq T^*$. Integrating both sides of (4.6) on the interval $[T^*, t]$, we get

$$L(t) + \int_{T^*}^t A_1(s)|x(s) - u(s)|ds + \left(\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u\right)\int_{T^*}^t |y_1(s) - v_1(s)|ds$$

$$+ \left(\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u\right)\int_{T^*}^t |y_2(s) - v_2(s)|ds + \int_{T^*}^t A_2(s)|y_3(s) - v_3(s)|ds \leq L(T^*)$$

$$\Longrightarrow L(t) + \alpha_1 \int_{T^*}^t |x(s) - u(s)|ds + \left(\delta_1^l + d^l C_0^l + 2m\beta_1^l + D_1^l - D_2^u\right)\int_{T^*}^t |y_1(s) - v_1(s)|ds$$

$$+ \left(\delta_2^l + 2m\beta_2^l + D_2^l - D_1^u\right)\int_{T^*}^t |y_2(s) - v_2(s)|ds + \alpha_2 \int_{T^*}^t |y_3(s) - v_3(s)|ds \leq L(T^*)$$

$$\Longrightarrow L(t) \text{ is bounded on } [T^*, \infty) \text{ and also } \int_{T^*}^\infty |x(s) - u(s)|ds < \infty$$

$$\int_{T^*}^\infty |y_i(s) - v_i(s)|ds < \infty \quad (i = 1, 2, 3).$$

$$(4.7)$$

By Theorem 3.3, $|x(s) - u(s)|$, $|y_i(s) - v_i(s)|$ $(i = 1, 2, 3)$ are bounded on $[T^*, \infty)$. On the other hand, it is easy to see that $\dot{x}(t)$, $\dot{u}(t)$, $\dot{y}_i(t)$, and $\dot{v}_i(t)$ $(i = 1, 2, 3)$ are bounded for $t \geq T^*$. Therefore, $|x(t) - u(t)|$, $|y_i(t) - v_i(t)|$ $(i = 1, 2, 3)$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's Lemma [42], we conclude that

$$|x(t) - u(t)| \longrightarrow 0, \quad |y_i(t) - v_i(t)| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \ i = 1, 2, 3. \tag{4.8}$$

This completes the proof. □

From Theorems 3.4 and 4.3 we observe that the diffusion coefficient functions and time delay have no effect on the uniformly persistent of the system (2.12) but they have some effect on the global asymptotic stability of this system.

5. Conclusions

In this paper, we have considered pollution in a nonautonomous Lotka-Volterra type predator-prey system together with diffusional migration among the immature predator population between protective and nonprotective patches. A constant delay among the matured predators due to the gestation of the predators has been incorporated. It is also assumed that the mature adult predators can only contribute to the reproduction of the predator biomass and that the change rate of predators depends on the number of the preys and of the predators presented at some previous time. Here we incorporate a stage-structure for the predators and assume that the toxicants have no effect on the mature species.

The toxic of polluted population comes from the environment. Suppose that environment of patch 1 is polluted nonprotective region, and patch 2 is the protective region for the immature predator population which is only affected by pollution. In order to protect the existence of the polluted population, we can use the artificial method, the one's own purification function of the population, toxin in the body of the individuals in nonprotective patch can be removed, and then we will bring them into protective patch. The scale of the protective zone can be regulated through diffusive coefficients $D_i(t)$, $i = 1, 2$.

We have obtained the survival threshold of each population of this system. Our mathematical results reveal the fact that lower values $\delta_1(t)$ (instantaneous per capita death rate function of nonprotective immature predator population), $\delta_2(t)$ (instantaneous per capita death rate function of protective immature predator population), $C_0(t)$ (concentration of toxicant in the immature predator organism in nonprotective patch at time t), $C_e(t)$ (concentration of toxicant in the environment at time t), and $d(t)$ (dose response parameter of nonprotective immature predator species to the organismal toxicant concentration at time t) lead to make the underlying system uniformly persistent. By constructing an appropriate Lyapunov function, we have derived sufficient conditions to confirm that if this system admits a positive periodic solution, then it is globally asymptotically stable. We have observed that the diffusion coefficient functions and time delay have no effect on the uniformly persistent of the system (2.12) but they have some effect on the global asymptotic stability of this system.

In future work, it would be interesting to expand on simulations by using some current realistic data to estimate the parameters. The collection of data relevant to key parameters by ecologists and environmentalists at a global level would be of significant use from an implementation and policy perspective.

Acknowledgments

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