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Research Article

Conditional Processes Induced by Birth and Death Processes

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For birth and death processes with finite state space, we consider stochastic processes induced by conditioning on hitting the right boundary point before hitting the left boundary point. We call the induced stochastic processes the conditional processes. We show that the conditional processes are again birth and death processes when the right boundary point is absorbing. On the other hand, it is shown that the conditional processes do not have Markov property and they are not birth and death processes when the right boundary point is reflecting.

1. Introduction

For one-dimensional diffusion processes on $[0,1]$ related to diffusion models in population genetics, Ewens [1] considered stochastic processes induced by conditioning on hitting the boundary point 1 before hitting the other boundary point 0. The boundary points 0 and 1 are accessible and absorbing boundaries for the diffusion processes that he considered and the induced stochastic processes are again diffusion processes. Then the induced stochastic processes are referred to as the conditional diffusion processes by Ewens [1] (see also [2]). Motivated by this work, Iizuka et al. [3] were concerned with one-dimensional generalized diffusion processes (ODGDPs for brief) on (l_1, l_2) whose speed measures are right-continuous and strictly increasing functions. They considered stochastic processes induced by conditioning on hitting the right boundary point l_2 before hitting the left boundary point l_1. The induced stochastic processes are called the conditional processes. They showed as Theorem 2.1 that the conditional processes are again ODGDPs when the boundary point l_2 is accessible with the absorbing boundary condition (Assertion 1). If the original process $x(t)$ is a one-dimensional diffusion process with the generator

$$L = \frac{a(x)}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \tag{1.1}$$

then the conditional process $x^*(t)$ induced by conditioning on hitting l_2 before hitting l_1 is again a one-dimensional diffusion process and its generator can be expressed as

$$L^* = \frac{a(x)}{2}\frac{d^2}{dx^2} + \left\{ b(x) + a(x)\frac{s_0(x)}{s(x) - s(l_1)} \right\}\frac{d}{dx}. \tag{1.2}$$

Here we put

$$\begin{gathered} s_0(x) = \exp\left\{ -2\int_c^x \frac{b(y)}{a(y)}dy \right\}, \\ s(x) = \int_c^x s_0(y)dy, \end{gathered} \tag{1.3}$$

where c is a point with $l_1 < c < l_2$ (see [4, 5]). On the other hand, Iizuka et al. [3] showed as Theorem 2.2 that the probability distributions of the conditional processes do not satisfy the Chapman-Kolmogorov equation when the boundary point l_2 is accessible with the reflecting boundary condition. Hence the conditional processes cannot be Markov processes when the boundary point l_2 is accessible with the reflecting boundary condition (Assertion 2).

An important class of ODGDPs which is used as stochastic models in various fields is that of birth and death processes. For example, Moran [6] introduced a birth and death process as one of fundamental stochastic models in population genetics called Moran model (we will consider this model in Section 5). However, the speed measure of any birth and death process is not a strictly increasing function (see [7]) and we cannot apply the results of [3] to birth and death processes.

In this paper we prove that Assertions 1 and 2 hold for the case that the speed measure is a nondecreasing step function. The motivation of this paper is to investigate the properties of the conditional processes induced by conditioning on hitting the right boundary point before hitting the left boundary point when the original processes are birth and death processes. The proof of Theorem 2.2 in [3] is analytical (nonprobabilistic) and it is not easy to see that the conditional processes do not satisfy Markov property when the right boundary point is accessible with the reflecting boundary condition. The proof presented in this paper for Assertion 2 is based on the fact that the state space is discrete. The proof is probabilistic and we can see intuitively that the conditional processes do not satisfy Markov property when the right boundary point is reflecting. It is our extra purpose to see this by considering birth and death processes.

In Section 2 we state our results more precisely. Section 3 is devoted to their proofs. In Section 4 we introduce a very simple birth and death process and present concrete expressions of its conditional processes considering all the boundary conditions. Finally we discuss some stochastic models in population genetics and their conditional processes in Section 5.

2. Main Results

Let $\mathfrak{e}$ be an exponentially distributed random variable with the mean 1 and let $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \ldots\}$ be a sequence of independent copies of $\mathfrak{e}$. We put $\tau_0 = 0$ and $\tau_k = \sum_{i=1}^{k} \mathfrak{e}_i$ $(k = 1, 2, \ldots)$. For $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \ldots\}$, an integer N $(N \geq 2)$, and points a_i $(i = 0, 1, \ldots, N)$ such that $a_0 < a_1 < a_2 < \cdots < a_N$, we consider a birth and death process $\mathbb{D} = [X(t), P_x]$ with the state space $\Sigma = \{a_0, a_1, a_2, \ldots, a_N\}$ satisfying the following conditions. For $a_i \in \Sigma$ and $\tau_k < t < \tau_{k+1}$, conditional probabilities conditional on $X(\tau_k) = a_i$ satisfy

$$\begin{gathered} P_x(X(t) = a_i \mid X(\tau_k) = a_i) = 1, \\ P_x(X(\tau_{k+1}) = a_{i+1} \mid X(\tau_k) = a_i) = p_i, \\ P_x(X(\tau_{k+1}) = a_{i-1} \mid X(\tau_k) = a_i) = q_i, \\ P_x(X(\tau_{k+1}) = a_i \mid X(\tau_k) = a_i) = 1 - p_i - q_i = r_i, \end{gathered} \tag{2.1}$$

where $0 \leq p_0 \leq 1$, $p_N = 0$, $q_0 = 0$, $0 \leq q_N \leq 1$, and $p_i > 0$, $q_i > 0$, $r_i \geq 0$ for $i = 1, 2, \ldots, N-1$. Here P_x denotes the probability measure concentrated at the event $\{X(0) = x\}$, that is, $P_x(X(0) = x) = 1$. The end (boundary) point a_0 [resp., a_N] is called to be absorbing or reflecting according to $p_0 = 0$ [resp., $q_N = 0$] or $p_0 > 0$ [resp., $q_N > 0$].

The generator $\mathcal{L}$ of $\mathbb{D}$ is given by

$$\mathcal{L}u(a_i) = p_i\{u(a_{i+1}) - u(a_i)\} - q_i\{u(a_i) - u(a_{i-1})\}, \quad i = 1, 2, \ldots, N-1, \tag{2.2}$$

for $u \in D(\mathcal{L})$, where $D(\mathcal{L})$ is the set of all functions u on Σ such that

$$\begin{gathered} u(a_0) = 0 \quad \text{if } a_0 \text{ is absorbing}, \\ \mathcal{L}u(a_0) = p_0\{u(a_1) - u(a_0)\} \quad \text{if } a_0 \text{ is reflecting}, \\ u(a_N) = 0 \quad \text{if } a_N \text{ is absorbing}, \\ \mathcal{L}u(a_N) = -q_N\{u(a_N) - u(a_{N-1})\} \quad \text{if } a_N \text{ is reflecting}. \end{gathered} \tag{2.3}$$

Here is a proof of (2.2). By means of (2.1), we find that

$$\begin{gathered} E_{a_i}[u(X(t))] = E_{a_i}[u(X(t)); t < \tau_1] + E_{a_i}[u(X(t)); \tau_1 \leq t < \tau_2] + E_{a_i}[u(X(t)); \tau_2 \leq t], \\ E_{a_i}[u(X(t)); t < \tau_1] = u(a_i) P_{a_i}(t < \tau_1) = u(a_i) e^{-t}, \\ E_{a_i}[u(X(t)); \tau_1 \leq t < \tau_2] = \{u(a_{i+1}) p_i + u(a_{i-1}) q_i + u(a_i) r_i\}(1 - e^{-t}) e^{-t}, \\ E_{a_i}[u(X(t)); \tau_2 \leq t] = o(t) \quad \text{as } t \downarrow 0. \end{gathered} \tag{2.4}$$

Therefore we obtain the following:

$$\begin{aligned}\mathcal{L}u(a_i) &= \lim_{t\downarrow 0}\frac{E_{a_i}[u(X(t))]-u(a_i)}{t}\\ &= -u(a_i)+u(a_{i+1})p_i+u(a_{i-1})q_i+u(a_i)r_i\\ &= p_i\{u(a_{i+1})-u(a_i)\}-q_i\{u(a_i)-u(a_{i-1})\}\end{aligned} \tag{2.5}$$

(see also [7]).

We show that the birth and death process $\mathbb{D}$ can be described as an ODGDP. We set

$$l_1=\begin{cases}a_0 & \text{if } p_0=0,\\ -\infty & \text{if } p_0>0,\end{cases}\qquad l_2=\begin{cases}a_N & \text{if } q_N=0,\\ \infty & \text{if } q_N>0,\end{cases} \tag{2.6}$$

$$s(x)=\begin{cases}\rho_0(x), & l_1<x\le a_0,\\ \dfrac{x-a_0}{a_1-a_0}, & a_0\le x\le a_1,\\ s(a_i)+\dfrac{q_1q_2\cdots q_i}{p_1p_2\cdots p_i}\cdot\dfrac{x-a_i}{a_{i+1}-a_i}, & a_i\le x\le a_{i+1},\quad i=1,\dots,N-1,\\ s(a_N)+\rho_N(x), & a_N\le x<l_2,\end{cases} \tag{2.7}$$

where ρ_0 [resp., ρ_N] is an increasing continuous function on $(l_1,a_0]$ [resp., $[a_N,l_2)$] such that $\rho_0(a_0)=0$ and $\rho_0(l_1)=-\infty$ [resp., $\rho_N(a_N)=0$ and $\rho_N(l_2)=\infty$]. Further we set

$$m(x)=\begin{cases}-\infty, & x<l_1,\\ -1/p_0, & l_1\le x<a_0,\\ 0, & a_0\le x<a_1,\\ 1/q_1, & a_1\le x<a_2,\\ m(a_i-)+\dfrac{p_1p_2\cdots p_{i-1}}{q_1q_2\cdots q_i}, & a_i\le x<a_{i+1},\quad i=2,\dots,N-1,\\ m(a_N-)+\dfrac{p_1p_2\cdots p_{N-1}}{q_1q_2\cdots q_N}, & a_N\le x<l_2,\\ \infty, & l_2\le x.\end{cases} \tag{2.8}$$

Note that $\{l_1\le x<a_0\}=\emptyset$ [resp., $\{a_N\le x<l_2\}=\emptyset$] if $p_0=0$ [resp., $q_N=0$]. Here s is a real-valued continuous increasing function on $S=(l_1,l_2)$, and m is a right-continuous nondecreasing function on $\mathbb{R}$. They are called the scale function and the speed measure, respectively. We set $m(\{x\})=m(x)-m(x-)$, $m_i=m(\{a_i\})$, and $s_i=s(a_i)$, $i=0,1,\dots,N$. We note that $m_0=\infty$ [resp., $m_N=\infty$] if a_0 [resp. a_N] is absorbing.

For a function f on S, we simply write $f(l_1)$ [resp., $f(l_2)$] in place of $f(l_1+)$ [resp., $f(l_2-)$] provided $f(l_1+)$ [resp., $f(l_2-)$] exists. Further, f^+ [resp., f^-] stands for the right [resp., left] derivative of f with respect to s if it exists, that is, $f^+(x)=\lim_{\varepsilon\downarrow 0}\{f(x+\varepsilon)-f(x)\}/\{s(x+\varepsilon)-s(x)\}$ [resp., $f^-(x)=\lim_{\varepsilon\downarrow 0}\{f(x-\varepsilon)-f(x)\}/\{s(x-\varepsilon)-s(x)\}$].

We set $\Sigma^* = \Sigma \cap S$. Let $D(\mathcal{G})$ be the space of all bounded continuous functions u on S satisfying the following conditions.

($\mathcal{G}$.1) There exist a function f on Σ^* and two constants A_1, A_2 such that

$$u(x) = A_1 + A_2\{s(x) - s(c)\} + \int_{(c,x]} \{s(x) - s(y)\} f(y)\,dm(y), \quad x \in S. \tag{2.9}$$

($\mathcal{G}$.2) For each $i = 1, 2$, $u(l_i) = 0$ if $|l_i| < \infty$.

Throughout this paper we denote by c an arbitrarily fixed point of Σ^*. The operator $\mathcal{G}$ is defined by the mapping from $u \in D(\mathcal{G})$ to f that appeared in (2.9). The operator $\mathcal{G}$ is called the one-dimensional generalized diffusion operator (ODGDO for brief) with (s, m). It is known that there exists a strong Markov process $\mathbb{D}^*$ with the generator $\mathcal{G}$, which is called an ODGDP on S (see [8, 9]). It is also known that $\mathbb{D}$ can be identified with $\mathbb{D}^*$ (see [7–9]). Indeed, it is easy to see that $u \in D(\mathcal{G})$ satisfies the following:

$$\begin{aligned}
u^+(a_i) = u^-(a_{i+1}) &= \frac{u(a_{i+1}) - u(a_i)}{s_{i+1} - s_i}, \quad i = 0, 1, \ldots, N-1, \\
\mathcal{G}u(a_i) &= \frac{u^+(a_i) - u^-(a_i)}{m_i} \\
&= p_i\{u(a_{i+1}) - u(a_i)\} - q_i\{u(a_i) - u(a_{i-1})\}, \quad i = 1, 2, \ldots, N-1, \\
u(a_0) &= 0 \quad \text{if } p_0 = 0, \\
\mathcal{G}u(a_0) &= \frac{u^+(a_0)}{m_0} = p_0\{u(a_1) - u(a_0)\} \quad \text{if } p_0 > 0, \\
u(a_N) &= 0 \quad \text{if } q_N = 0, \\
\mathcal{G}u(a_N) &= -\frac{u^-(a_N)}{m_N} = -q_N\{u(a_N) - u(a_{N-1})\} \quad \text{if } q_N > 0.
\end{aligned} \tag{2.10}$$

In order to make the boundary conditions at a_0 and a_N clear, we use $\mathbb{D}^{IJ}$ and P_x^{IJ} in place of $\mathbb{D}$ and P_x, respectively. Here $I, J \in \{A, R\}$, and $I = A$ [resp., $J = A$] means that a_0 [resp., a_N] is absorbing (i.e., $p_0 = 0$ [resp., $q_N = 0$]) and $I = R$ [resp., $J = R$] means that a_0 [resp., a_N] is reflecting (i.e., $p_0 > 0$ [resp., $q_N > 0$]). It is known that there is the transition probability density $p^{IJ}(t, x, y)$ of $\mathbb{D}^{IJ}$ with respect to m, that is,

$$P_x^{IJ}(X(t) = y) = p^{IJ}(t, x, y)m(\{y\}), \quad t > 0,\ x, y \in \Sigma^* \tag{2.11}$$

(see [8, 10]).

Let $\Sigma_o = \{a_1, \ldots, a_N\}$ and let σ_a be the first hitting time at a, that is, $\sigma_a = \inf\{t > 0 : X(t) = a\}$. In this paper we consider stochastic processes induced by the following conditional probability:

$$Q_x^{IJ}(\,\cdot\,) = P_x^{IJ}(\,\cdot \mid \sigma_{a_N} < \sigma_{a_0}), \quad x \in \Sigma_o. \tag{2.12}$$

We set

$$h(x) = P_x^{IJ}(\sigma_{a_N} < \sigma_{a_0}), \quad x \in \Sigma. \tag{2.13}$$

It is known that

$$h(x) = \frac{s(x) - s(a_0)}{s(a_N) - s(a_0)} = \frac{s(x)}{s_N}, \quad x \in \Sigma \tag{2.14}$$

(see [8]). We note that h is independent of boundary conditions $I, J \in \{A, R\}$.

First we show that Q^{IA} induces a birth and death process for $I \in \{A, R\}$.

Theorem 2.1. *Assume that a_N is absorbing. Then $Q_x^{IA}(X(t) = y)$ is independent of $I \in \{A, R\}$, and it is represented as*

$$Q_x^{IA}(X(t) = y) = \frac{h(y)}{h(x)} P_x^{AA}(X(t) = y) = \frac{1}{s(x)} p^{AA}(t, x, y) s(y) m(\{y\}), \tag{2.15}$$

for $t > 0$ and $x, y \in \Sigma_o \setminus \{a_N\}$. Further Q_x^{IA} induces a birth and death process $\mathbb{D}_o$ on Σ_o for which the end point a_1 is reflecting, the end point a_N is absorbing, and the generator $\mathcal{L}_o$ is given by

$$\begin{aligned} \mathcal{L}_o u(a_i) &= p_i\{u(a_{i+1}) - u(a_i)\} - q_i\{u(a_i) - u(a_{i-1})\} \\ &\quad + \frac{q_1 \cdots q_i}{p_1 \cdots p_i} \cdot \frac{p_i}{s_i}\{u(a_{i+1}) - u(a_{i-1})\}, \quad i = 2, \dots, N-1, \end{aligned} \tag{2.16}$$

for $u \in D(\mathcal{L}_o)$, where $D(\mathcal{L}_o)$ is the set of all functions u on Σ_o such that

$$\mathcal{L}_o u(a_1) = (p_1 + q_1)\{u(a_2) - u(a_1)\}, \tag{2.17}$$

$$u(a_N) = 0. \tag{2.18}$$

Theorem 2.1 shows that the relation between (1.1) and (1.2) for diffusion processes corresponds to the relation between (2.2) and (2.16) for birth and death processes. We note that the generator is given by (2.17) and the boundary condition (2.18) when $N = 2$.

Remark 2.2. By means of (2.7),

$$\begin{aligned} \frac{p_i s_{i+1}}{s_i} &= p_i + \frac{q_1 \cdots q_i}{p_1 \cdots p_i} \cdot \frac{p_i}{s_i}, \qquad \frac{q_i s_{i-1}}{s_i} = q_i - \frac{q_1 \cdots q_{i-1}}{p_1 \cdots p_{i-1}} \cdot \frac{q_i}{s_i}, \\ p_i \frac{s_{i+1}}{s_i} + q_i \frac{s_{i-1}}{s_i} &= p_i + q_i \leq 1, \qquad 1 - p_i \frac{s_{i+1}}{s_i} - q_i \frac{s_{i-1}}{s_i} = r_i. \end{aligned} \tag{2.19}$$

Combining these with (2.16), we find that Q_x^{IA} satisfies the following. For $x \in \Sigma_o \setminus \{a_N\}$, $\tau_k \quad (k = 1, 2, \ldots)$, and $\tau_k < t < \tau_{k+1}$,

$$\begin{aligned} Q_x^{IA}(X(t) = a_i \mid X(\tau_k) = a_i) &= 1, \\ Q_x^{IA}(X(\tau_{k+1}) = a_{i+1} \mid X(\tau_k) = a_i) &= p_i \frac{s_{i+1}}{s_i}, \\ Q_x^{IA}(X(\tau_{k+1}) = a_{i-1} \mid X(\tau_k) = a_i) &= q_i \frac{s_{i-1}}{s_i}, \\ Q_x^{IA}(X(\tau_{k+1}) = a_i \mid X(\tau_k) = a_i) &= r_i. \end{aligned} \tag{2.20}$$

We turn to the case that a_N is reflecting. When $m(x)$ is strictly increasing, a representation of Q_x^{IR} is given by (2.11) of [3]. We note that this representation is available even if $m(x)$ is not strictly increasing. Therefore we obtain the following representation for birth and death processes:

$$\begin{aligned} Q_x^{IR}(X(t) = y) = \frac{s(y)}{s(x)} P_x^{AA}(X(t) = y) &+ \frac{s_N}{s(x)} \mathcal{M}(t, x, y) \\ &+ \frac{s_N}{s(x)} \int_0^t \mu_x(u) \mathcal{N}^I(t - u, a_N, y) du, \end{aligned} \tag{2.21}$$

for $I \in \{A, R\}$, $t > 0$, and $x, y \in \Sigma_o \setminus \{a_N\}$. Here $\mu_x(u)$, $\mathcal{M}(t, x, y)$ and $\mathcal{N}^I(t, x, y)$ are given as follows. For $t > 0$ and $x \in \Sigma_o \setminus \{a_N\}$, let

$$\mu_x(t) = -\lim_{y \uparrow a_N} \frac{p^{AA}(t, x, a_N) - p^{AA}(t, x, y)}{s(a_N) - s(y)}. \tag{2.22}$$

For $t > 0$ and $x \in \Sigma_o$, let

$$\nu_x(t) = \lim_{y \downarrow a_0} \frac{p^{AR}(t, x, y) - p^{AR}(t, x, a_0)}{s(y) - s(a_0)}. \tag{2.23}$$

It is known that μ_x and ν_x are nonnegative density functions such as

$$P_x^{IJ}(\sigma_{a_N} < t, \sigma_{a_N} < \sigma_{a_0}) = \int_0^t \mu_x(u) du, \tag{2.24}$$

$$P_x^{IR}(\sigma_{a_0} < t) = \int_0^t \nu_x(u) du \tag{2.25}$$

(see [8]). Note that μ_x is independent of $I, J \in \{A, R\}$ and ν_x is independent of $I \in \{A, R\}$. By virtue of [11], we see that

$$\mu_x(t) = \frac{p^{AA}(t, x, a_{N-1})}{s_N - s_{N-1}}, \qquad \nu_x(t) = \frac{p^{AR}(t, x, a_1)}{s_1}. \tag{2.26}$$

Then we set

$$\mathcal{M}(t,x,y) = \int_0^t \mu_x(u) P_{a_N}^{AR}(X(t-u) = y)\,du,$$
$$\mathcal{N}^I(t,x,y) = \int_0^t \nu_x(u) P_{a_0}^{IR}(X(t-u) = y)\,du. \tag{2.27}$$

We note that $\mathcal{N}^A(\cdot,\cdot,\cdot) = 0$.

The second and the third terms of the right-hand side of (2.21) come from sample path's behavior after hitting the boundary a_N. This representation suggests that Q^{IR} does not satisfy Markov property. Indeed we obtain the following theorem.

Theorem 2.3. *Let $I \in \{A,R\}$ and $0 < t_1 < t_2$. Then*

$$Q_x^{IR}(X(t_2) = y \mid X(t_1) = z) = Q_z^{IR}(X(t_2 - t_1) = y) \tag{2.28}$$

does not hold for some $x, y, z \in \Sigma_o$. This implies that Q_x^{IR} does not satisfy Markov property.

This theorem is proved by applying the following simple proposition for sample path's behavior after hitting the boundary a_N.

Proposition 2.4. *Let $I, J \in \{A,R\}$ and $t > 0$. Then*

$$P_x^{IJ}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) = P_x^{IJ}(X(t) = y) P_x^{IJ}(\sigma_{a_N} < \sigma_{a_0}) \tag{2.29}$$

does not hold for some $x \in \Sigma_o \setminus \{a_N\}$ and $y \in \Sigma_o$.

We prove this proposition in the following section.

3. Proofs of Theorems

We use the same notations as those in Section 2.

3.1. Proof of Theorem 2.1

First we prepare the following lemma. The proof of this lemma is easy and we omit it.

Lemma 3.1. *Let $I, J \in \{A, R\}$, $t > 0$, and $x, y \in \Sigma_o \setminus \{a_N\}$. Then it holds true that*

$$P_x^{IJ}(X(t) = y, t < \sigma_{a_0} \wedge \sigma_{a_N}) = P_x^{AA}(X(t) = y), \tag{3.1}$$

$$P_x^{IJ}(X(t) = y) = P_x^{AA}(X(t) = y) + P_x^{IJ}(X(t) = y, t > \sigma_{a_0} \wedge \sigma_{a_N}), \tag{3.2}$$

$$\begin{aligned} &P_x^{IJ}(X(t) = y, t < \sigma_{a_0} \wedge \sigma_{a_N}, \sigma_{a_N} < \sigma_{a_0}) \\ &\quad = P_x^{IJ}(X(t) = y, t < \sigma_{a_0} \wedge \sigma_{a_N}) P_y^{IJ}(\sigma_{a_N} < \sigma_{a_0}) \\ &\quad = P_x^{AA}(X(t) = y) h(y). \end{aligned} \tag{3.3}$$

Proof of Theorem 2.1. We assume that a_N is absorbing. Let $I \in \{A, R\}$ and $t > 0$. Then

$$Q_{a_N}^{IA}(X(t) = y) = P_{a_N}^{IA}(X(t) = y) = \begin{cases} 1 & \text{if } \ y = a_N, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $x \in \Sigma_o \setminus \{a_N\}$. Then by means of (2.13), and (2.24),

$$\begin{aligned} Q_x^{IA}(X(t) = a_N) &= \frac{P_x^{IA}(X(t) = a_N, \sigma_{a_N} \le t < \sigma_{a_0})}{P_x^{IA}(\sigma_{a_N} < \sigma_{a_0})} \\ &= \frac{1}{h(x)} P_x^{IA}(\sigma_{a_N} \le t < \sigma_{a_0}) = \frac{1}{h(x)} \int_0^t \mu_x(u) du. \end{aligned} \tag{3.5}$$

Let $x, y \in \Sigma_o \setminus \{a_N\}$. Then by using Markov property of $\mathbb{D}^{IA}$, (2.13) and (3.1), we see that

$$\begin{aligned} Q_x^{IA}(X(t) = y) &= \frac{1}{h(x)} P_x^{IA}(X(t) = y, t < \sigma_{a_N} \wedge \sigma_{a_0}, \sigma_{a_N} < \sigma_{a_0}) \\ &= \frac{1}{h(x)} P_x^{IA}(X(t) = y, t < \sigma_{a_N} \wedge \sigma_{a_0}) P_y^{IA}(\sigma_{a_N} < \sigma_{a_0}) \\ &= \frac{h(y)}{h(x)} P^{AA}(X(t) = y). \end{aligned} \tag{3.6}$$

The formulas (3.4), (3.5), and (3.6) show that $Q_x^{IA}(X(t) = y)$ is independent of $I \in \{A, R\}$. The formula (2.15) follows from (2.11), (2.14), and (3.6).

It follows from Theorem 2.2 and Propositions 3.1 and 3.4 of [12] that Q_x^{IA} induces an ODGDP $\mathbb{D}_o$ on (a_0, a_N), the boundary a_0 is entrance in the sense of Feller (see [8, 13]), the boundary a_N is absorbing, and the generator is the ODGDO $\mathcal{G}_o$ with (s_o, m_o), where

$$s_o(x) = \int_c^x s(y)^{-2} ds(y) = \frac{1}{s(c)} - \frac{1}{s(x)}, \tag{3.7}$$

$$m_o(x) = \int_{(c,x]} s(y)^2 dm(y). \tag{3.8}$$

Therefore,

$$\mathcal{G}_o u(a_i) = \left\{ \frac{u(a_{i+1}) - u(a_i)}{s_o(a_{i+1}) - s_o(a_i)} - \frac{u(a_i) - u(a_{i-1})}{s_o(a_i) - s_o(a_{i-1})} \right\} \{m_o(a_i) - m_o(a_i-)\}^{-1}, \tag{3.9}$$

for a function u on $\Sigma_o \setminus \{a_N\}$ and $i = 2, 3, \ldots, N-1$. By means of (2.7), (2.8), and (3.7), we see that

$$\{s_o(a_{i+1}) - s_o(a_i)\}\{m_o(a_i) - m_o(a_i-)\} = \left(\frac{1}{s_i} - \frac{1}{s_{i+1}}\right) s_i^2 m_i = (s_{i+1} - s_i)\frac{s_i m_i}{s_{i+1}} = \frac{s_i}{s_{i+1} p_i}. \tag{3.10}$$

In the same way, we have

$$\{s_o(a_i) - s_o(a_{i-1})\}\{m_o(a_i) - m_o(a_i-)\} = \frac{s_i}{s_{i-1} q_i}. \tag{3.11}$$

Therefore we get

$$\begin{aligned} \mathcal{G}_o u(a_i) &= \frac{s_{i+1}}{s_i}\, p_i \{u(a_{i+1}) - u(a_i)\} - \frac{s_{i-1}}{s_i}\, q_i \{u(a_i) - u(a_{i-1})\} \\ &= p_i \{u(a_{i+1}) - u(a_i)\} - q_i \{u(a_i) - u(a_{i-1})\} + \frac{q_1 \ldots q_i}{p_1 \ldots p_i} \cdot \frac{p_i}{s_i} \{u(a_{i+1}) - u(a_{i-1})\}, \end{aligned} \tag{3.12}$$

for $i = 2, 3, \ldots, N-1$. Since a_0 is entrance, we see that

$$\begin{aligned} \mathcal{G}_o u(a_1) &= \frac{u(a_2) - u(a_1)}{s_o(a_2) - s_o(a_1)} \{m_o(a_i) - m_o(a_i-)\}^{-1} \\ &= \frac{s_2}{s_1} p_1 \{u(a_2) - u(a_1)\} = (p_1 + q_1)\{u(a_2) - u(a_1)\}, \end{aligned} \tag{3.13}$$

by virtue of general theory on ODGDOs. Thus we find that $\mathbb{D}_o$ is a birth and death process on Σ_o, the generator is given by (2.16), the end point a_1 is reflecting with (2.17), and the end point a_N is absorbing. The proof is completed. □

3.2. Proof of Theorem 2.3

We introduce the Green function corresponding to $\mathbb{D}^{IJ}$. For $I, J \in \{A, R\}$, $k = 1, 2$, and $\alpha > 0$, let $g_k^{IJ}(\cdot, \alpha)$ be a continuous function on S satisfying the following properties:

$$g_k^{IJ}(\cdot, \alpha) > 0 \text{ on } S, \tag{3.14}$$

$$g_1^{IJ}(\cdot, \alpha) \text{ is nondecreasing and } g_2^{IJ}(\cdot, \alpha) \text{ is nonincreasing on } S, \tag{3.15}$$

$$g_1^{AJ}(a_0, \alpha) = 0, \tag{3.16}$$

$$g_1^{RJ,+}(a_0, \alpha) = g_1^{RJ}(a_1, \alpha) - g_1^{RJ}(a_0, \alpha) = \alpha g_1^{RJ}(a_0, \alpha) m_0, \tag{3.17}$$

$$g_2^{IA}(a_N, \alpha) = 0, \tag{3.18}$$

$$g_2^{IR,-}(a_N, \alpha) = \frac{g_2^{IR}(a_N, \alpha) - g_2^{IR}(a_{N-1}, \alpha)}{s_N - s_{N-1}} = -\alpha g_2^{IR}(a_N, \alpha) m_N, \tag{3.19}$$

$$\begin{aligned} g_k^{IJ}(x, \alpha) &= g_k^{IJ}(c, \alpha) + g_k^{IJ,+}(c, \alpha)\{s(x) - s(c)\} \\ &\quad + \alpha \int_{(c,x]} \{s(x) - s(y)\} g_k^{IJ}(y, \alpha) dm(y), \quad x \in S. \end{aligned} \tag{3.20}$$

Here $g_k^{IJ,\pm}(x, \alpha) = \lim_{\varepsilon \downarrow 0} \{g_k^{IJ}(x \pm \varepsilon, \alpha) - g_k^{IJ}(x, \alpha)\} / \{s(x \pm \varepsilon) - s(x)\}$. It is known that there exist such functions $g_k^{IJ}(\cdot, \alpha), k = 1, 2$ (see [8]). We set $W^{IJ}(\alpha) = g_1^{IJ,+}(x, \alpha) g_2^{IJ}(x, \alpha) - g_1^{IJ}(x, \alpha) g_2^{IJ,+}(x, \alpha)$. Note that $W^{IJ}(\alpha)$ is a positive number independent of $x \in S$. We put

$$G^{IJ}(\alpha, x, y) = G^{IJ}(\alpha, y, x) = W^{IJ}(\alpha)^{-1} g_1^{IJ}(x, \alpha) g_2^{IJ}(y, \alpha), \tag{3.21}$$

for $\alpha > 0$ and $l_1 < x \le y < l_2$, which is the Green function corresponding to $\mathbb{D}^{IJ}$. It is also known that

$$G^{IJ}(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p^{IJ}(t, x, y) dt, \tag{3.22}$$

for $\alpha > 0$ and $x, y \in \Sigma$ (see [8, 14]).

First we prove Proposition 2.4.

Proof of Proposition 2.4. We divide the proof into four cases.

Case 1. $I = J = A$. Since a_N is absorbing, we find that

$$\begin{aligned} P_x^{AA}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) &= P_x^{AA}(X(t) = y, t < \sigma_{a_0} \wedge \sigma_{a_N}, \sigma_{a_N} < \sigma_{a_0}) \\ &= P_x^{AA}(X(t) = y) h(y), \end{aligned} \tag{3.23}$$

by means of (3.3). Since there are $x, y \in \Sigma_o \setminus \{a_N\}$ such that $x \ne y$ for $N \ge 3$, (3.23) shows that (2.29) does not hold true for $x \ne y$ for $N \ge 3$.

Let $N = 2$. Then

$$P_{a_1}^{AA}(X(t) = a_2, \sigma_{a_2} < \sigma_{a_0}) = P_{a_1}^{AA}(\sigma_{a_2} < \sigma_{a_0}, \sigma_{a_2} < t),$$
$$P_{a_1}^{AA}(X(t) = a_2) P_{a_1}^{AA}(\sigma_{a_2} < \sigma_{a_0}) = P_{a_1}^{AA}(\sigma_{a_2} < \sigma_{a_0}, \sigma_{a_2} < t) P_{a_1}^{AA}(\sigma_{a_2} < \sigma_{a_0}), \tag{3.24}$$

which imply that (2.29) is not valid for $x = a_1$ and $y = a_2$. Thus (2.29) does not hold for some $x, y \in \Sigma_o$.

Case 2. $I = R$ and $J = A$. By means of (3.2), we also see that

$$\begin{aligned} &P_x^{RA}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) - P_x^{RA}(X(t) = y) h(x) \\ &\quad = P_x^{AA}(X(t) = y)\{h(y) - h(x)\} - P_x^{RA}(X(t) = y, t > \sigma_{a_0} \wedge \sigma_{a_N}) h(x). \end{aligned} \tag{3.25}$$

The right-hand side of this formula is negative if $x \geq y$. This implies that (2.29) does not hold true for $x \geq y$.

Case 3. $I = A$ and $J = R$. By means of (3.3),

$$P_x^{AR}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) = P_x^{AA}(X(t) = y) h(y) + P_x^{AR}(X(t) = y, t > \sigma_{a_0} \wedge \sigma_{a_N}, \sigma_{a_N} < \sigma_{a_0}). \tag{3.26}$$

Since a_0 is absorbing, we get

$$\begin{aligned} &P_x^{AR}(X(t) = y, t > \sigma_{a_0} \wedge \sigma_{a_N}, \sigma_{a_N} < \sigma_{a_0}) \\ &\quad = P_x^{AR}(X(t) = y, \sigma_{a_N} < t < \sigma_{a_0}) \\ &\quad = P_x^{AR}(X(t) = y, t > \sigma_{a_0} \wedge \sigma_{a_N}). \end{aligned} \tag{3.27}$$

Combining these equalities with (3.2) and (3.3), we see that

$$\begin{aligned} &P_x^{AR}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) - P_x^{AR}(X(t) = y) h(x) \\ &\quad = P_x^{AA}(X(t) = y)\{h(y) - h(x)\} + P_x^{AR}(X(t) = y, \sigma_{a_N} < t < \sigma_{a_0})\{1 - h(x)\}. \end{aligned} \tag{3.28}$$

The right-hand side of this formula is positive if $x \leq y$. This implies that (2.29) does not hold true for $x \leq y$.

Case 4. $I = J = R$. Suppose that (2.29) holds true for $t > 0$, $x \in \Sigma_o \setminus \{a_N\}$, and $y \in \Sigma_o$. Then

$$\int_0^\infty e^{-\alpha t} H(t, x, y)\, dt = 0, \tag{3.29}$$

for $\alpha > 0$, $x \in \Sigma_o \setminus \{a_N\}$, and $y \in \Sigma_o$, where

$$H(t,x,y) = \frac{1}{m(\{y\})}\Big\{P_x^{RR}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) - P_x^{RR}(X(t) = y)h(x)\Big\}. \tag{3.30}$$

By means of (3.3),

$$\begin{aligned}
&P_x^{RR}(X(t) = y, \sigma_{a_N} < \sigma_{a_0}) \\
&\quad = P_x^{AA}(X(t) = y)h(y) + P_x^{RR}(X(t) = y, t > \sigma_{a_0} \wedge \sigma_{a_N}, \sigma_{a_N} < \sigma_{a_0}) \\
&\quad = P_x^{AA}(X(t) = y)h(y) + \int_0^t P_x^{RR}(\sigma_{a_N} \in du, u < \sigma_{a_0}) P_{a_N}^{RR}(X(t-u) = y),
\end{aligned} \tag{3.31}$$

for $\alpha > 0$ and $x, y \in \Sigma_o \setminus \{a_N\}$.

Combining this with (3.22), we see that

$$\begin{aligned}
0 &= \int_0^\infty e^{-\alpha t} H(t,x,y)dt \\
&= G^{AA}(\alpha,x,y)h(y) + E_x^{RR}[e^{-\alpha\sigma_{a_N}}, \sigma_{a_N} < \sigma_{a_0}]G^{RR}(\alpha,a_N,y) \\
&\quad - G^{RR}(\alpha,x,y)h(x),
\end{aligned} \tag{3.32}$$

for $\alpha > 0$ and $x, y \in \Sigma_o \setminus \{a_N\}$, where E_x^{IJ} stands for the expectation with respect to P_x^{IJ}. Here we note that (3.32) is valid for $y = a_N$. Indeed,

$$\begin{aligned}
P_x^{RR}(X(t) = a_N, \sigma_{a_N} < \sigma_{a_0}) &= P_x^{RR}(X(t) = a_N, \sigma_{a_N} \le t, \sigma_{a_N} < \sigma_{a_0}) \\
&= \int_0^t P_x^{RR}(\sigma_{a_N} \in du, u < \sigma_{a_0}) P_{a_N}^{RR}(X(t-u) = a_N).
\end{aligned} \tag{3.33}$$

Combining this with (3.22), we see that

$$\begin{aligned}
0 &= \int_0^\infty e^{-\alpha t} H(t,x,a_N)dt \\
&= E_x^{RR}[e^{-\alpha\sigma_{a_N}}, \sigma_{a_N} < \sigma_{a_0}]G^{RR}(\alpha,a_N,a_N) - G^{RR}(\alpha,x,a_N)h(x),
\end{aligned} \tag{3.34}$$

which implies (3.32) with $y = a_N$.

Since (3.32) holds true for $x \in \Sigma_o \setminus \{a_N\}$ and $y \in \Sigma_o$, we have

$$\begin{aligned} 0 = &-G^{AA}(\alpha, x, a_{N-1}) h(a_{N-1}) \\ &+ E_x^{RR}\left[e^{-\alpha\sigma_{a_N}}, \sigma_{a_N} < \sigma_{a_0}\right]\left\{G^{RR}(\alpha, a_N, a_N) - G^{RR}(\alpha, a_N, a_{N-1})\right\} \\ &- \left\{G^{RR}(\alpha, x, a_N) - G^{RR}(\alpha, x, a_{N-1})\right\} h(x), \end{aligned} \tag{3.35}$$

for $x \in \Sigma_o \setminus \{a_N\}$. By virtue of (3.17), (3.20), and (3.21), we see that

$$\begin{aligned} &G^{RR}(\alpha, a_N, a_N) - G^{RR}(\alpha, a_N, a_{N-1}) \\ &\quad = W^{RR}(\alpha)^{-1}\left\{g_1^{RR}(a_N, \alpha) - g_1^{RR}(a_{N-1}, \alpha)\right\} g_2^{RR}(a_N, \alpha) \\ &\quad = \alpha W^{RR}(\alpha)^{-1}(s_N - s_{N-1}) g_2^{RR}(a_N, \alpha) \int_{[a_0, a_{N-1}]} g_1^{RR}(z, \alpha)\, dm(z). \end{aligned} \tag{3.36}$$

We take a point $x \in \Sigma_o \setminus \{a_N\}$ such that $x \le a_{N-1}$. Then by virtue of (3.19), (3.20), and (3.21),

$$\begin{aligned} &G^{RR}(\alpha, x, a_N) - G^{RR}(\alpha, x, a_{N-1}) \\ &\quad = W^{RR}(\alpha)^{-1} g_1^{RR}(x, \alpha)\left\{g_2^{RR}(a_N, \alpha) - g_2^{RR}(a_{N-1}, \alpha)\right\} \\ &\quad = -\alpha W^{RR}(\alpha)^{-1}(s_N - s_{N-1}) g_1^{RR}(x, \alpha) \int_{(a_{N-1}, a_N]} g_2^{RR}(z, \alpha)\, dm(z). \end{aligned} \tag{3.37}$$

Thus we obtain that

$$\begin{aligned} 0 = &-G^{AA}(\alpha, x, a_{N-1}) h(a_{N-1}) + E_x^{RR}\left[e^{-\alpha\sigma_{a_N}}, \sigma_{a_N} < \sigma_{a_0}\right] \\ &\times \alpha W^{RR}(\alpha)^{-1}(s_N - s_{N-1}) \sum_{l=0}^{N-1} g_1^{RR}(a_l, \alpha) g_2^{RR}(a_N, \alpha) m_l \\ &+ h(x) \alpha W^{RR}(\alpha)^{-1}(s_N - s_{N-1}) g_1^{RR}(x, \alpha) g_2^{RR}(a_N, \alpha) m_N. \end{aligned} \tag{3.38}$$

It is known that

$$\begin{aligned} \lim_{\alpha \downarrow 0} G^{AA}(\alpha, \xi, \eta) &= \frac{\{s(\xi) - s(a_0)\}\{s(a_N) - s(\eta)\}}{s(a_N) - s(a_0)}, \quad a_0 \le \xi \le \eta \le a_N, \\ \lim_{\alpha \downarrow 0} \alpha G^{RR}(\alpha, \xi, \eta) &= \frac{1}{m(l_2) - m(l_1)}, \quad \xi, \eta \in S \end{aligned} \tag{3.39}$$

(see [8, 9, 14, 15]). Therefore letting $\alpha \downarrow 0$ in (3.38) leads us to

$$0 = \frac{s(x)}{s_N}\{-(s_N - s_{N-1})h(a_{N-1})\} + \frac{h(x)(s_N - s_{N-1})}{m(l_2) - m(l_1)} \sum_{l=0}^{N} m_l$$
$$= \frac{s(x)}{s_N^2}(s_N - s_{N-1})^2. \tag{3.40}$$

This contradicts the fact that the last term is positive. Thus (2.29) does not hold true for $t > 0$, $x \in \Sigma_o \setminus \{a_N\}$, and $y \in \Sigma_o$. □

Remark 3.2. Let $N = 2$, $I = J \in \{A, R\}$, $q_0 = 0$, $p_2 = 0$, and $p_1 = q_1 = 1/2$. Then we see that

$$P_{a_1}^{II}(\sigma_{a_2} < \sigma_{a_0}) = \frac{1}{2},$$
$$P_{a_1}^{II}(X(t) = a_1, \sigma_{a_2} < \sigma_{a_0}) = P_{a_1}^{II}(X(t) = a_1, \sigma_{a_0} < \sigma_{a_2}), \tag{3.41}$$

where $I \in \{A, R\}$. Therefore

$$P_{a_1}^{II}(X(t) = a_1, \sigma_{a_2} < \sigma_{a_0}) = \frac{1}{2}P_{a_1}^{II}(X(t) = a_1) = P_{a_1}^{II}(X(t) = a_1)P_{a_1}^{II}(\sigma_{a_2} < \sigma_{a_0}), \tag{3.42}$$

that is, (2.29) is valid for $x = y = a_1$. Proposition 2.4 implies, however, that (2.29) does not hold for $x = a_1$ and $y = a_2$.

Proof of Theorem 2.3. Let $I \in \{A, R\}$, $0 < t_1 < t_2$, $x, z \in \Sigma_o \setminus \{a_N\}$, and $y \in \Sigma_o$. Then, by using Markov property of $\mathbb{D}^{IR}$, we obtain that

$$Q_x^{IR}(X(t_2) = y \mid X(t_1) = z) = \frac{Q_x^{IR}(X(t_1) = z, X(t_2) = y)}{Q_x^{IR}(X(t_1) = z)}$$
$$= \frac{P_x^{IR}(X(t_1) = z, X(t_2) = y, \sigma_{a_N} < \sigma_{a_0})}{P_x^{IR}(X(t_1) = z, \sigma_{a_N} < \sigma_{a_0})}$$
$$= \Big\{P_x^{AA}(X(t_1) = z)P_z^{IR}(X(t_2 - t_1) = y \mid \sigma_{a_N} < \sigma_{a_0})h(z)$$
$$+P_x^{IR}(X(t_1) = z, \sigma_{a_N} < t_1, \sigma_{a_N} < \sigma_{a_0})P_z^{IR}(X(t_2 - t_1) = y)\Big\}$$
$$\times \Big\{P_x^{AA}(X(t_1) = z)h(z) + P_x^{IR}(X(t_1) = z, \sigma_{a_N} < \mathrm{t}_1, \sigma_{a_N} < \sigma_{a_0})\Big\}^{-1}. \tag{3.43}$$

Therefore (2.28) is equivalent to the following:

$$\begin{aligned}
&P_x^{AA}(X(t_1)=z)P_z^{IR}\left(X(t_2-t_1)=y \mid \sigma_{a_N}<\sigma_{a_0}\right)h(z)\\
&\quad + P_x^{IR}(X(t_1)=z,\sigma_{a_N}<t_1,\sigma_{a_N}<\sigma_{a_0})P_z^{IR}\left(X(t_2-t_1)=y\right)\\
&\quad = \left\{P_x^{AA}(X(t_1)=z)h(z)+P_x^{IR}(X(t_1)=z,\sigma_{a_N}<t_1,\sigma_{a_N}<\sigma_{a_0})\right\}\\
&\quad \times P_z^{IR}\left(X(t_2-t_1)=y \mid \sigma_{a_N}<\sigma_{a_0}\right).
\end{aligned} \tag{3.44}$$

Again (3.44) is equivalent to the following:

$$\begin{aligned}
&P_x^{IR}(X(t_1)=z,\sigma_{a_N}<t_1,\sigma_{a_N}<\sigma_{a_0})P_z^{IR}\left(X(t_2-t_1)=y\right)\\
&\quad = P_x^{IR}(X(t_1)=z,\sigma_{a_N}<t_1,\sigma_{a_N}<\sigma_{a_0})P_z^{IR}\left(X(t_2-t_1)=y \mid \sigma_{a_N}<\sigma_{a_0}\right).
\end{aligned} \tag{3.45}$$

Since $P_x^{IR}(X(t_1)=z,\ \sigma_{a_N}<t_1,\ \sigma_{a_N}<\sigma_{a_0})>0$, (3.45) is equivalent to

$$P_z^{IR}\left(X(t_2-t_1)=y\right)=P_z^{IR}\left(X(t_2-t_1)=y \mid \sigma_{a_N}<\sigma_{a_0}\right). \tag{3.46}$$

However (3.46) does not hold true for some $y\in\Sigma_o$ and $z\in\Sigma_o\setminus\{a_N\}$ by virtue of Proposition 2.4. Thus (2.28) does not hold true for some $x,z\in\Sigma_o$ and $y\in\Sigma_o$. □

4. Examples

In this section, we consider a simple birth and death process. Let $N=3,\ a_i=i\ \ (i=0,1,2,3)$, $q_0=0,\ p_3=0$, and $p_i=q_i=1/K\ \ (i=1,2)$, where $K\geq 2$. For $\tau_k\ \ (k=1,2,\ldots)$ given in Section 2, the transition law of this birth and death process $\mathbb{D}$ satisfies that

$$\begin{aligned}
&P_x(X(\tau_{k+1})=0 \mid X(\tau_k)=1)=P_x(X(\tau_{k+1})=2 \mid X(\tau_k)=1)=\frac{1}{K},\\
&P_x(X(\tau_{k+1})=1 \mid X(\tau_k)=2)=P_x(X(\tau_{k+1})=3 \mid X(\tau_k)=2)=\frac{1}{K}.
\end{aligned} \tag{4.1}$$

4.1. Case That the End Point 1 Is Absorbing

We first consider $\mathbb{D}^{AA}$, that is,

$$\begin{aligned}
&P_x(X(\tau_{k+1})=0 \mid X(\tau_k)=0)=1,\\
&P_x(X(\tau_{k+1})=3 \mid X(\tau_k)=3)=1.
\end{aligned} \tag{4.2}$$

Then $l_1 = 0$ and $l_2 = 3$. Further (2.7) and (2.8) are reduced to

$$s(x) = x, \qquad m(x) = \begin{cases} -\infty, & x < 0, \\ 0, & 0 \le x < 1, \\ K, & 1 \le x < 2, \\ 2K, & 2 \le x < 3, \\ \infty, & 3 \le x. \end{cases} \tag{4.3}$$

By virtue of [11], we obtain that

$$\begin{gathered} P_x^{AA}(X(t) = y) = p^{AA}(t, x, y) m(\{y\}), \\ p^{AA}(t, x, y) = \frac{1}{2K} \chi_{\{1,2\}}(x) \chi_{\{1,2\}}(y) \left\{ e^{-t/K} + (-1)^{x+y} e^{-3t/K} \right\}, \end{gathered} \tag{4.4}$$

where $\chi_\Lambda(\xi) = 1$ [resp., $\chi_\Lambda(\xi) = 0$] if $\xi \in \Lambda$ [resp., $\xi \notin \Lambda$]. Further by virtue of Theorem 2.1,

$$Q_x^{IA}(X(t) = y) = \frac{Ky}{x} p^{AA}(t, x, y), \tag{4.5}$$

for $x, y \in \{1, 2\}$, where $I \in \{A, R\}$. Note that $Q_x^{RA}(X(t) = y)$ is expressed by using $p^{AA}(t, x, y)$. As in Remark 2.2, this induces a birth and death process $\mathbb{D}_o$ on $\{1, 2, 3\}$ with the transition law

$$\begin{gathered} Q_x^{IA}(X(\tau_{k+1}) = 2 \mid X(\tau_k) = 1) = 1, \\ Q_x^{IA}(X(\tau_{k+1}) = 3 \mid X(\tau_k) = 2) = \frac{3}{2K}, \\ Q_x^{IA}(X(\tau_{k+1}) = 1 \mid X(\tau_k) = 2) = \frac{1}{2K}, \\ Q_x^{IA}(X(\tau_{k+1}) = 3 \mid X(\tau_k) = 3) = 1. \end{gathered} \tag{4.6}$$

4.2. Case That the End Point 0 Is Absorbing and the End Point 3 Is Reflecting

We next consider $\mathbb{D}^{AR}$ with $q_3 = 1$, that is,

$$\begin{gathered} P_x(X(\tau_{k+1}) = 0 \mid X(\tau_k) = 0) = 1, \\ P_x(X(\tau_{k+1}) = 2 \mid X(\tau_k) = 3) = 1. \end{gathered} \tag{4.7}$$

For simplicity, we put $K = 2$. Then $l_1 = 0$ and $l_2 = \infty$. Further (2.7) and (2.8) are reduced to

$$s(x) = x, \qquad m(x) = \begin{cases} -\infty, & x < 0, \\ 0, & 0 \le x < 1, \\ 2, & 1 \le x < 2, \\ 4, & 2 \le x < 3, \\ 5, & 3 \le x. \end{cases} \tag{4.8}$$

By virtue of [11], we obtain that

$$\begin{aligned} P_x^{AR}(X(t) = y) &= p^{AR}(t,x,y)m(\{y\}), \quad x,y \in \{1,2,3\}, \\ p^{AR}(t,x,y) &= \frac{1}{12}e^{-(2-\sqrt{3})t/2}\psi_1^{AR}(x)\psi_1^{AR}(y) + \frac{1}{3}e^{-t}\psi_2^{AR}(x)\psi_2^{AR}(y) \\ &\quad + \frac{1}{12}e^{-(2+\sqrt{3})t/2}\psi_3^{AR}(x)\psi_3^{AR}(y), \\ \psi_1^{AR}(x) &= \begin{cases} x, & x = 0,1, \\ \sqrt{3}, & x = 2, \\ 2, & x = 3, \end{cases} \qquad \psi_2^{AR}(x) = \begin{cases} x, & x = 0,1, \\ 0, & x = 2, \\ -1, & x = 3, \end{cases} \\ \psi_3^{AR}(x) &= \begin{cases} x, & x = 0,1, \\ -\sqrt{3}, & x = 2, \\ 2, & x = 3, \end{cases} \qquad m(\{y\}) = \begin{cases} 2, & y = 1,2, \\ 1, & y = 3. \end{cases} \end{aligned} \tag{4.9}$$

By means of (2.21), we obtain that

$$\begin{aligned} Q_x^{AR}(X(t) = y) &= \frac{y}{x}P_x^{AA}(X(t) = y) + \frac{3}{x}\mathcal{M}(t,x,y) \\ &= \frac{y}{x}p^{AA}(t,x,y)m(\{y\}) + \frac{3}{x}M(t,x,y)m(\{y\}), \end{aligned} \tag{4.10}$$

for $x, y \in \{1,2\}$, where $p^{AA}(t,x,y)$ is given by (4.4) with $K = 2$, and

$$\begin{aligned} M(t,x,y) &= \int_0^t \mu_x(u)p^{AR}(t-u,3,y)\,du, \\ \mu_x(t) &= p^{AA}(t,x,2) = \frac{1}{4}\left\{e^{-t/2} + (-1)^x e^{-3t/2}\right\}, \\ p^{AR}(t,3,y) &= \frac{1}{6}\left\{e^{-(2-\sqrt{3})t/2}\psi_1^{AR}(y) - 2e^{-t}\psi_2^{AR}(y) + e^{-(2+\sqrt{3})t/2}\psi_3^{AR}(y)\right\}. \end{aligned} \tag{4.11}$$

4.3. Case That the End Points 0 and 3 Are Reflecting

We finally consider $\mathbb{D}^{RR}$ with $p_0 = 1$ and $q_3 = 1$, that is,

$$\begin{aligned} &P_x(X(\tau_{k+1}) = 1 \mid X(\tau_k) = 0) = 1, \\ &P_x(X(\tau_{k+1}) = 2 \mid X(\tau_k) = 3) = 1. \end{aligned} \tag{4.12}$$

For simplicity, we put $K = 2$. Then $l_1 = -\infty$ and $l_2 = \infty$. Further (2.7) and (2.8) are reduced to

$$s(x) = x, \qquad m(x) = \begin{cases} -1, & x < 0, \\ 0, & 0 \le x < 1, \\ 2, & 1 \le x < 2, \\ 4, & 2 \le x < 3, \\ 5, & 3 \le x. \end{cases} \tag{4.13}$$

By virtue of [11], we obtain that

$$\begin{aligned} &P_x^{RR}(X(t) = y) = p^{RR}(t,x,y)m(\{y\}), \quad x,y \in \{0,1,2,3\}, \\ &p^{RR}(t,x,y) = \frac{1}{6}\varphi_1^{RR}(x)\varphi_1^{RR}(y) + \frac{1}{3}e^{-t/2}\varphi_2^{RR}(x)\varphi_2^{RR}(y) \\ &\qquad + \frac{1}{3}e^{-3t/2}\varphi_3^{RR}(x)\varphi_3^{RR}(y) + \frac{1}{6}e^{-2t}\varphi_4^{RR}(x)\varphi_4^{RR}(y), \\ &\varphi_1^{RR}(x) = 1, \quad x = 0,1,2,3, \quad \varphi_2^{RR}(x) = \begin{cases} (-1)^x, & x = 0,3, \\ \dfrac{(-1)^{x-1}}{2}, & x = 1,2, \end{cases} \\ &\varphi_3^{RR}(x) = \begin{cases} 1, & x = 0,3, \\ -\dfrac{1}{2}, & x = 1,2, \end{cases} \qquad \varphi_4^{RR}(x) = (-1)^x, \ x = 0,1,2,3, \\ &m(\{y\}) = \begin{cases} 1, & y = 0,3, \\ 2, & y = 1,2. \end{cases} \end{aligned} \tag{4.14}$$

By means of (2.21), we obtain that

$$\begin{aligned} Q_x^{RR}(X(t)=y) &= \frac{y}{x}P_x^{AA}(X(t)=y) + \frac{3}{x}\mathcal{M}(t,x,y) + \frac{3}{x}\int_0^t \mu_x(u)\mathcal{N}^R(t-u,3,y)du \\ &= \frac{y}{x}p^{AA}(t,x,y)m(\{y\}) + \frac{3}{x}M(t,x,y)m(\{y\}) \\ &\quad + \frac{3}{x}\int_0^t \mu_x(u)N^R(t-u,3,y)m(\{y\})du, \end{aligned} \tag{4.15}$$

for $x, y \in \{1,2\}$, where $p^{AA}(t,x,y)$ is given by (4.4) with $K = 2$, and $M(t,x,y)$ and $\mu_x(t)$ are given by (4.11). Further $N^R(t,3,y)$ is given as follows:

$$\begin{aligned} N^R(t,x,y) &= \int_0^t \nu_3(u)p^{RR}(t-u,0,y)du, \\ \nu_3(t) = p^{AR}(t,3,1) &= \frac{1}{6}\left\{e^{-(2-\sqrt{3})t/2} - 2e^{-t} + e^{-(2+\sqrt{3})t/2}\right\}, \\ p^{RR}(t,0,y) &= \frac{1}{6}\left\{1 - (-1)^y e^{-t/2} - e^{-3t/2} + (-1)^y e^{-2t}\right\}. \end{aligned} \tag{4.16}$$

5. Conditional Processes in Population Genetics

Here we consider two stochastic models in population genetics and their conditional processes. In this section, we use notations different from those of the previous sections to emphasize the difference between the original models and the induced models of conditional processes. We denote the conditional process by $x^*(t)$ [resp., $X^*(t)$] when the original process is $x(t)$ [resp., $X(t)$] as we did in Section 1.

5.1. Diffusion Model

We consider the following diffusion model for a randomly mating population consisting of N haploid individuals with two types (alleles) A_1 and A_2. Let $x(t)$ be the relative frequency of A_1 at time t. Then $x(t)$ is a one-dimensional diffusion process on $[0,1]$ with the generator

$$L = \frac{x(1-x)}{2N}\frac{d^2}{dx^2} + \{u_2(1-x) - u_1 x\}\frac{d}{dx}, \tag{5.1}$$

where u_1 [resp., u_2] is mutation rate from A_1 [resp., A_2] to A_2 [resp., A_1] (see [4]).

First we consider the case that $u_1 = 0$. The point 1 is accessible and exit boundary if $u_1 = 0$ (see [16]). For this diffusion process, consider a stochastic process $x^*(t)$ induced by conditioning on hitting the boundary point 1 before hitting the other boundary point 0. The induced stochastic process is again a diffusion process with the generator

$$L^* = \frac{x(1-x)}{2N}\frac{d^2}{dx^2} + \left(u_2 + \frac{1}{N}\right)(1-x)\frac{d}{dx} \tag{5.2}$$

by (1.2) and Theorem 2.1 of [3]. Note that the effect of conditioning is that it inflates the mutation rate u_2 to $u_2 + 1/N$. Ewens [1] considered the case that $u_1 = u_2 = 0$ and the induced diffusion process is referred to as the conditional diffusion process by Ewens [1] (see also [2]).

Next we consider the case that $0 < 2Nu_1 < 1$. The point 1 is regular boundary in this case (see [16]) and we can pose various boundary conditions there. If we pose the absorbing boundary condition, then the induced process is again a diffusion process with the generator

$$L^* = \frac{x(1-x)}{2N}\frac{d^2}{dx^2} + \left\{\left(u_2 + \frac{1}{N}\right)(1-x) - u_1 x\right\}\frac{d}{dx} \tag{5.3}$$

by (1.2) and Theorem 2.1 of [3]. On the other hand, if we pose the reflecting boundary condition as it is usually done in population genetics (see [17–19]), then the induced conditional process does not satisfy the Chapman-Kolmogorov equation and this process is not a diffusion process due to Theorem 2.2 of [3]. These results imply that we cannot use the diffusion model whose generator is given by (5.3) as the conditional process when we pose the reflecting boundary condition at the boundary point 1.

5.2. Moran Model

Moran [6] introduced the following birth and death process as one of the fundamental stochastic models in population genetics called continuous-time Moran model (see [4] for discrete time Moran model). We refer to this model as Moran model for brief. Let N be the number of individuals in a haploid population with two types A_1 and A_2, where N is an integer greater than 2. Let $\tau_0 = 0$ and τ_k, $k = 1, 2, \cdots$ be a sequence of random times introduced in Section 2. At time τ_k an individual is chosen randomly and it reproduces a new individual ($k \geq 1$). The type of the newborn individual is A_1 [resp., A_2] with probability $1 - v_1$ [resp., $1 - v_2$] and it is A_2 [resp., A_1] with probability v_1 [resp., v_2] if the parent is A_1 [resp., A_2], where $0 \leq v_1,\ v_2 \leq 1$. Then at this time τ_k an individual except newborn individual is chosen randomly to die. There is no change at time $t \neq \tau_k$ ($k \geq 1$). Denoting by $X(t)$ the relative frequency of A_1 at time t, $X(t)$ is a birth and death process on $\{0, 1/N, 2/N, \ldots, (N-1)/N, 1\}$ with the transition law

$$\begin{gathered}
\mathrm{P}\left(X(\tau_{k+1}) = \frac{j+1}{N} \mid X(\tau_k) = \frac{j}{N}\right) = \frac{(1-v_1)j(N-j) + v_2(N-j)^2}{N^2} = p_j, \\
\mathrm{P}\left(X(\tau_{k+1}) = \frac{j-1}{N} \mid X(\tau_k) = \frac{j}{N}\right) = \frac{(1-v_2)(N-j)j + v_1 j^2}{N^2} = q_j, \\
\mathrm{P}\left(X(\tau_{k+1}) = \frac{j}{N} \mid X(\tau_k) = \frac{j}{N}\right) = r_j,
\end{gathered} \tag{5.4}$$

where $p_j + q_j + r_j = 1$ ($0 \leq j \leq N$). Note that $p_0 = v_2$, $q_0 = 0$, $r_0 = 1 - v_2$, $p_N = 0$, $q_N = v_1$, and $r_N = 1 - v_1$. Note also that $p_j > 0$ unless $v_1 = 1$ and $v_2 = 0$, $q_j > 0$ unless $v_1 = 0$ and $v_2 = 1$, and $r_j > 0$ for $0 < j < N$. The process $X(t)$ does not jump at time τ_k if the types of newborn individual and the dead are the same even though a "birth and death" event occurs at time τ_k. One of the end points 0 is absorbing [resp., reflecting] when $v_2 = 0$ [resp., $v_2 > 0$] and the other end point 1 is absorbing [resp., reflecting] when $\nu_1 = 0$ [resp., $\nu_1 > 0$]. Let σ_i be the first hitting time to i $(i = 0, 1)$.

First we consider the case that $\nu_1 = \nu_2 = 0$. This is the case without mutation and both boundary points are absorbing with

$$\begin{aligned} p_j = q_j &= \frac{j(N-j)}{N^2}, \\ r_j = 1 - p_j - q_j &= 1 - \frac{2j(N-j)}{N^2}. \end{aligned} \tag{5.5}$$

By (2.7) and (2.8) we have

$$\begin{aligned} s(x) &= x, \\ m(x) &= \begin{cases} -\infty, & x < 0, \\ 0, & 0 \le x < \dfrac{1}{N}, \\ \dfrac{N^2}{N-1} + \dfrac{N^2}{2(N-2)} + \cdots + \dfrac{N^2}{i(N-i)}, & \dfrac{i}{N} \le x < \dfrac{i+1}{N}, \quad i = 1, \dots, N-1, \\ \infty, & 1 \le x. \end{cases} \end{aligned} \tag{5.6}$$

Then Theorem 2.1 implies that the conditional process $X^*(t)$ conditional on $\{\sigma_1 < \sigma_0\}$ is again a birth and death process on $\{1/N, 2/N, \dots, (N-1)/N, 1\}$ with the transition law

$$\begin{aligned} \mathrm{P}\left(\mathrm{X}^*(\tau_{k+1}) = \frac{j+1}{N} \mid \mathrm{X}^*(\tau_k) = \frac{j}{N}\right) &= p_j^*, \\ \mathrm{P}\left(\mathrm{X}^*(\tau_{k+1}) = \frac{j-1}{N} \mid \mathrm{X}^*(\tau_k) = \frac{j}{N}\right) &= q_j^*, \\ \mathrm{P}\left(\mathrm{X}^*(\tau_{k+1}) = \frac{j}{N} \mid \mathrm{X}^*(\tau_k) = \frac{j}{N}\right) &= r_j^*, \end{aligned} \tag{5.7}$$

where $p_N^* = q_N^* = 0$ and

$$\begin{aligned} p_j^* &= \frac{j+1}{j} p_j = \frac{(j+1)(N-j)}{N^2}, \\ q_j^* &= \frac{j-1}{j} q_j = \frac{(j-1)(N-j)}{N^2}, \\ r_j^* &= r_j = 1 - \frac{2j(N-j)}{N^2}, \end{aligned} \tag{5.8}$$

for $1 \leq j < N$. The end point $1/N$ is reflecting since $p_1^* = 2(N-1)/N^2 > 0$. Note that $m(x)$ of the original Moran model $X(t)$ reduces to

$$m(x) = \begin{cases} -\infty, & x < 0, \\ 0, & 0 \leq x < \dfrac{1}{3}, \\ \dfrac{9}{2}, & \dfrac{1}{3} \leq x < \dfrac{2}{3}, \\ 9, & \dfrac{2}{3} \leq x < 1 \\ \infty, & 1 \leq x, \end{cases} \tag{5.9}$$

if $N = 3$ and this is essentially the same as the simple birth and death process discussed in Section 4.1.

Next we consider the case that $0 < \nu_1 < 1$. The boundary point 1 is reflecting. Then the induced conditional process does not satisfy Markov property and this conditional process is not a birth and death process by Theorem 2.3.

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