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Research Article

Linear Independence of q-Logarithms over the Eisenstein Integers

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For fixed complex q with $|q| > 1$, the q-logarithm L_q is the meromorphic continuation of the series $\sum_{n>0} z^n/(q^n-1)$, $|z| < |q|$, into the whole complex plane. If K is an algebraic number field, one may ask if $1, L_q(1), L_q(c)$ are linearly independent over K for $q, c \in K^\times$ satisfying $|q| > 1, c \neq q, q^2, q^3, \ldots$. In 2004, Tachiya showed that this is true in the Subcase $K = \mathbb{Q}$, $q \in \mathbb{Z}$, $c = -1$, and the present authors extended this result to arbitrary integer q from an imaginary quadratic number field K, and provided a quantitative version. In this paper, the earlier method, in particular its arithmetical part, is further developed to answer the above question in the affirmative if K is the Eisenstein number field $\mathbb{Q}(\sqrt{-3})$, q an integer from K, and c a primitive third root of unity. Under these conditions, the linear independence holds also for $1, L_q(c), L_q(c^{-1})$, and both results are quantitative.

1. Introduction and Results

For fixed complex q of absolute value greater than 1, the q-logarithm L_q is defined by the power series

$$L_q(z) := \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1}, \tag{1.1}$$

which converges in $|z| < |q|$ and has the meromorphic continuation

$$z \sum_{n=1}^{\infty} \frac{1}{q^n - z} \tag{1.2}$$

into the whole complex plane. In the early 1990s, the irrationality investigations on this q-logarithm got a fresh impetus by two papers of Borwein [1, 2], where he introduced new analytic tools to demonstrate quantitative versions of the following. If $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and if $c \in \mathbb{Q}^\times \setminus q^{\mathbb{N}}$, then both numbers $L_q(c)$ and $L_{q^2}(c) - L_{q^2}(cq)$ are irrational, with the second result appearing only in [2].

The next important step was made by Tachiya [3], who succeeded in proving, for $q \in \mathbb{Z} \setminus \{0, \pm 1\}$, the linear independence of $1, L_q(1), L_q(-1)$ over $\mathbb{Q}$ using Borwein's function theoretic method from [2]. Shortly later, quantitative refinements of this result and also of the linear independence of $1, L_q(1), L_q'(1)$ were obtained independently by Zudilin [4] and by the present authors [5]; here the dash indicates differentiation with respect to z. Somehow related to Tachiya's above-mentioned theorem is the linear independence over $\mathbb{Q}$ of $1, L_q(\sqrt{q}), L_q(-\sqrt{q})$ for squares $q \in \mathbb{Z} \setminus \{0, 1\}$, which was established in [6]. Another result, proved in [7], is the linear independence of $1, L_q(1), L_{-q}(1)$ for any $q \in \mathbb{Z} \setminus \{0, 1\}$. It should be noted that all these linear independence statements remain true if one replaces $\mathbb{Q}$ by an arbitrary imaginary quadratic number field and if one supposes q to be in its ring of integers.

One starting point of our present work was the question whether we can replace in Tachiya's result the primitive second root of unity (-1, of course) by a primitive third root of unity. As we will see in Theorem 1.2 below, this is indeed true if we study linear independence over the particular quadratic number field $\mathbb{Q}(\sqrt{-3})$. The parameter q has to be from its ring of integers, which is sometimes called ring of Eisenstein integers since Eisenstein (1844) was the first to thoroughly investigate its algebraic properties in the course of his proof of a cubic reciprocity law.

Another interesting question concerns the linear independence of 1 and the values of L_q at both primitive third roots of unity is to be answered quantitatively as follows.

Theorem 1.1. *Let c be a primitive third root of unity, and let O_K denote the ring of integers of $K := \mathbb{Q}(c)$. Then, for any $q \in O_K$ with $|q| > 1$, the numbers $1, L_q(c), L_q(c^{-1})$ are linearly independent over K. Moreover, there exists a constant $\gamma \in \mathbb{R}_+$ depending at most on $|q|$ such that, for any $Q = (Q_0, Q_1, Q_2) \in O_K^3$ with $|Q| := \max(|Q_1|, |Q_2|)$ large enough, the inequality*

$$\left| Q_0 + Q_1 L_q(c) + Q_2 L_q\left(c^{-1}\right) \right| \geq |Q|^{-\eta - \gamma (\log\log|Q|)/(\log|Q|)^{1/2}} \tag{1.3}$$

holds with $\eta := 39.9475\ldots$.

This question arose when preparing our recent work [8], where, as a very particular application, we obtained a quantitative version of the irrationality of the series

$$S_q(a) := \sum_{n=1}^{\infty} \frac{q^n}{q^{2n} + aq^n + 1} \tag{1.4}$$

for $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and rational a (with some necessary exceptions). Namely, it is easily seen that

$$S_q(1) = \frac{L_q(c) - L_q\left(c^{-1}\right)}{c - c^{-1}} \tag{1.5}$$

holds for c as in Theorem 1.1.

Next, we formulate our analogue of Tachiya's result for third roots of unity.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, the numbers* $1, L_q(1), L_q(c)$ *are linearly independent over* K*. Moreover, there exists a constant* $\gamma \in \mathbb{R}_+$ *depending at most on* $|q|$ *such that, for any* $Q = (Q_0, Q_1, Q_2) \in O_K^3$ *with* $|Q| := \max(|Q_1|, |Q_2|)$ *large enough, the inequality*

$$\left|Q_0 + Q_1 L_q(1) + Q_2 L_q(c)\right| \geq |Q|^{-\eta-\gamma(\log\log|Q|)/(\log|Q|)^{1/2}} \tag{1.6}$$

holds with $\eta := 56.6026\ldots$*.*

It should be noted that here the value of η can be slightly decreased using more involved considerations, on which we will briefly comment at the end of Section 4 (see Remarks 4.1 and 4.2).

Of course, Theorems 1.1 and 1.2 together suggest the following problem. Is it true that the four numbers $1, L_q(1), L_q(c), L_q(c^{-1})$ are linearly independent over K assuming the hypotheses of our above results? We have to admit that, at least at the moment, we are not in a position to prove this statement. Another even more tantalizing problem is the natural question if it is possible to prove analogues of Theorems 1.1 and 1.2 for c a primitive fourth or sixth root of unity, the other two cases, where $\mathbb{Q}(c)$ is simultaneously a cyclotomic and an imaginary quadratic number field. On the difficulties with this problem we will make some comments at the end of Section 3 (see Remark 3.4).

To prove Theorems 1.1 and 1.2, we will essentially use our generalization [5] of Borwein's function theoretical method from [2]. In Section 2, the analytical tools are presented in a way suitable for both situations. Sections 3 and 4 contain the necessary arithmetic considerations to conclude the proofs of Theorems 1.1 and 1.2, respectively.

2. The Analytic Construction

The following extensive lemma contains all analytic information we need for the proofs of our main results.

Lemma 2.1. *With* $q, c_1, c_2, Q_1, Q_2 \in \mathbb{C}$ *satisfying* $|q| > \max(1, |c_1|, |c_2|)$ *define the meromorphic function*

$$V(z) := \sum_{k=1}^{\infty}\left(\frac{Q_1}{q^k - c_1 z} + \frac{Q_2}{q^k - c_2 z}\right) \tag{2.1}$$

and consider, for large parameters $L_1, L_2, M, N \in \mathbb{N}$*, the (positively oriented) integral*

$$J := \frac{1}{2\pi i}\oint_{|z|=1} \frac{\prod_{t=1}^{2}\prod_{\ell=1}^{L_t}(q^\ell - c_t z)}{z^M \prod_{n=1}^{N}(1 - q^n z)} V(z)\,dz. \tag{2.2}$$

(a) *Then the following explicit formula holds:*

$$J = \sum_{n=1}^{N} R_n \left(V(1) - \sum_{k=1}^{n} \sum_{t=1}^{2} \frac{Q_t}{q^k - c_t} \right) + \sum_{\kappa+\lambda+\nu=M-1} P_{\kappa,\lambda} \cdot \frac{Q_1 c_1^{\nu} + Q_2 c_2^{\nu}}{q^{\nu+1} - 1}, \tag{2.3}$$

where, for $n = 1, \dots, N$,

$$R_n := (-1)^{N+1-n} q^{(M-L_1-L_2)n+n(n-1)/2} \frac{\prod_{t=1}^{2} \prod_{n<\ell\le n+L_t} (q^{\ell} - c_t)}{\prod_{\nu=1}^{n-1} (q^{\nu} - 1) \cdot \prod_{\nu=1}^{N-n} (q^{\nu} - 1)} \tag{2.4}$$

and all $P_{\kappa,\lambda}$ *in the above triple sum over all* (κ, λ, ν) *of nonnegative integers with* $\kappa + \lambda + \nu = M - 1$ *are in* $\mathbb{Z}[q, c_1, c_2]$.

(b) *Supposing additionally that* $|c_1| = |c_2| = 1, |L_1 - L_2| = 1$ *and defining* $L := \max(L_1, L_2)$, *one has the evaluation*

$$\left| \sum_{n=1}^{N} R_n \right| = |q|^{L^2+MN+O(1)} \tag{2.5}$$

for M *large enough and, the O-constant depending on* $|q|$ *at most.*

(c) *Supposing, moreover, that* $c_1/c_2 \notin q^{\mathbb{Z}}, L_2 = L_1 - 1$ *if* $|Q_1| \le |Q_2|$ *but* $L_2 = L_1 + 1$ *if* $|Q_1| > |Q_2|$, *then the asymptotic formula*

$$|J| = \max(|Q_1|, |Q_2|) \cdot |q|^{-(M+N-2L)L-N^2/2+O(N)} \tag{2.6}$$

holds as soon as L, M, N *have the same order of magnitude and* $M + N - 2L$ *is large enough. Here the O-constant depends on* $|q|$ *and* $|c_1 - c_2|$ *at most.*

Proof. (a) We apply the residue theorem to the integral J defined in (2.2) and use the poles of the integrand in $|z| < 1$ noting that $V(z)$ is holomorphic in $|z| \le 1$, by (2.1) and the hypothesis $|c_1|, |c_2| < |q|$. Thus, we obtain

$$\begin{aligned} J = &-\sum_{n=1}^{N} \frac{q^{(M-L_1-L_2)n} \prod_{t=1}^{2} \prod_{\ell=1}^{L_t} (q^{\ell+n} - c_t)}{\prod_{\nu=1}^{n-1} (1 - q^{-\nu}) \cdot \prod_{\nu=1}^{N-n} (1 - q^{\nu})} \cdot \frac{V(q^{-n})}{q^n} \\ &+ \sum_{\kappa+\lambda+\nu=M-1} \frac{1}{\kappa!} \left(\frac{d}{dz} \right)^{\kappa} \left\{ \prod_{t=1}^{2} \prod_{\ell=1}^{L_t} \left(q^{\ell} - c_t z \right) \right\} \Bigg|_{z=0} \cdot \frac{1}{\lambda!} \left(\frac{d}{dz} \right)^{\lambda} \prod_{n=1}^{N} (1 - q^n z)^{-1} \Bigg|_{z=0} \cdot \frac{V^{(\nu)}(0)}{\nu!}, \end{aligned} \tag{2.7}$$

the first sum coming from the simple poles at q^{-n} $(n = 1, \ldots, N)$ and the second one from the M-fold pole at the origin. The above first sum leads immediately to the n-sum in (2.3) using the definition of R_n in (2.4) and the fact

$$V\left(q^{-n}\right) = q^n \sum_{k=1}^{\infty} \sum_{t=1}^{2} \frac{Q_t}{q^{k+n} - c_t} = q^n \sum_{k>n} \sum_{t=1}^{2} \frac{Q_t}{q^k - c_t} = q^n \left(V(1) - \sum_{k=1}^{n} \sum_{t=1}^{2} \frac{Q_t}{q^k - c_t} \right) \tag{2.8}$$

for any $n \in \mathbb{N}$. In the above triple sum, the factor in front of $V^{(\nu)}(0)/\nu!$ is just what we denoted in (2.3) by $P_{\kappa,\lambda}$, hence $P_{\kappa,\lambda} \in \mathbb{Z}[q, c_1, c_2]$ is evident. From $(d/dz)^{\nu}(q^k - c_t z)^{-1} = \nu! c_t^{\nu} (q^k - c_t z)^{-\nu-1}$ for any $\nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and from (2.1) we simply deduce that

$$\frac{1}{\nu!} V^{(\nu)}(0) = \sum_{k=1}^{\infty} \left(Q_1 c_1^{\nu} + Q_2 c_2^{\nu}\right) q^{-k(\nu+1)}, \tag{2.9}$$

whence, the triple sum in (2.3).

(b) If $c \in \mathbb{C}$ satisfies $|c| = 1$, then we have for any $\ell \in \mathbb{N}$

$$1 - |q|^{-\ell} \leq \left|1 - cq^{-\ell}\right| \leq 1 + |q|^{-\ell} < \left(1 - |q|^{-\ell}\right)^{-1}, \tag{2.10}$$

whence, by (2.4),

$$|R_n| < \gamma_0^4 |q|^{(M-L_1-L_2)n + \sum_{t=1}^{2} \sum_{\ell=n+1}^{n+L_t} \ell - (1/2)(N-n)(N+1-n)} \tag{2.11}$$

with $\gamma_0 := \prod_{\ell=1}^{\infty} (1 - |q|^{-\ell})^{-1}$. Notice that $|R_n|$ can be bounded below by γ_0^{-4} times the same $|q|$-power as in (2.11). Our additional hypothesis on the L_t's means that either $L_1 = L, L_2 = L - 1$ or $L_1 = L - 1, L_2 = L$; hence the exponent of $|q|$ in (2.11) equals

$$(M + 1 - 2L)n + \frac{1}{2} \sum_{t=1}^{2} L_t(L_t + 2n + 1) - \binom{N + 1 - n}{2} = Mn + L^2 - \binom{N + 1 - n}{2}. \tag{2.12}$$

Thus, we have for $n = 2, \ldots, N$

$$\left| \frac{R_{n-1}}{R_n} \right| < \gamma_0^8 |q|^{-M-N-1+n} \leq \gamma_0^8 |q|^{-M-1} \tag{2.13}$$

and the right-hand side is $\leq 1/3$ for M large enough (in terms of $|q|$ only). Under this condition we find

$$\left| \frac{R_{N-1}}{R_N} + \frac{R_{N-1}}{R_N} \cdot \frac{R_{N-2}}{R_{N-1}} + \cdots + \frac{R_{N-1}}{R_N} \cdot \cdots \cdot \frac{R_1}{R_2} \right| < \frac{1}{2}, \tag{2.14}$$

whence, $(1/2)|R_N| < |R_N + R_{N-1} + \cdots + R_1| < (3/2)|R_N|$ and the inequalities $\gamma_0^{-4}|q|^{L^2+MN} < |R_N| < \gamma_0^4 |q|^{L^2+MN}$ (compare (2.11) and (2.12)) establish (2.5).

(c) We may assume that $(Q_1, Q_2) \neq (0,0)$ since otherwise (2.6) is trivial. In contrast to the situation in (a), to evaluate $|J|$ asymptotically from (2.2), we use the poles of the integrand outside the unit circle. As a matter of fact, one can easily show that $-J$ is just the sum of the residues at all these (simple) poles appearing precisely at the points q^{k_t}/c_t with $k_t > L_t$ $(t = 1,2)$. To justify this equality, the estimate $|V(z)| = |Q| \cdot O(H|q|^{-H})$ on $|z| = |q|^{H+(1/2)}$ for large $H \in \mathbb{N}$ is useful; the O-constant depends only on $|q|$. The distinctness of the before-mentioned poles is guaranteed by our hypothesis $c_1/c_2 \notin q^{\mathbb{Z}}$. Thus, we are led to an expression of J as sum

$$Q_1 c_1^{M+N-1-L_2} \sum_{k>L_1} \frac{\prod_{\ell=1}^{L_1}(q^\ell - q^k) \cdot \prod_{\ell=1}^{L_2}(c_1 q^\ell - c_2 q^k)}{q^{Mk} \prod_{n=1}^{N}(c_1 - q^{n+k})} \tag{2.15}$$

plus the same sum, where the subscripts 1 and 2 are interchanged.

Denoting the kth summand in (2.15) by S_k, similar considerations as for (2.11) show the existence of a constant $\gamma_1 > 1$ depending only on $|q|$ and $|c_1 - c_2|$ such that

$$\gamma_1^{-1} |q|^{(2L-1-M-N)k-N(N+1)/2} \leq |S_k| \leq \gamma_1 |q|^{(2L-1-M-N)k-N(N+1)/2} \tag{2.16}$$

holds for every $k > L_1$. This implies that

$$\left|\frac{S_{k+1}}{S_k}\right| \leq \gamma_1^2 |q|^{2L-1-M-N} \tag{2.17}$$

for the same k, and here the right-hand side is bounded by $1/3$, say, since $M + N - 2L$ is supposed to be large enough. As in (b), this leads to

$$\frac{1}{2}|S_{L_1+1}| \leq \left|\sum_{k>L_1} S_k\right| \leq \frac{3}{2}|S_{L_1+1}| \tag{2.18}$$

for the sum in (2.15). Thus, the absolute value of term (2.15) is bounded above by

$$\left(\frac{3\gamma_1}{2}\right)|Q_1||q|^{-(2L-1-M-N)(L_1+1)-N(N+1)/2} \tag{2.19}$$

and below by the same expression with $3\gamma_1/2$ replaced by $1/(2\gamma_1)$ (compare (2.16)).

If $Q_1 Q_2 = 0$, we may assume that $Q_1 \neq 0, Q_2 = 0$ without loss of generality. Then we have $L_1 = L - 1, L_2 = L$, by one of our additional hypotheses in (c), and (2.15) and (2.19) together with the remark after (2.19) lead to (2.6) noting that all quotients of any two of L, M, N are bounded above and below by certain absolute constants.

Suppose finally that $Q_1Q_2 \neq 0$. Denoting term (2.15) by J_1 and term (2.15) with subscripts 1 and 2 interchanged by J_2, we know that $J_1J_2 \neq 0$ (see (2.19) and the remark thereafter). Then, in the case $|Q_1| \leq |Q_2|$, we have $|J| = |J_2||1 + (J_1/J_2)|$ and

$$\left|\frac{J_1}{J_2}\right| \leq \frac{(3\gamma_1/2)|Q_1|}{(1/(2\gamma_2))|Q_2|}|q|^{(2L-1-M-N)(L_1-L_2)} \leq 3\gamma_1\gamma_2|q|^{2L-1-M-N}, \tag{2.20}$$

by (2.19) and the corresponding lower bound for $|J_2|$, where the constant γ_1 from (2.19) is replaced by γ_2. Hence $|J_1/J_2| \leq 1/2$ as soon as $M + N - 2L$ is large enough, and this leads to $(1/2)|J_2| \leq |J| \leq (3/2)|J_2|$ giving (2.6) for $|Q_1| \leq |Q_2|$ if we use for $|J_2|$ evaluation (2.19) and thereafter with subscript 1 replaced by 2. The case $|Q_1| > |Q_2|$ is treated analogously. □

3. Proof of Theorem 1.1

For c as in Theorem 1.1, we have $(X - c)(X - c^{-1}) = X^2 + X + 1$, which is the cyclotomic polynomial $\Phi_3(X)$. The main arithmetical tool for the proof of this theorem concerns certain divisibility properties of the polynomial $\Phi_3(X^k)$ and is contained in Lemma 3.2, the proof of which will be prepared in the following auxiliary result.

Lemma 3.1. *For any $k \in \mathbb{N}$, one has*

$$\Phi_3\left(X^k\right) = X^{2k} + X^k + 1 = \prod_{d|k,3d\nmid k} \Phi_{3d}(X). \tag{3.1}$$

Proof. From the well-known formula $X^k - 1 = \prod_{d|k}\Phi_d(X)$ we find

$$X^{3k} - 1 = \prod_{d|3k,3|d} \Phi_d(X) \cdot \prod_{d|3k,3\nmid d} \Phi_d(X) = \prod_{\delta|k}\Phi_{3\delta}(X) \cdot \prod_{d|k,3\nmid d} \Phi_d(X) \tag{3.2}$$

whence,

$$\Phi_3\left(X^k\right) = \frac{X^{3k} - 1}{X^k - 1} = \frac{\prod_{d|k}\Phi_{3d}(X) \cdot \prod_{d|k,3\nmid d}\Phi_d(X)}{\prod_{d|k,3\nmid d}\Phi_d(X) \cdot \prod_{d|k,3|d}\Phi_d(X)}, \tag{3.3}$$

and after cancellation we obtain the desired result. □

Lemma 3.2. *For any $n \in \mathbb{N}$, all polynomials $\Phi_3(X^k)$ $(k = 1, \dots, n)$ divide (in $\mathbb{Z}[X]$) the product*

$$\prod_{n<\ell\leq 2n} \Phi_3\left(X^\ell\right). \tag{3.4}$$

Proof. According to Lemma 3.1, product (3.4) equals

$$\prod_{n<\ell\leq 2n}\prod_{\delta|\ell,3\delta\nmid\ell} \Phi_{3\delta}(X). \tag{3.5}$$

Since the factors on the right-hand side of (3.1) are pairwise coprime in $\mathbb{Z}[X]$, it suffices to show that each $\Phi_{3d}(X)$ with $d|k, 3d \nmid k$ appears in product (3.5).

Clearly, $\Phi_{3k}(X)$ divides $\Phi_3(X^k)$. If $k > n/2$, then we put $\ell^* := 2k$ and have $n < \ell^* \leq 2n$, and we consider the contribution

$$\prod_{\delta|\ell^*, 3\delta\nmid\ell^*} \Phi_{3\delta}(X) \tag{3.6}$$

to product (3.5), in which $\Phi_{3k}(X)$ obviously occurs. We now suppose that $d \mid k, d \neq k$, or $d = k$ but $k \leq n/2$. In the first case, we have $dd' = k$ with some integer $d' \geq 2$, hence $d \leq k/2$. Thus, in both remaining situations, we have $d \leq n/2$. Then at least two successive multiples of d are in the ℓ-set $\{n+1, \ldots, 2n\}$ appearing in (3.4). Take two successive multiples of those multiples, ρd and $(\rho+1)d$. Then $\rho, \rho+1$ are not both divisible by 3, and with such a number ρ^* we put $\ell^* := \rho^* d$. Clearly, $\Phi_{3d}(X)$ appears in product (3.6), whence it appears also in (3.5). □

We are now in a position to prove Theorem 1.1. With c, q, Q_1, Q_2 as there, we apply Lemma 2.1 to $c_1 = c, c_2 = c^{-1}$, and obtain from (2.1)

$$V(1) = Q_1 L_q(c) + Q_2 L_q\left(c^{-1}\right), \tag{3.7}$$

the "interesting" part of the linear form to be bounded below in Theorem 1.1. On the parameters L_1, L_2 (hence on L), M, N we assume that all conditions mentioned in Lemma 2.1 (a), (b), (c). Rewriting (2.3) as

$$J = Q^* V(1) + P^* \quad \text{with } Q^* := \sum_{n=1}^{N} R_n \tag{3.8}$$

and obvious definition of P^*, it is clear that P^* and Q^* are both in K. Furthermore, $|Q^*|$ and $|J|$ are asymptotically evaluated in (2.5) and (2.6), respectively. Our next aim is to determine a $D \in O_K \setminus \{0\}$ such that $Q := D \cdot Q^*$ and $P := D \cdot P^*$ are both in O_K.

To this purpose, we first remark that the double sum over k and t appearing in (2.3) equals

$$\sum_{k=1}^{n} \frac{Q_1(q^k - c^{-1}) + Q_2(q^k - c)}{\Phi_3(q^k)} \tag{3.9}$$

and R_n from (2.4) equals

$$\pm q^{E(n)} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix}_q \frac{(q^{n+L} - c^*) \prod_{n<l\leq n+L-1} \Phi_3(q^\ell)}{\prod_{\nu=1}^{N-1} (q^\nu - 1)} \tag{3.10}$$

with $E(n) := (M+1-2L)n + n(n-1)/2$, with a suitable $c^* \in \{c, c^{-1}\}$ and the q-binomial coefficient $\begin{bmatrix} N-1 \\ n-1 \end{bmatrix}_q$ being in $\mathbb{Z}[q]$. Assuming furthermore that $L > N$, we see that, for every $n \in \{1, \ldots, N\}$, all numbers $\Phi_3(q^k)$ $(k \leq n)$ appearing in the denominators of (3.9) divide the product $\prod_{n<\ell\leq 2n}\Phi_3(q^\ell)$ (in $\mathbb{Z}[q]$, thus in O_K), by Lemma 3.2, whence they divide the product appearing in the numerators of (3.10).

Secondly, considering the triple sum in (2.3), it is clear that D must contain the factor

$$\operatorname{lcm}\left\{q-1,q^2-1,\ldots,q^M-1\right\}=\prod_{d\leq M}\Phi_d(q)\quad \left(\text{up to a unit in } O_K\right); \tag{3.11}$$

see, for example, [6, Lemma 4(i)]. According to [6, Lemma 3(i)], we have for the denominator appearing in (3.10)

$$\prod_{\nu<N}(q^\nu-1)=\prod_{d<N}\Phi_d(q)^{[(N-1)/d]}=P_N(q)\cdot\prod_{d<N}\Phi_d(q), \tag{3.12}$$

a formula by which $P_N(q)\in O_K$ is defined. Note that, therefore,

$$|P_N(q)|=|q|^{(1/2-3/\pi^2)N^2+O(N\log N)} \tag{3.13}$$

holds with an O-constant depending only on $|q|$.

To sum up, our above considerations and a comparison of (3.11) and (3.12) show that we may take

$$D:=q^{E^*}\cdot P_N(q)\cdot\prod_{d\leq M}\Phi_d(q) \tag{3.14}$$

assuming additionally that $M\geq N-1$, where $E^*\in\mathbb{N}_0$ has to satisfy the inequalities $E(n)+E^*\geq 0\ \ (n=1,\ldots,N)$; see after (3.10). Clearly, $E(n)\geq 0$ holds for all $n\in\mathbb{N}$ if $M\geq 2L-1$. But if $M\leq 2L-2$, then $\min_{n\geq 1}E(n)=-(1/2)(2L-M)(2L-M-1)$ and this minimum is attained at one of the successive positive integers $2L-M-1$ or $2L-M$, which, by $M+N>2L-1\Leftrightarrow N\geq 2L-M$, are both in $\{1,\ldots,N\}$. Therefore we may choose in (3.14)

$$E^*:=\begin{cases}0 & \text{if } M\geq 2L-1,\\ \dfrac{1}{2}(2L-M)(2L-M-1) & \text{if } M\leq 2L-2.\end{cases} \tag{3.15}$$

Collecting all inequalities on L,M,N we met so far, we now choose $L=N+1, M=[\alpha N]$ with suitable $\alpha>1$ to be fixed later. In particular, if $1<\alpha\leq 2$ holds, then $E^*=(1/2)(2-\alpha)^2N^2+O(N)$, by the second alternative in (3.15). Thus, we may write (3.15) as

$$E^*=\varepsilon^*N^2+O(N)\quad\text{with } \varepsilon^*:=\begin{cases}0 & \text{if } \alpha\geq 2,\\ \dfrac{1}{2}(2-\alpha)^2 & \text{if } 1<\alpha\leq 2.\end{cases} \tag{3.16}$$

This, (3.13), and the well-known asymptotic formula for the product in (3.14) lead to

$$|D|=|q|^{\tau N^2+O(N\log N)}\quad\text{with } \tau:=\varepsilon^*+\frac{1}{2}+\frac{3}{\pi^2}\left(\alpha^2-1\right). \tag{3.17}$$

Further, we deduce from (3.8) and (2.5) that

$$|Q^*| = |q|^{\sigma N^2+O(N)} \quad \text{with } \sigma := \alpha + 1 \tag{3.18}$$

and from (2.6)

$$|J| = |Q||q|^{-\beta N^2+O(N)} \quad \text{with } \beta := \alpha - \frac{1}{2} \tag{3.19}$$

with all O-constants depending at most on $|q|$.

With $Q, P \in O_K$ as defined after (3.8), (3.8) is equivalent to

$$D \cdot J = QV(1) + P. \tag{3.20}$$

In the sequel, we have to be sure that this O_K-linear form $QV(1) + P$ is "very small", that is, $D \cdot J$ is "very small". From (3.17) and (3.19) we have the inequalities

$$|Q||q|^{-(\beta-\tau)N^2-\gamma_3 N\log N} \le |D \cdot J| \le |Q||q|^{-(\beta-\tau)N^2+\gamma_3 N\log N}, \tag{3.21}$$

where γ_3 (and all subsequent γ's) is a positive constant depending only on $|q|$. To guarantee the "smallness" of $D \cdot J$, we have to suppose

$$\beta > \tau. \tag{3.22}$$

We now take our linear form $Q_0 + Q_1 L_q(c) + Q_2 L_q(c^{-1}) =: \mathcal{L}$ with large $|Q| = \max(|Q_1|, |Q_2|)$ and define $N \in \mathbb{N}$ uniquely by

$$|q|^{(\beta-\tau)(N-1)^2-\gamma_3(N-1)\log(N-1)} < 2|Q| \le |q|^{(\beta-\tau)N^2-\gamma_3 N\log N}. \tag{3.23}$$

Clearly N becomes large exactly if $|Q|$ does. Combination of the right-hand sides of (3.21) and (3.23) yields the right half of

$$|q|^{-\gamma_4 N\log N} \le |D \cdot J| \le \frac{1}{2}, \tag{3.24}$$

whereas the left half comes from the left-hand inequalities in (3.21) and (3.23).

From our above definition of $\mathcal{L}$ and from (3.7), we see that $\mathcal{L} = Q_0 + V(1)$, whence

$$Q\mathcal{L} = QQ_0 - P + DJ, \tag{3.25}$$

by (3.20). Thus, if $P \ne QQ_0$, then $|Q\mathcal{L}| \ge 1/2$ using the right-hand side of (3.24), whereas $|Q\mathcal{L}| \ge |q|^{-\gamma_4 N\log N}$ if $P = QQ_0$. This remark and the asymptotic evaluation of $|Q| = |DQ^*|$ from (3.17) and (3.18) give us

$$|\mathcal{L}| \ge |q|^{-(\sigma+\tau)N^2-\gamma_5 N\log N}. \tag{3.26}$$

Since (3.23) implies that $\log|Q| = (\beta-\tau)N^2\log|q| + O(N\log N)$, we can eliminate N from the right-hand side of (3.26) with the final result

$$|\mathcal{L}| \geq |Q|^{-\eta-\gamma(\log\log|Q|)/(\log|Q|)^{1/2}} \quad \text{with } \eta := \frac{\sigma+\tau}{\beta-\tau}. \tag{3.27}$$

For the numerical evaluation of η, we have to ensure (3.22), which, by (3.17) and (3.19), is equivalent to

$$\alpha - 1 > \varepsilon^* + \frac{3}{\pi^2}\left(\alpha^2-1\right). \tag{3.28}$$

If $\alpha \geq 2$ this is equivalent to $\alpha < (\pi^2/3) - 1 = 2.28986\cdots =: \alpha_2$; see the definition of ε^* in (3.16). If $1 < \alpha \leq 2$, then (3.28) reads as

$$\alpha - 1 > \frac{1}{2}(2-\alpha)^2 + \frac{3}{\pi^2}\left(\alpha^2-1\right), \tag{3.29}$$

which, after some calculation, yields $\alpha > 1.50852\cdots =: \alpha_1$.

Finally, we minimize η in terms of $\alpha \in]\alpha_1, \alpha_2[=: I$ or, more conveniently,

$$\eta + 1 = \frac{\beta+\sigma}{\beta-\tau}, \tag{3.30}$$

where τ occurs only once. As a function of α, this is positive continuously differentiable in I and tends to $+\infty$ as $\alpha \uparrow \alpha_2$ and $\alpha \downarrow \alpha_1$. Furthermore, its derivative vanishes in I exactly at $\alpha_0 := (1/4)(\sqrt{(121\pi^2-90)/(\pi^2+6)} - 1) = 1.8353799\ldots$, whence (3.30) reaches its minimal value in I exactly at α_0. Evaluation of (3.30) at α_0 yields the value $39.9475\ldots$ for η given in Theorem 1.1.

Remark 3.3. It should be pointed out that evaluation of (3.30) simply at $\alpha = 2$ would lead to the much weaker result $(7\pi^2+18)/(2\pi^2-18) = 50.0729\ldots$ for η.

Remark 3.4. Here we will try to explain to some extent the difficulties, may be unexpected at first glance, of proving an analogue of Theorem 1.1 for primitive fourth or sixth roots of unity c. Let us restrict ourselves to the first case, where K is the Gaussian field $\mathbb{Q}(i)$ and $(X-c)(X-c^{-1}) = X^2+1 = \Phi_4(X)$. In this situation, we have the following analogue of Lemma 3.2: All polynomials $\Phi_4(X^k)$ $(k = 1,\ldots,n)$ divide the product

$$\prod_{n<\ell\leq 3n} \Phi_4\left(X^\ell\right); \tag{3.31}$$

moreover, the upper bound $3n$ under the product sign cannot be replaced by something smaller. (Remember that in (3.4) this upper bound was $2n$.) Thus, we have to assume $L > 2N$ (instead of $L > N$ before) to guarantee that, for every $n \in \{1,\ldots,N\}$, all $\Phi_4(q^k)$ $(k \leq n)$

divide $\prod_{n<\ell\leq 3n}\Phi_4(q^\ell)$. Then the denominator D from (3.14) remains unchanged, and we try it with the parameter choice

$$L = 2N + 1, \qquad M = [\alpha N], \tag{3.32}$$

where we need $\alpha > 3$ since $M + N - 2L$ has to be large (see (2.6)). The crucial question is, of course, if we can ensure inequality (3.22) by a suitable choice of α. Clearly, comparing (2.6) and (3.19), we now have to work with $\beta = 2\alpha - (11/2)$ and with τ from (3.17) but with $\varepsilon^* = 0$ if $\alpha \geq 4$ and $\varepsilon^* = (1/2)(4-\alpha)^2$ if $4 \geq \alpha > 3$ (see (3.15) and (3.16)). It is easily checked that, unfortunately, there is no $\alpha > 3$, for which (3.22) holds.

It should be added that the situation becomes even worse if c is a primitive sixth root of unity. Again the analogue of Lemma 3.2 is the main obstacle.

4. Proof of Theorem 1.2

With c, q, Q_1, Q_2 as there, we now apply Lemma 2.1 to $c_1 = 1, c_2 = c$ and obtain $V(1) = Q_1 L_q(1) + Q_2 L_q(c)$ from (2.1), and this is again the main part of the O_K-linear form to be estimated from below in Theorem 1.2. Plainly, (3.8) remains valid but with new $P^*, Q^* \in K$, namely,

$$Q^* = \sum_{n=1}^{N} R_n = \sum_{n=1}^{N} (-1)^{N+1-n} q^{E(n)} \frac{\prod_{\nu=n+1}^{n+L_1}(q^\nu - 1)}{\prod_{\nu=1}^{n-1}(q^\nu - 1)\cdot\prod_{\nu=1}^{N-n}(q^\nu - 1)} \cdot \prod_{\ell=n+1}^{n+L_2}\left(q^\ell - c\right), \tag{4.1}$$

with $E(n)$ after (3.10), and

$$P^* = -\sum_{n=1}^{N} R_n \sum_{k=1}^{n}\left(\frac{Q_1}{q^k - 1} + \frac{Q_2}{q^k - c}\right) + \sum_{\kappa+\lambda+\nu=M-1} P_{\kappa,\lambda}\cdot\frac{Q_1 + Q_2 c^\nu}{q^{\nu+1} - 1}, \tag{4.2}$$

with all $P_{\kappa,\lambda} \in \mathbb{Z}[q, c]$.

Next, we determine a "denominator" D for these P^*, Q^*. Assuming $L \geq N$ it is clear, by $L_1 \geq L - 1 \geq N - 1$, that the quotient appearing on the right-hand side of (4.1) is in $\mathbb{Z}[q]$, whence q^{E^*} is a denominator for any R_n and so for Q^*, if E^* is defined as in (3.15). To multiply away the denominators from the triple sum in (4.2), D must contain also factor (3.11), that is,

$$\prod_{d\leq M}\Phi_d(q). \tag{4.3}$$

Assuming that $M \geq N$, this factor takes also care of the $q^k - 1\ \ (k \leq n)$ appearing in the denominators of the first sum in (4.2). What about the additional $q^k - c\ \ (k \leq n)$ appearing there? Since $q^k - c$ divides $q^{3k} - 1$ in O_K for any $k \in \mathbb{N}$, it is clear: Because all $q^{3k} - 1$ with $k \leq M/3$ divide the above product (3.11), it is enough to take

$$D = q^{E^*}\cdot\prod_{d\leq M}\Phi_d(q)\cdot\prod_{M/3<k\leq N}\left(q^k - c\right), \tag{4.4}$$

where, as always, empty products have to be interpreted as 1.

Choosing L, M exactly as in Section 3 (after (3.15)), we have to suppose again $\alpha > 1$. If $\alpha \geq 3$, then the second product in (4.4) is empty and we obtain $|D| = |q|^{(3\alpha^2/\pi^2)N^2+O(N\log N)}$ hence (3.17) with $\tau = 3\alpha^2/\pi^2$. Since the value of β is as in (3.19), inequality (3.22) would mean $\alpha - 1/2 > 3\alpha^2/\pi^2$, which is never satisfied if $\alpha \geq 3$. Hence we have to check the interval $1 < \alpha < 3$, where we obtain formula (3.17) with

$$\tau = \varepsilon^* + \frac{3}{\pi^2}\alpha^2 + \frac{1}{2}\left(1 - \frac{\alpha^2}{9}\right), \tag{4.5}$$

ε^* being as in (3.16). Omitting the calculations, it is easily seen that the decisive inequality (3.22) holds exactly in the subinterval $]1.9119132\ldots, 2.1734402\ldots[$ of $]1,3[$. So we have to minimize the function (3.30) in this subinterval. Calculation shows that this minimum appears exactly at $\alpha_0 := (1/4)(\sqrt{(54+359\pi^2)/(54-\pi^2)} - 1) = 2.0071097\ldots$ yielding the numerical value $704.2655\ldots$ for η, much worse than our claim in Theorem 1.2.

Of course, our procedure to include in (4.4) the second product to take care of the denominators $q^k - c$ with "large" k's in (4.2) is too trivial. To decrease the last value of η considerably, we consider the even k with $(M/3) < k \leq N$. Writing $k = 2j$, $(M/6) < j \leq (N/2)$ and taking $c = c^4$ into account, we find

$$q^k - c = \left(q^j - c^2\right)\left(q^j + c^2\right). \tag{4.6}$$

If $(N/2) \leq (M/3)$ (or equivalently $\alpha \geq 3/2$), then we know that, for every even k with $(M/3) < k \leq N$, the corresponding factor $q^j - c^2$ in (4.6) is already considered in the first product of (4.4) since it divides $q^{3j} - 1$ in O_K. Hence we may replace the second product in (4.4) by

$$\prod_{(M/3)<k\leq N,\ k\text{ odd}} \left(q^k - c\right) \cdot \prod_{(M/6)<j\leq(N/2)} \left(q^j + c^2\right), \tag{4.7}$$

with the absolute value of which being asymptotically

$$|q|^{(3/8)(1-\alpha^2/9)N^2+O(N)}. \tag{4.8}$$

Thus, for our refinement of formula (4.4), we obtain again (3.17) but τ from (4.5) is now replaced by

$$\tau = \varepsilon^* + \frac{3}{\pi^2}\alpha^2 + \frac{3}{8}\left(1 - \frac{\alpha^2}{9}\right). \tag{4.9}$$

After a bit of calculation, we see that (3.22) is satisfied here exactly in the larger subinterval $]1.6507301\ldots, 2.4519699\ldots[$ of $[3/2, 3[$. The minimum of the function (3.30) in this subinterval occurs exactly at $\alpha_0 = (1/4)(\sqrt{(72+1403\pi^2)/(72+11\pi^2)} - 1) = 1.9449636\ldots$ and the corresponding η-value is just the one given in Theorem 1.2.

Remark 4.1. Step by step we can further refine this procedure. One possibility is to extract from the first product in (4.7) certain factors, namely, from those with k divisible by 5. For $k = 5(2j-1)$ we note, by $c = c^{10}$,

$$q^k - c = q^{5(2j-1)} - c^{5\cdot 2} = \left(q^{2j-1} - c^2\right)\left(q^{4(2j-1)} + q^{3(2j-1)}c^2 + \cdots\right). \tag{4.10}$$

Since the factor $q^{2j-1} - c^2$ divides $q^{3(2j-1)} - 1$ in O_K, we know that this factor of $q^k - c$, for odd k divisible by 5, appears already in the product (3.11). Thus, we may refine (4.7) to

$$\prod_{\substack{(M/3)<k\leq N\\ (k,10)=1}} \left(q^k - c\right) \cdot \prod_{\substack{(M/3)<k\leq N\\ k \text{ odd},\, 5|k}} \left(q^{4k/5} + q^{3k/5}c^2 + \cdots\right) \cdot \prod_{(M/6)<j\leq(N/2)} \left(q^j + c^2\right). \tag{4.11}$$

The numerical calculations show that this step reduces the η-value from $56.6026\ldots$ to $52.5205\ldots$.

Remark 4.2. The "limit case" of reducing the k-product in (4.4) still further, possibly to a factor $\Psi \in O_K \setminus \{0\}$ with asymptotic $|\Psi| = |q|^{o(N^2)}$, would yield the following. Let $\alpha > 1$. Then (3.22) holds if and only if $\alpha \in]1.2562872\ldots, 2.6749215\ldots[$ and the optimal choice is $\alpha_0 = (1/4)(\sqrt{(6+105\pi^2)/(6+\pi^2)} - 1) = 1.776073\ldots$ leading to $\eta = 12.8692\ldots$.

References

[1] P. B. Borwein, "On the irrationality of $\sum(1/(q^n + r))$," *Journal of Number Theory*, vol. 37, no. 3, pp. 253–259, 1991.

[2] P. B. Borwein, "On the irrationality of certain series," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 112, no. 1, pp. 141–146, 1992.

[3] Y. Tachiya, "Irrationality of certain Lambert series," *Tokyo Journal of Mathematics*, vol. 27, no. 1, pp. 75–85, 2004.

[4] W. Zudilin, "Approximations to q-logarithms and q-dilogarithms, with applications to q-zeta values," *Zapiski Nauchnykh Seminarov Sankt-Peterburgskoe Otdelenie. Matematicheskiĭ Institut im. V. A. Steklova (POMI)*, vol. 322, pp. 107–124, 2005.

[5] P. Bundschuh and K. Väänänen, "Linear independence of q-analogues of certain classical constants," *Results in Mathematics*, vol. 47, no. 1-2, pp. 33–44, 2005.

[6] P. Bundschuh, "Linear independence of values of a certain Lambert series," *Results in Mathematics*, vol. 51, no. 1-2, pp. 29–42, 2007.

[7] P. Bundschuh, "Remarks on linear independence of q-harmonic series" (Russian), to appear in *Fundamental'naya i Prikladnaya Matematika*.

[8] P. Bundschuh and K. Väänänen, "Quantitative linear independence of an infinite product and its derivatives," *Manuscripta Mathematica*, vol. 129, no. 4, pp. 423–436, 2009.