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Research Article

Generalized Fuzzy Interior Ideals in Abel Grassmann's Groupoids

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Using the notion of a fuzzy point and its belongness to and quasicoincidence with a fuzzy subset, some new concepts of a fuzzy interior ideal in Abel Grassmann's groupoids S are introduced and their interrelations and related properties are invesitigated. We also introduce the notion of a strongly belongness and strongly quasicoincidence of a fuzzy point with a fuzzy subset and characterize fuzzy interior ideals of S in terms of these relations.

1. Introduction

The idea of a quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [1, 2], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation ($\in$) and "quasicoincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and they introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Davvaz [3, 4] introduced the concept of $(\in, \in \vee q)$-fuzzy sub-near-rings (R-subgroups, ideals) of a near-ring and investigated some of their interesting properties. Jun and Song [5] discussed general forms of fuzzy interior ideals in semigroups. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal [6]. Also Davvaz and many others used this concept in several other algebraic structures (see [7–16]). Jun [13, 17], gave the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebras. In [18], Luo introduced the concept of a strong neighborhood. According to him, a fuzzy point x_λ $(0 < \lambda < 1)$ is said to be strongly belong to a fuzzy subset F, denoted by $x_\lambda \underline{\in} F$, if and only if $F(x) > \lambda$. λ-strong cut set $F_{\underline{\lambda}}$ of F is given by $F_{\underline{\lambda}} = \{x \in X \mid F(x) > \lambda\}$, where X is a nonempty set. The idea of Q-neighborhood in fuzzy topology was introduced by Pu and Liu in [19]. According to them, a fuzzy point x_λ is said to be strongly quasicoincident with F, denoted by $x_\lambda \underline{q} F$, if and only if $\lambda + F(x) > 1$.

An *Abel Grassmann's groupoid*, abbreviated as *AG-groupoid*, is a *groupoid S* whose elements satisfy the left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in S$. An AG-groupoid is the midway structure between a *commutative semigroup* and a groupoid. It is a useful non-associative structure with wide applications in theory of flocks. In an AG-groupoid the *medial law*, $(ab)(cd) = (ac)(bd)$ for all $a, b, c \in S$ (see [20]). If there exists an element e in an AG-groupoid S such that $ex = x$ for all $x \in S$ then S is called an *AG-groupoid with left identity e*. If an AG-groupoid S has the *right identity* then S is a *commutative monoid*. If an AG-groupoid S contains left identity then $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c \in S$. Also $a(bc) = b(ac)$ holds for all $a, b, c \in S$.

In this paper, we define (α, β)-fuzzy interior ideals of an AG-groupoid and give some interesting characterizations of an AG -groupoids in terms of (α, β)-fuzzy interior ideals. We also introduce the notion of $(\underline{\alpha}, \overline{\beta})$-fuzzy interior ideals of an AG-groupoid.

2. Preliminaries

For subsets A, B of an AG-groupoid S, we denote $AB = \{ab \in S \mid a \in A, b \in B\}$. A nonempty subset A of an AG-groupoid S is called an *AG-subgroupoid* of S if $A^2 \subseteq A$. A is called an *interior ideal* of S if $(SA)S \subseteq A$.

Let S be an AG-groupoid. By a fuzzy subset F of S, we mean a mapping, $F : S \to [0,1]$.

For fuzzy subsets F_1 and F_2 of S, define

$$F_1 \circ F_2 : S \longrightarrow [0,1],\ a \longmapsto F_1 \circ F_2(a) = \begin{cases} \displaystyle\bigvee_{a=yz} \min\{F_1(y), F_2(z)\}, & \text{if } a = yz\ (\forall a, x, y \in S), \\ 0, & \text{if } a \neq yz. \end{cases} \tag{2.1}$$

We denote by $\mathcal{F}(S)$ the set of all fuzzy subsets of S. One can easily see that $(\mathcal{F}(S), \circ)$ becomes an AG-groupoid as shown in [21]. The order relation "$\subseteq$" on $\mathcal{F}(S)$ is defined as follows:

$$F_1 \subseteq F_2 \quad \text{iff } F_1(x) \leq F_2(x) \quad \forall x \in S, \quad \forall F_1, F_2 \in \mathcal{F}(S). \tag{2.2}$$

For a nonempty family of fuzzy subsets $\{F_i\}_{i \in I}$, of an AG-groupoid S, the fuzzy subsets $\bigcup_{i \in I} F_i$ and $\bigcap_{i \in I} F_i$ of S are defined as follows:

$$\begin{gathered} \bigcup_{i \in I} F_i : G \longrightarrow [0,1], a \longmapsto \left(\bigcup_{i \in I} F_i\right)(a) := \sup_{i \in I}\{F_i(a)\}, \\ \bigcap_{i \in I} F_i : G \longrightarrow [0,1], a \longmapsto \left(\bigcap_{i \in I} F_i\right)(a) := \inf_{i \in I}\{F_i(a)\}. \end{gathered} \tag{2.3}$$

If I is a finite set, say $I = \{1, 2, \ldots, n\}$, then clearly

$$\bigcup_{i\in I} F_i(a) = \max\{F_1(a), F_2(a), \ldots, F_n(a)\},$$
$$\bigcap_{i\in I} F_i(a) = \min\{F_1(a), F_2(a), \ldots, F_n(a)\}. \tag{2.4}$$

Definition 2.1 (cf. [21]). Let S be an AG-groupoid and F a fuzzy subset of S. Then F is called a *fuzzy interior ideal* of S, if it satisfies the following conditions.

$(B1)$ (for all $x, y \in S$) $(F(xy) \geq \min\{F(x), F(y)\})$.

$(B2)$ (for all $x, y, a \in S$) $(F((xa)y) \geq F(a))$.

Let F be a fuzzy subset of S and $\emptyset \neq A \subseteq S$, then the *characteristic function* χ_A of A is defined as

$$\chi_A : S \longrightarrow [0,1], a \longmapsto \chi_A(a) := \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{if } a \notin A. \end{cases} \tag{2.5}$$

Lemma 2.2 (cf. [21]). *Let S be an AG-groupoid and F a fuzzy subset of S. Then F is a fuzzy interior ideal of S if and only if χ_A is a fuzzy interior ideal of S.*

Let S be an AG-groupoid and F a fuzzy subset of S. Then for every $\lambda \in (0,1]$ the set

$$U(F;\lambda) := \{x \mid x \in S, F(x) \geq \lambda\} \tag{2.6}$$

is called a level set of F.

The proof of the following lemma is easy and we omit it.

Lemma 2.3. *Let S be an AG-groupoid and F a fuzzy subset of S. Then F is a fuzzy interior ideal of S if and only if $U(F;\lambda)(\neq \emptyset)$ is an interior ideal of S for every $\lambda \in (0,1]$.*

3. (α, β)-Fuzzy Interior Ideal

In what follows let S denote an AG-groupoid and let α, β denote any one of $\in, q, \in \vee q, \in \wedge q$.

Let S be an AG-groupoid and F a fuzzy subset of S, then the set of the form

$$F(y) := \begin{cases} \lambda(\neq 0), & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases} \tag{3.1}$$

is called a *fuzzy point* with support x and value λ and is denoted by x_λ. A fuzzy point x_λ is said to *belong to* (resp., *quasicoincident with*) a fuzzy set F, written as $x_\lambda \in F$ (resp., $x_\lambda q F$) if $F(x) \geq \lambda$ (resp., $F(x) + \lambda \geq 1$). If $x_\lambda \in F$ or $x_\lambda q F$, then $x_\lambda \in \vee q F$. The symbol $\overline{\in \vee q}$ means $\in \vee q$

does not hold. A fuzzy point x_λ is said to be *strongly belong to* (resp., *strongly quasicoincident with*) a fuzzy set F, written as $x_\lambda \underline{\in} F$ (resp., $x_\lambda \underline{q} F$) if $F(x) > \lambda$ (resp., $\lambda + F(x) > 1$). If $x_\lambda \underline{\in} F$ or $x_\lambda \underline{q} F$, then $x_\lambda \underline{\in} \vee \underline{q} F$. The symbol $\overline{\underline{\in} \vee \underline{q}}$ means that $\underline{\in} \vee \underline{q}$ does not hold.

Every fuzzy interior ideal of S is an $(\in, \in)$-fuzzy interior ideal of S, as shown in the following theorem.

Theorem 3.1. *For any fuzzy subset F of S. The conditions $(B1)$ and $(B2)$ of Definition 2.1, are equivalent to the following.*

$(B3)$ $(\forall x, y \in S)$ $(\forall \lambda_1, \lambda_2 \in (0,1])$ $(x_{\lambda_1} \in F, y_{\lambda_2} \in F \rightarrow (xy)_{\min\{\lambda_1,\lambda_2\}} \in F)$.

$(B4)$ $(\forall x, y, a \in S)$ $(\forall \lambda \in (0,1])(a_\lambda \in F \rightarrow ((xa)y)_\lambda \in F)$.

Proof. $(B1) \rightarrow (B3)$. Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \in F$ and $y_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(y) \geq \lambda_2$. By $(B1)$ we have

$$F(xy) \geq \min\{F(x), F(y)\} \geq \min\{\lambda_1, \lambda_2\}, \tag{3.2}$$

and so $(xy)_{\min\{\lambda_1,\lambda_2\}} \in F$.

$(B3) \rightarrow (B1)$. Let $x, y \in S$. Since $x_{F(x)} \in F$ and $y_{F(y)} \in F$. Then by $(B3)$, we have $(xy)_{\min\{F(x),F(y)\}} \in F$ and so $F(xy) \geq \min\{F(x), F(y)\}$.

$(B2) \rightarrow (B4)$. Let $x, y, a \in S$ and $\lambda \in (0,1]$ be such that $a_\lambda \in F$. Then $F(a) \geq \lambda$. By $(B2)$ we have

$$F((xa)y) \geq F(a) \geq \lambda, \tag{3.3}$$

and so $((xa)y)_\lambda \in F$.

$(B4) \rightarrow (B2)$. Let $x, y \in S$. Since $a_{F(a)} \in F$, by $(B4)$, we have $((xa)y)_{F(a)} \in F$ and so $F((xa)y) \geq F(a)$. □

4. $(\in, \in \vee q)$-Fuzzy Interior Ideals

In [5], Jun and Song introduced the concept of a generalized fuzzy interior ideal of a semigroup. In [12], Jun et al. introduced the concept an (α, β)-fuzzy bi-ideal of an ordered semigroup and characterized ordered semigroups in terms of (α, β)-fuzzy bi-ideals. In this section we define the notions of $(\in, \in \vee q)$-fuzzy interior ideals of an Abel Grassmann's groupoid and investigate some of their properties in terms of $(\in, \in \vee q)$-fuzzy interior ideals.

Let F be a fuzzy subset of S and $F(x) \leq 0.5$ for all $x \in S$. Let $x \in S$ and $\lambda \in (0,1]$ be such that $x_\lambda \in \wedge \mathrm{q} F$. Then $x_\lambda \in F$ and $x_\lambda q F$ and so $F(x) \geq \lambda$ and $F(x) + \lambda \geq 1$. It follows that $1 < F(x) + \lambda \leq F(x) + F(x) = 2F(x)$, and so $F(x) > 0.5$, which is a contradiction. This means that $\{x \in S \mid x_\lambda \in \wedge q F\} = \emptyset$.

Definition 4.1. A fuzzy subset F of S is called an(α, β)-*fuzzy interior ideal* of S, where $\alpha \neq \in \wedge q$, if it satisfies the following conditions:

$(B5)$ $(\forall x, y \in S)$ $(\forall \lambda_1, \lambda_2 \in (0,1])$ $(x_{\lambda_1} \alpha F, y_{\lambda_2} \alpha F \rightarrow (xy)_{\min\{\lambda_1,\lambda_2\}} \beta F)$.

$(B6)$ $(\forall x, y, a \in S)$ $(\forall \lambda \in (0,1])$ $(a_\lambda \alpha F \rightarrow ((xa)y)_\lambda \beta F)$.

Proposition 4.2. *Let F be a fuzzy subset of S. If $\alpha = \in$ and $\beta = \in \vee q$ in Definition 4.1. Then* $(B5)$*, and* $(B6)$*, respectively, of Definition 4.1, are equivalent to the following conditions:*

$(B7)$ $(\forall x, y \in S)(F(xy) \geq \min\{F(x), F(y), 0.5\})$.

$(B8)$ $(\forall x, y, a \in S)(F((xa)y) \geq \min\{F(a), 0.5\})$.

Remark 4.3. A fuzzy subset F of an AG-groupoid S is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if it satisfies conditions (B7), and (B8) of the above proposition.

Using Proposition 4.2, we have the following characterization of $(\in, \in \vee q)$-fuzzy interior ideals of an AG-groupoid.

Lemma 4.4. *Let S be an AG-groupoid and $\emptyset \neq I \subseteq S$. Then I is an interior ideal of S if and only if the characteristic function χ_I of I is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

The converse of Theorem 3.1 is not true in general, as shown in the following example.

Example 4.5. Let $S = \{a, b, c, d, e\}$ be an AG-groupoid with the following multiplication:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

(4.1)

The $(S, \cdot)$ is an AG-groupoid. The interior ideals of S are $\{a\}$ and $\{a, c, d, e\}$. Define a fuzzy subset $F : S \to [0, 1]$ by

$$F(a) = 0.8, \quad F(c) = 0.6, \quad F(d) = 0.4, \quad F(e) = 0.2, \quad F(b) = 0.1. \tag{4.2}$$

Then

$$U(F; \lambda) := \begin{cases} S, & \text{if } \lambda \in (0, 0.1], \\ \{a, c, d, e\}, & \text{if } \lambda \in (0.1, 0.2], \\ \{a\}, & \text{if } \lambda \in (0.6, 1], \\ \emptyset, & \text{if } \lambda \in (0.8, 1]. \end{cases} \tag{4.3}$$

Obviously, F is an $(\in, \in \vee q)$-fuzzy interior ideal of S by Lemma 4.4. But we have the following.

(i) F is not an $(\in, \in)$-fuzzy interior ideal of S, since $d_{0.38} \in F$ but

$$(dd)_{\min\{0.38, 0.38\}} = e_{0.38} \overline{\in} F. \tag{4.4}$$

(ii) F is not an $(\in,q)$-fuzzy interior ideal of S, since $d_{0.36} \in F$ but

$$(dd)_{\min\{0.36,0.36\}} = e_{0.36}\overline{q}F. \tag{4.5}$$

(iii) F is not a $(q,\in)$-fuzzy interior ideal of S, since $c_{0.52}qF$ and $e_{0.82}qF$ but

$$(ce)_{\min\{0.52,0.82\}} = d_{0.52}\overline{\in}F. \tag{4.6}$$

(iv) F is not a $(q,\in\vee q)$-fuzzy bi-ideal of S, since $c_{0.52}qF$ and $e_{0.82}qF$ but

$$(ce)_{\min\{0.52,0.82\}} = d_{0.52}\overline{\in\vee q}F. \tag{4.7}$$

(v) F is not an $(\in\vee q,\in\wedge q)$-fuzzy interior ideal of S, since $d_{0.38} \in \vee qF$ but

$$(dd)_{\min\{0.38,0.38\}} = e_{0.38}\overline{\in\wedge q}F. \tag{4.8}$$

(vi) F is not an $(\in\vee q,q)$-fuzzy interior ideal of S, since $c_{0.56} \in \vee qF$ and $e_{0.18} \in \vee \mathrm{q}F$ but

$$(ce)_{\min\{0.56,0.18\}} = d_{0.18}\overline{q}F. \tag{4.9}$$

(vii) F is not an $(\in\vee q,\in)$-fuzzy interior ideal of S, since $d_{0.38} \in \vee qF$, but

$$(dd)_{\min\{0.38,0.38\}} = d_{0.38}\overline{\in}F. \tag{4.10}$$

(viii) F is not $(\in\wedge q,\in)$-fuzzy interior ideal of S, $d_{0.38} \in \wedge qF$, but

$$(dd)_{\min\{0.38,0.38\}} = d_{0.38}\overline{\in}F. \tag{4.11}$$

(ix) F is not a (q,q)-fuzzy interior ideal of S, since $c_{0.52}qF$ and $e_{0.82}qF$ but

$$(ce)_{\min\{0.52,0.82\}} = d_{0.52}\overline{q}F. \tag{4.12}$$

(x) F is not an $(\in, \in \vee q)$-fuzzy interior ideal of S, since $c_{0.52} \in F$ and $e_{0.82} \in F$ but

$$(ce)_{\min\{0.52,0.82\}} = d_{0.52}\overline{\in \vee q}F. \tag{4.13}$$

(xi) F is not an $(\in \vee q, \in \vee q)$-fuzzy interior ideal of S, $c_{0.58} \in F$ and $e_{0.86} \in F$ but

$$(ce)_{\min\{0.58,0.86\}} = d_{0.58}\overline{\in \vee q}F. \tag{4.14}$$

Remark 4.6. By Remark 4.3, every fuzzy interior ideal of an AG-groupoid S is an $(\in, \in \vee q)$-fuzzy interior ideal of S. However, the converse is not true, in general.

Example 4.7. Consider the AG-groupoid given in Example 4.5, and define a fuzzy subset $F : S \to [0,1]$ by

$$F(a) = 0.8, \quad F(c) = 0.6, \quad F(d) = 0.4, \quad F(e) = 0.2, \quad F(b) = 0.1. \tag{4.15}$$

Clearly F is an $(\in, \in \vee q)$-fuzzy interior ideal of S. But F is not an (α, β)-fuzzy interior ideal of S as shown in Example 4.5.

Theorem 4.8. *Every $(\in, \in)$-fuzzy interior ideal of S is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. It is straightforward. □

Theorem 4.9. *Every $(\in \vee q, \in \vee q)$-fuzzy interior ideal of S is $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. Let F be an $(\in \vee q, \in \vee q)$-fuzzy interior ideal of S. Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1}, y_{\lambda_2} \in F$. Then $x_{\lambda_1}, y_{\lambda_2} \in \vee qF$, which implies that $(xy)_{\min\{\lambda_1,\lambda_2\}} \in \vee qF$. Let $x, y, a \in S$ and $\lambda \in (0,1]$ be such that $a_\lambda \in F$. Then $a_\lambda \in \vee qF$, and we have $((xa)y)_\lambda \in \vee qF$. □

Theorem 4.10. *Let F be a nonzero (α, β)-fuzzy interior ideal of S. Then the set $F_0 := \{x \in S \mid F(x) > 0\}$ is an interior ideal of S.*

Proof. Let $x, y \in F_0$. Then $F(x) > 0$ and $F(y) > 0$. Assume that $F(xy) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{F(x)}\alpha F$ and $y_{F(y)}\alpha F$ but $(xy)_{\min\{F(x),F(y)\}}\overline{\beta}F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that $x_1 qF$ and $y_1 qF$ but $(xy)_{\min\{1,1\}} = (xy)_1\overline{\beta}F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $F(xy) > 0$, that is, $xy \in F_0$. Let $a \in F_0$ and $x, y \in S$. Then $F(a) > 0$. Assume that $F((xa)y) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then, $a_{F(a)}\alpha F$ but $((xa)y)_{F(a)}\overline{\beta}F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that a_1qF but $((xa)y)_{\min\{1,1\}} = ((xa)y)_1\overline{\beta}F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $F((xa)y) > 0$, that is, $(xa)y \in F_0$. Consequently, F_0 is an interior ideal of S. □

Theorem 4.11. *Let I be an interior ideal and F a fuzzy subset of S such that*

(1) $(\forall x \in S \setminus I)(F(x) = 0)$,

(2) $(\forall x \in I)(F(x) \geq 0.5)$.

Then

(a) *F is a $(q, \in \vee q)$-fuzzy interior ideal of S,*

(b) *F is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} qF$ and $y_{\lambda_2} qF$. Then $x, y \in I$ and we have $xy \in I$. If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F(xy) \geq 0.5 \geq \min\{\lambda_1, \lambda_2\}$ and hence $(xy)_{\min\{\lambda_1,\lambda_2\}} \in F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then

$$F(xy) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1, \tag{4.16}$$

and so $(xy)_{\min\{\lambda_1,\lambda_2\}} qF$. Therefore $(xy)_{\min\{\lambda_1,\lambda_2\}} \in \vee qF$. Let $x, y, a \in S$ and $\lambda \in (0,1]$ be such that $a_{\lambda_1} qF$. Then $a \in I$ and we have $(xa)y \in (SI)S \subseteq I$. If $\lambda_1 \leq 0.5$, then $F((xa)y) \geq 0.5 \geq \lambda$ and hence $((xa)y)_\lambda \in F$. If $\lambda_1 > 0.5$, then

$$F\big((xa)y\big) + \lambda \geq 0.5 + 0.5 = 1, \tag{4.17}$$

and so $((xa)y)_\lambda qF$. Therefore $((xa)y)_\lambda \in \vee qF$. Therefore F is a $(q, \in \vee q)$-fuzzy interior ideal of S.

Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \in F$ and $y_{\lambda_2} \in F$. Then $x, y \in I$ and we have $xy \in I$. If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F(xy) \geq 0.5 \geq \min\{\lambda_1, \lambda_2\}$ and hence $(xy)_{\min\{\lambda_1,\lambda_2\}} \in F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then

$$F(xy) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1, \tag{4.18}$$

and so $(xy)_{\min\{\lambda_1,\lambda_2\}} qF$. Therefore $(xy)_{\min\{\lambda_1,\lambda_2\}} \in \vee qF$. Now let $x, y, a \in S$ and $\lambda \in (0,1]$ be such that $a_\lambda \in F$. Then $a \in I$ and we have $(xa)y \in I$. If $\lambda \leq 0.5$, then $F((xa)y) \geq 0.5 \geq \lambda$ and hence $((xa)y)_\lambda \in F$. If $\lambda > 0.5$, then

$$F\big((xa)y\big) + \lambda > 0.5 + 0.5 = 1, \tag{4.19}$$

and so $((xa)y)_\lambda qF$. Therefore

$$\big((xa)y\big)_\lambda \in \vee qF, \tag{4.20}$$

and so F is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

From Example 4.5, we see that an $(\in, \in \vee q)$-fuzzy interior ideal is not a $(q, \in \vee q)$-fuzzy interior ideal (Example 4.5, iv).

In the following theorem we give a condition for an $(\in, \in \vee q)$-fuzzy interior ideal to be an $(\in, \in)$-fuzzy interior ideal of S.

Theorem 4.12. *Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of S such that $F(x) < 0.5$ for all $x \in S$. Then F is an $(\in, \in)$-fuzzy interior ideal of S.*

Proof. Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1}, y_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(y) \geq \lambda_2$ and so $F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \min\{\lambda_1, \lambda_2, 0.5\} = \min\{\lambda_1, \lambda_2\}$ and hence $(xy)_{\min\{\lambda_1,\lambda_2\}} \in F$. Now, let $x, y, a \in S$ and $\lambda \in (0,1]$ be such that $a_\lambda \in F$. Then $F(a) \geq \lambda$ and we have

$$F((xa)y) \geq F(a) \geq \lambda; \tag{4.21}$$

consequently, $((xa)y)_\lambda \in F$. Therefore F is an $(\in, \in)$-fuzzy interior ideal of S. □

For any fuzzy subset F of an AG-groupoid S and $\lambda \in (0,1]$, we denote

$$Q(F;\lambda) := \{x \in S \mid x_\lambda q F\}, \qquad [F]_\lambda := \{x \in S \mid x_\lambda \in \vee q F\}. \tag{4.22}$$

Obviously, $[F]_\lambda = U(F;\lambda) \cup Q(F;\lambda)$.

We call $[F]_\lambda$ an $(\in \vee q)$-level interior ideal of F and $Q(F;\lambda)$ a q-level interior ideal of F.

We have given a characterization of $(\in, \in \vee q)$-fuzzy interior ideals by using level subsets (see Proposition 4.2). Now we provide another characterization of $(\in, \in \vee q)$-fuzzy interior ideals by using the set $[F]_\lambda$.

Theorem 4.13. *Let S be an AG-groupoid and F a fuzzy subset of S. Then A is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if $[F]_\lambda$ is an interior ideal of S for all $\lambda \in (0,1]$.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of S. Let $x, y \in [F]_\lambda$ for $\lambda \in (0,1]$. Then $x_\lambda \in \vee q F$ and $y_\lambda \in \vee q F$, that is, $F(x) \geq t$ or $F(x) + t \geq 1$, and $F(y) \geq t$ or $F(y) + t \geq 1$. Since F is an $(\in, \in \vee q)$-fuzzy interior ideal of S, we have

$$F(xy) \geq \min\{F(x), F(y), 0.5\}. \tag{4.23}$$

We discuss the following cases.

Case 1. Let $F(x) \geq \lambda$ and $F(y) \geq \lambda$. If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} = 0.5, \tag{4.24}$$

and hence $(xy)_\lambda \in F$. If $\lambda \leq 0.5$. Then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \lambda, \tag{4.25}$$

and so $(xy)_\lambda \in F$. Hence $(xy)_\lambda \in \vee q F$.

Case 2. Let $F(x) \geq \lambda$ and $F(y) + \lambda \geq 1$. If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}, \tag{4.26}$$

and we discuss the following cases.

If $\lambda > 0.5$, then

$$\begin{aligned} F(xy) &\geq \min\{F(x), F(y), 0.5\} \\ &= \min\{F(y), 0.5\} \\ &\geq \min\{1-\lambda, 0.5\} = 1-\lambda, \end{aligned} \tag{4.27}$$

that is, $F(xy) + t \geq 1$ and thus $(xy)_t qF$. If $\lambda \leq 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \min\{\lambda, 1-\lambda, 0.5\} = \lambda, \tag{4.28}$$

and so $(xy)_\lambda \in F$. Hence $(xy)_\lambda \in \vee qF$.

Case 3. Let $F(x) + \lambda \geq 1$ and $F(y) \geq \lambda$. If $\lambda < 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \min\{F(x), 0.5\} \geq \min\{1-\lambda, 0.5\} = 1-\lambda, \tag{4.29}$$

that is, $F(xy) + \lambda \geq 1$ and hence $(xy)_\lambda qF$. If $\lambda < 0.5$, then

$$\begin{aligned} F(xy) &\geq \min\{F(x), F(y), 0.5\} \\ &\geq \min\{1-\lambda, \lambda, 0.5\} = \lambda, \end{aligned} \tag{4.30}$$

and so $(xy)_\lambda \in F$. Hence $(xy)_\lambda \in \vee qF$.

Case 4. Let $F(x) + \lambda \geq 1$ and $F(y) + \lambda \geq 1$. If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} > \min\{1-\lambda, 0.5\} = 1-\lambda, \tag{4.31}$$

that is, $F(xy) + \lambda \geq 1$ and thus $(xy)_\lambda qF$. If $\lambda \leq 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \min\{1-\lambda, 0.5\} = 0.5 \geq \lambda, \tag{4.32}$$

and so $(xy)_\lambda \in F$. Thus in any case, we have $(xy)_\lambda \in \vee qF$. Therefore $xy \in [F]_\lambda$. Now, let $a \in [F]_\lambda$ for $\lambda \in (0,1]$. Then $a_\lambda \in \vee qF$, that is, $F(x) \geq \lambda$ or $F(x) + \lambda \geq 1$. Since F is an $(\in, \in \vee q)$-fuzzy interior ideal of S, we have

$$F((xa)y) \geq F(a). \tag{4.33}$$

Case 1. Let $F(a) \geq \lambda$. If $\lambda \geq 0.5$, then

$$F((xa)y) \geq F(a) \geq 0.5 \tag{4.34}$$

and hence $((xa)y)_\lambda qF$. If $\lambda < 0.5$, then

$$F\big((xa)y\big) \geq F(a) \geq \lambda, \tag{4.35}$$

and so $((xa)y)_\lambda \in F$. Hence $((xa)y)_\lambda \in \vee qF$.

Case 2. Let $F(a) \geq \lambda$ and $F(a) + \lambda \geq 1$. If $\lambda \geq 0.5$, then

$$F\big((xa)y\big) \geq F(a)\lambda \geq 0.5. \tag{4.36}$$

If $\lambda < 0.5$, then

$$F\big((xa)y\big) \geq F(a) \geq \min\{1-\lambda, 0.5\} = 1-\lambda, \tag{4.37}$$

that is, $F((xa)y) + \lambda \geq 1$ and thus $((xa)y)_\lambda qF$. If $\lambda \leq 0.5$, then

$$F\big((xa)y\big) \geq F(a) \geq \min\{\lambda, 1-\lambda, 0.5\} = \lambda, \tag{4.38}$$

and so $((xa)y)_\lambda \in F$. Hence $((xa)y)_\lambda \in \vee qA$.

Thus in any case, we have $((xa)y)_\lambda \in \vee qF$. Therefore $(xa)y \in [F]_\lambda$.

Conversely, let F be a fuzzy subset of S and let $x, y, a \in S$ be such that $F(xy) < \lambda < \min\{F(x), F(y), 0.5\}$ for some $\lambda \in (0, 0.5]$. Then $x, y \in U(F;\lambda) \subseteq [F]_\lambda$, it implies that $xy \in [F]_\lambda$. Hence $F(xy) \geq \lambda$ or $F(xy) + \lambda \geq 1$, a contradiction. Hence $F(xy) \geq \min\{F(x), F(y), 0.5\}$ for all $x, y \in S$. Now let $F((xa)y) < F(a)$ for some $x, y, a \in S$. Choose λ such that $F((xa)y) < \lambda < F(a)$. Then $a \in U(F;\lambda) \subseteq [F]_\lambda$. It follows that $(xa)y \in [F]_\lambda$. This implies that $F((xa)y) \geq \lambda$ or $F((xa)y) + \lambda > 1$, a contradiction. Hence $F((xa)y) \geq \min\{F(a), 0.5\}$ for all $x, y, a \in S$. By Proposition 4.2, it follows that F is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

$U(F;\lambda)$ and $[F]_\lambda$ are interior ideals of S for all $\lambda \in (0, 1]$, but $Q(F;\lambda)$ is not an interior ideal of G for all $\lambda \in (0, 1]$, in general. As shown in the following example.

Example 4.14. Consider the *AG*-groupoid as given in Example 4.5. Define a fuzzy subset F by

$$F(a) = 0.8, \quad F(c) = 0.6, \quad F(d) = 0.4, \quad F(e) = 0.2, \quad F(b) = 0.1. \tag{4.39}$$

Then $Q(F;\lambda) = \{a, c, d\}$ for all $0.2 < \lambda \leq 0.4$. Since $c_{0.56} \in F$ and $e_{0.18} \in F$ but $(ce)_{\min\{0.56, 0.18\}} = d_{0.18}\overline{q}F$, hence $Q(F;\lambda)$ is not an interior ideal of S for all $\lambda \in (0.2, 0.4]$.

Proposition 4.15. *If $\{F_i\}_{i\in I}$ is a family of $(\in, \in \vee q)$-fuzzy bi-ideals of an AG-groupoid S, then $\bigcap_{i\in I} F_i$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of $(\in, \in \vee q)$-fuzzy bi-ideals of S. Let $x, y \in S$. Then

$$\begin{aligned}\left(\bigcap_{i\in I} F_i\right)(xy) &= \bigwedge_{i\in I} F_i(xy) \geq \bigwedge_{i\in I}\left(F_i(x) \wedge F_i(y)\right)\\ &= \left(\bigwedge_{i\in I} F_i(x) \wedge \bigwedge_{i\in I} F_i(y)\right)\\ &= \left(\bigcap_{i\in I} F_i\right)(x) \wedge \left(\bigcap_{i\in I} F_i\right)(y).\end{aligned} \tag{4.40}$$

Let $x, y, a \in G$. Then

$$\begin{aligned}\left(\bigcap_{i\in I} F_i\right)((xa)y) &= \bigwedge_{i\in I} F_i((xa)y) \geq \bigwedge_{i\in I}\left(F_i(a)\right)\\ &= \left(\bigcap_{i\in I} F_i\right)(a).\end{aligned} \tag{4.41}$$

Thus $\bigcap_{i\in I} F_i$ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

Definition 4.16. Let S be an AG-groupoid and F a fuzzy subset of S. Then F is called a *strongly fuzzy interior ideal* of S, if it satisfies the following conditions.

($B9$) $(\forall x, y \in S)(F(xy) > \min\{F(x), F(y)\})$.

($B10$) $(\forall x, y, a \in S)((F(xa)y) > F(a))$.

Every fuzzy interior ideal of an AG-groupoid S is strongly fuzzy interior ideal of S.

Theorem 4.17. *For any fuzzy subset F of S. The conditions* ($B9$) *and* ($B10$) *of Definition 4.16 are equivalent to the following.*

($B11$) $(\forall x, y \in S)$ $(\forall \lambda_1, \lambda_2 \in (0,1])(x_{\lambda_1} \underline{\in} F, y_{\lambda_2} \underline{\in} F \rightarrow (xy)_{\min\{\lambda_1,\lambda_2\}} \underline{\in} F)$.

($B12$) $(\forall x, y, a \in S)$ $(\forall \lambda \in (0,1])(a_\lambda \underline{\in} F \rightarrow ((xa)y)_\lambda \underline{\in} F)$.

Proof. $(B9) \rightarrow (B11)$. Let F be a fuzzy subset of S. Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \underline{\in} F$, $y_{\lambda_2} \underline{\in} F$. Then $F(x) > \lambda_1$ and $F(y) > \lambda_2$. Using (B9)

$$F(xy) > \min\{F(x), F(y)\} > \min\{\lambda_1, \lambda_2\}, \tag{4.42}$$

and so $(xy)_{\min\{\lambda_1,\lambda_2\}} \in F$.

$(B11) \rightarrow (B9)$. Let $x, y \in S$. Since $x_{F(x)} \underline{\in} F$ and $y_{F(y)} \underline{\in} F$. Then by ($B9$), we have $(xy)_{\min\{F(x),F(y)\}} \underline{\in} F$ and so $F(xy)$ >$\min\{F(x), F(y)\}$.

$(B10) \rightarrow (B11)$. Let $x, y, a \in S$ and $\lambda \in (0,1]$ be such that $a_\lambda \underline{\in} F$. Then $F(a) > \lambda$. By $(B10)$ we have

$$F\big((xa)y\big) > F(a) > \lambda, \tag{4.43}$$

and so $((xa)y)_\lambda \underline{\in} F$.

$(B11) \rightarrow (B10)$. Let $x, y \in S$. Since $a_{F(a)} \underline{\in} F$, by $(B11)$, we have $((xa)y)_{F(a)} \underline{\in} F$ and so $F((xa)y) > F(a)$. □

5. $(\underline{\in}, \underline{\in} \vee \underline{q})$-Fuzzy Interior Ideals

In this section we define the notions of $(\underline{\in}, \underline{\in} \vee \underline{q})$-fuzzy interior ideals of an Abel Grassmann's groupoid and investigate some of their properties in terms of $(\underline{\in}, \underline{\in} \vee \underline{q})$-fuzzy interior ideals.

Let F be a fuzzy subset of S and $F(x) < 0.5$ for all $x \in S$. Let $x \in S$ and $\lambda \in (0,1]$ be such that $x_\lambda \underline{\in} \wedge \underline{q} F$. Then $x_\lambda \underline{\in} F$ and $x_\lambda \underline{q} F$ and so $F(x) > \lambda$ and $F(x) + \lambda > 1$. It follows that $1 < F(x) + \lambda < F(x) + F(x) = 2F(x)$, and so $F(x) > 0.5$, which is a contradiction. This means that $\{x \in S \mid x_\lambda \underline{\in} \wedge \underline{q} F\} = \emptyset$.

Definition 5.1. A fuzzy subset F of S is called an *$(\underline{\alpha}, \underline{\beta})$-fuzzy interior ideal* of S, where $\alpha \neq \underline{\in} \wedge \underline{q}$, if it satisfies the following conditions.

$(B13)$ $(\forall x, y \in S)$ $(\forall \lambda_1, \lambda_2 \in (0,1])(x_{\lambda_1} \underline{\alpha} F, y_{\lambda_2} \underline{\alpha} F \rightarrow (xy)_{\min\{\lambda_1, \lambda_2\}} \underline{\beta} F)$.

$(B14)$ $(\forall x, y, a \in S)$ $(\forall \lambda \in (0,1])(a_\lambda \underline{\alpha} F \rightarrow ((xa)y)_\lambda \underline{\beta} F)$.

Proposition 5.2. *Let F be a fuzzy subset of S. If $\underline{\alpha} = \underline{\in}$ and $\underline{\beta} = \underline{\in} \vee \underline{q}$ in Definition 5.1. Then $(B13)$, and $(B14)$, respectively, of Definition 5.1, are equivalent to the following conditions.*

$(B15)$ $(\forall x, y \in S)(F(xy) \geq \min\{F(x), F(y), 0.5\})$.

$(B16)$ $(\forall x, y, a \in S)(F((xa)y) \geq \min\{F(a), 0.5\})$.

Remark 5.3. A fuzzy subset F of an AG-groupoid S is an $(\underline{\in}, \underline{\in} \vee \underline{q})$-fuzzy interior ideal of S if and only if it satisfies conditions (B15) and (B16) of the above proposition.

Using Proposition 5.2, we have the following characterization of $(\underline{\in}, \underline{\in} \vee \underline{q})$-fuzzy interior ideals of an AG-groupoid.

Lemma 5.4. *Let S be an AG-groupoid and $\emptyset \neq I \subseteq S$. Then I is an interior ideal of S if and only if the characteristic function χ_I of I is an $(\underline{\in}, \underline{\in} \vee \underline{q})$-fuzzy interior ideal of S.*

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