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Research Article

Derived Categories and the Analytic Approach to General Reciprocity Laws: Part III

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Building on the scaffolding constructed in the first two articles in this series, we now proceed to the geometric phase of our sheaf (-complex) theoretic quasidualization of Kubota's formalism for n-Hilbert reciprocity. Employing recent work by Bridgeland on stability conditions, we extend our yoga of t-structures situated above diagrams of specifically designed derived categories to arrangements of metric spaces or complex manifolds. This prepares the way for proving n-Hilbert reciprocity by means of singularity analysis.

1. Introduction

After developing topological and derived sheaf-categorical aspects of our quasidualization of Kubota's formalism [1] for n-Hilbert reciprocity, in [2, 3], we now proceed to the geometric aspect of our construct. Our goal in the present paper is to exploit recent work by Bridgeland [4–6] to produce an arrangement of $7n$ complex manifolds constituting the next level of our architecture, with each such manifold sitting above a particular derived category. In [7] we developed what we have called a calculus (or yoga) of t-structures on each of the indicated 7-"vertex" diagrams (with one such diagram for each element of the group of nth roots of unity, μ_n); see also [3] in this connection. We now go on to collect these t-structures, or, rather, the bounded ones (see below), into sets that admit topologization in accord with the aforementioned contributions by Bridgeland

Most significantly, tactically speaking, t-structures are first off replaced by so-called stability conditions. Indeed, a single t-structure can have any number of stability conditions associated to it by coupling it to certain $\mathbf{C}$-valued homomorphisms from the Grothendieck group of the underlying derived category (of coherent sheaves). These mappings are additionally supposed to satisfy a certain Harder-Narasimhan (HN) condition. The salient point here is that, *qua* data, a Bridgeland stability condition is a bounded t-structure together with a suitable HN "central charge."

Dealing these sets of stability conditions, the structure of a metric space is one of Bridgeland's most exciting results [5]. Moreover, under some special assumptions these metric spaces, whose points are after all stability conditions, acquire the structure of complex manifolds, and, if these assumptions are strong enough, even finite-dimensional (f.d.) $\mathbf{C}$-manifolds. This marvelous state of affairs is the principal motivation for our shifting our focus from t-structures to stability conditions, in which context we presently delineate classes of the latter belonging to a single t-structure; the idea is to cap off the architecture of sheaf constructs we developed in Parts One and Two of the present series with an arrangement of spaces which permit a certain kind of singularity analysis. The salient point here is that, as we have shown in the first two parts of this series (cf., Proposition 5.1 of Part One; Proposition 6.1 of Part Two), nothing less than n-Hilbert reciprocity will follow if, in the aforementioned arrangement of $7n$ spaces (with replication, there is a common join above $\widetilde{X}^2_{\mathbf{A}}$, "in the middle"), certain $n-1$ "vertices," indexed on the nontrivial nth roots of 1, evince degeneracies, that is to say, singularities.

Grafting the geometric structures coming out of Bridgeland's work of just a few years ago onto our (already multileveled) construct accordingly sets the stage for an endgame *vis-à-vis* our approach to general reciprocity, which is after all our justification for this entire series of papers. The projected final tactics will doubtlessly be informed by, for instance, homology (perhaps even intersection homology as per Goresky-MacPherson [8–10]), suitable attendant cohomological approaches, Morse theory, or index theory. But these choices will be made in our next paper; our present purpose is, so to speak, geometrical, what with Part One having a topological orientation and Part Two being concerned with homological algebra (in the broad, modern sense).

2. Background from Parts One and Two

The *raison d' être* for all these considerations is Hecke's eighty-year-old challenge to generalize his analytic proof of quadratic reciprocity for an algebraic number field [11] to higher degrees. We gave detailed accounts of this foundational material in the introductory sections of the two predecessors to this article and refer the reader to those remarks for all the relevant details. However, for the reader's convenience we present a compact sketch of the current status of this open problem in the appendix; suffice it to say for now that our point of departure (namely, Part One) is the work done by Weil [12] and Kubota [1, 13] in the 1960s.

In the present context we take the liberty merely to sketch this background quickly so as to be able to proceed in this section with an expeditious rendering of what we have come to call quasidualization.

One of the main results of Part One [2] is Proposition 5.1 where, among other things, the splitting of $\widetilde{SL}_2(k)^{(n)}_{\mathbf{A}}$ (Kubota's n-fold cover of $SL_2(k)_{\mathbf{A}}$) on $SL_2(k)$ is cast in terms of the existence and behavior of certain morphisms in a set of diagrams in the category $\mathfrak{Top}$ of topological spaces. Specifically, writing μ for μ_n (the nth roots of unity), X_0 for $SL_2(k)$, $\widetilde{\mathrm{X}}_{\mathbf{A}}$ for $\widetilde{SL}_2(k)^{(n)}_{\mathbf{A}}$, and, in contrast to our choice of ξ_0 in Parts One and Two [2, 3] for a typical element of μ, setting $\mu = \langle\zeta\rangle$, so that $\zeta^n = 1$, $\zeta^\nu \neq 1$ if $0 < \nu < n$, we get the diagram

$$\begin{tikzcd}
\mu & \widetilde{\mathrm{X}}_{\mathbf{A}} \arrow[l, "j^0"'] & \coprod_{\ell=1}^{\infty} \mathrm{X}_{\zeta^\nu;\ell} \arrow[l, "m_{\zeta^\nu;c^{(n)}_{\mathbf{A}}}"'] \\
\mathrm{X}_0 \arrow[u, dotted, "s_{\mathbf{A}}"] & & \mathrm{X}_0^2 \arrow[ll, "m_0"] \arrow[u, dotted, "\Omega_{\zeta^\nu}"']
\end{tikzcd} \tag{2.1}$$

(cf., [2, (5.9)], [3, (6.3)]), for each $0 \leq \nu \leq n-1$. (From now on we adopt the convention of writing $\mu = \mu_n$ as $\{1 = \zeta^0, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$, with ζ being a primitive nth root of unity.) We refer the reader to our earlier papers for the exact definitions of the morphisms, except to note that, predictably, j^0 comes from a natural projection $s_{\mathbf{A}}$ from a putative splitting map, and we have chosen "m" to correspond to the according group laws, even as, in $\mathfrak{Top}$, we have taken care to cloak these. Additionally each $X_{\zeta^\nu;\ell}$ is locally closed [2, Corollary 4.6] and the existence of the Ω_{ζ^ν}, or their construction, is the centerpiece of one of the reformulations of Kubota's formalism for n-Hilbert reciprocity developed in [2].

However, it is also the case that Hecke's challenge can be settled along slightly different lines, with the same objects in place. We state this in our updated notation.

Proposition 2.1. *Setting $\widetilde{X}_0 = SL_2(k) \times \mu$ and $Y_{\zeta^\nu} = \widetilde{X}_0^2 \cap \coprod_{\ell=1}^{\infty} X_{\zeta^\nu;\ell}$, n-Hilbert reciprocity follows if one has that if $\nu \neq 0$ then $Y_{\zeta^\nu} = \emptyset$.*

Proof. This is Proposition 6.1 of [3]. (The salient point is that the sets $X_{\zeta^\nu;\ell}$ are carefully defined in terms of the inverse images of Kubota 2-cocycle $c_{\mathbf{A}}^{(n)} \in H^2(SL_2(k)_{\mathbf{A}}, \mu)$ at the ζ^ν; see [2, Section 4].) □

In this setting we gave, in [3, Sections 6, 7, and 8], a systematic development of successive layers of categorical objects located in tiers above a base diagram in $\mathfrak{Top}$ of the type

$$\begin{tikzcd}
\overline{Y}_{\zeta^\nu} \arrow[d, "\hat{i}_{\zeta^\nu}"'] \arrow[dr, "i_{\zeta^\nu}"] & & \\
\breve{X}_{\zeta^\nu} \arrow[r, "\breve{i}_{\zeta^\nu}"] & \widetilde{X}^2_{\mathbf{A}} & \breve{U}_{\zeta^\nu} \arrow[l, "\breve{j}_{\zeta^\nu}"'] \arrow[d, "\hat{j}_{\zeta^\nu}"] \\
W_{\zeta^\nu} \arrow[u, "j"] & & U_{\zeta^\nu} \arrow[ul, "j_{\zeta^\nu}"] \\
& & Z_{\zeta^\nu} \arrow[u, "i"']
\end{tikzcd} \tag{2.2}$$

(again, one for each $0 \leq \nu \leq n-1$). Here, critically, the i's and j's are all meant to convey that each according arrangement is an instance of the decomposition of an $X \in \mathfrak{Top}$ as

$$Y^{\text{closed}} \xrightarrow{i} X \xleftarrow{j} (X \setminus Y)^{\text{open}}. \tag{2.3}$$

Thus, (2.2) realizes a linking of four inclusion triples of the type (2.3).

It is standard that the stratification of X given in (2.3) gives rise to an exact triple of derived categories

$$\mathfrak{D}_Y = D^+(\mathfrak{Sh}/_Y) \xrightarrow{i_*} \mathfrak{D}_X = D^+(\mathfrak{Sh}/_X) \xrightarrow{j^*} \mathfrak{D}_U = D^+(\mathfrak{Sh}/_U), \tag{2.4}$$

where we have written U for $X \setminus Y$. As we proved in [7], this state of affairs supports the construction of a diagram of four corresponding linked exact triples of derived categories, respectively situated above the four inclusion triples constituting (2.2). Accordingly, we obtain for each $0 \leq \nu \leq n-1$ a diagram of linked exact triples of derived categories of the form

$$\begin{tikzcd}
\mathfrak{D}_{\overline{Y}_{\zeta^\nu}} \arrow[d, "\hat{i}_{\zeta^\nu,*}"'] \arrow[dr, "i_{\zeta^\nu,*}"] & & \\
\mathfrak{D}_{\check{X}_{\zeta^\nu}} \arrow[r, "\check{i}_{\zeta^\nu,*}"] \arrow[d, "j^*"'] & \mathfrak{D}_{\widetilde{X}^2_{\mathbf{A}}} \arrow[r, "\check{j}^*_{\zeta^\nu}"] \arrow[dr, "j^*_{\zeta^\nu}"'] & \mathfrak{D}_{\check{U}_{\zeta^\nu}} \\
\mathfrak{D}_{W_{\zeta^\nu}} & & \mathfrak{D}_{U_{\zeta^\nu}} \arrow[u, "\hat{j}^*_{\zeta^\nu}"'] \\
& & \mathfrak{D}_{Z_{\zeta^\nu}} \arrow[u, "i_*"']
\end{tikzcd} \tag{2.5}$$

Thus, *in toto*, we have n diagrams of the type (2.5), with shared vertices, or objects, at $\widetilde{X}^2_{\mathbf{A}}$.

Finally, once again using our results from [7], we presented in the last section of [3] a well-defined arrangement of t-structures on the vertex objects of (2.5), taking into account the yoga of *recollement* introduced in [14]. Along these lines we introduced in [3, 7] the following notation for *recollement* of t-structures, that is, (resp.) gluing and ungluing:

$$\begin{tikzcd}
t(\mathfrak{D}_y) \arrow[d] & t(\mathfrak{D}_y) \wedge t(\mathfrak{D}_U) \arrow[d, dotted] & t(\mathfrak{D}_U) \arrow[d] \\
\mathfrak{D}_Y \arrow[r, "i_*"] & \mathfrak{D}_X \arrow[r, "j^*"] & \mathfrak{D}_U
\end{tikzcd} \tag{2.6}$$

$$\begin{tikzcd}
{}^{\lambda}t(\mathfrak{D}_X) \arrow[d, dotted] & t(\mathfrak{D}_X) \arrow[d] & {}^{\varrho}t(\mathfrak{D}_X) \arrow[d, dotted] \\
\mathfrak{D}_Y \arrow[r, "i_*"] & \mathfrak{D}_X \arrow[r, "j^*"] & \mathfrak{D}_U
\end{tikzcd} \tag{2.7}$$

Here (cf., [14–16]) we have also that, as regards ungluing, ${}^{\lambda}t(\mathfrak{D}_X) = t(\mathfrak{D}_X) \cap \mathfrak{D}_Y$ while ${}^{\varrho}t(\mathfrak{D}_X) = j^*(t(\mathfrak{D}_X))$, using the same obvious conventions employed in [3]. All this makes, for the currently ultimate layer of the architecture at hand, to wit:

$$
\begin{tikzcd}
{}^{\lambda}t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \arrow[r, dotted] & \mathfrak{D}_{\overline{Y}_{\xi^\nu}} \arrow[dd] \arrow[ddr] & & & \\
& & t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \wedge t(\mathfrak{D}_{\breve{U}_{\xi^\nu}}) \arrow[d, dotted] & & \\
t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \arrow[r] & \mathfrak{D}_{\breve{X}_{\xi^\nu}} \arrow[r] \arrow[d] & \mathfrak{D}_{\widetilde{X}^2_{\mathbf{A}}} \arrow[r] \arrow[dr] & \mathfrak{D}_{\breve{U}_{\xi^\nu}} & t(\mathfrak{D}_{\breve{U}_{\xi^\nu}}) \arrow[l] \\
{}^{\varrho}t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \arrow[r, dotted] & \mathfrak{D}_{W_{\xi^\nu}} & & \mathfrak{D}_{U_{\xi^\nu}} \arrow[u] & t(\mathfrak{D}_{Z_{\xi^\nu}}) \wedge t(\mathfrak{D}_{\breve{U}_{\xi^\nu}}) \arrow[l, dotted] \\
& & {}^{\lambda}t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \wedge \wedge (t(\mathfrak{D}_{Z_{\xi^\nu}}) \wedge t(\mathfrak{D}_{\breve{U}_{\xi^\nu}})) \arrow[uu, dotted] & & \\
& & & \mathfrak{D}_{Z_{\xi^\nu}} \arrow[uu] & t(\mathfrak{D}_{Z_{\xi^\nu}}) \arrow[l]
\end{tikzcd}
\tag{2.8}
$$

Here the main result is Proposition 8.1 of [3] asserting that $t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \wedge t(\mathfrak{D}_{\breve{U}_{\xi^\nu}}) = {}^{\lambda}t(\mathfrak{D}_{\breve{X}_{\xi^\nu}}) \wedge (t(\mathfrak{D}_{Z_{\xi^\nu}}) \wedge t(\mathfrak{D}_{\breve{U}_{\xi^\nu}}))$, making for the lion's share of the aforementioned well definition of these arrangements.

With (2.8), specifically with the n diagrams of this sort joined together at the $\widetilde{X}^2_{\mathbf{A}}$ locale, we are in a position to bring some sort of singularity theory into play, the term obviously being understood in a particularly broad preliminary sense at this point.

3. Motivation for Using Bridgeland Stability Conditions

With a burgeoning "calculus" of t-structures available (cf., [7]), we can indeed bring a particular sort of singularity analysis to bear on our construct, courtesy of recent work by Bridgeland [4–6] already alluded to earlier. The main idea is to "inflate" the seven t-structures in (2.8) into equivalence classes of Bridgeland stability conditions, in view of the fact that suitable classes of such equivalence conditions carry a metric topological structure, and sometimes even the structure of a finite-dimensional complex manifold. Thus, bearing in mind, first, that *in toto*, with $\nu \in \{0, 1, 2, \ldots, n-1\}$, the data afforded by (2.8) provides for a collection of $6n + 1$ vertices, and, second, that each of these vertices will be made to support a class of Bridgeland stability conditions carrying a good deal of topological or even f.d. $\mathbf{C}$-manifold structure, we can realize at this level of our architecture something of an *Übermannigfaltigkeit*. (We ask the reader's indulgence regarding this linguistic whimsy, given that the phrase "supermanifolds" has already been taken.)

It is this *Übermannigfaltigkeit*, be it located in $\mathfrak{Top}$, $\mathbf{C}$-$\mathfrak{Mfd}$, or f.d. $\mathbf{C}$-$\mathfrak{Mfd}$, that will dictate the specific form our pending singularity analysis will take. An important feature in this regard is the nature and status of the morphisms that should be defined using the yoga of *recollement* as a point of departure. Thus, in (2.8), the t-structures $t(\mathfrak{D}_{\breve{X}_{\xi^\nu}})$, $t(\mathfrak{D}_{\breve{U}_{\xi^\nu}})$, and $t(\mathfrak{D}_{Z_{\xi^\nu}})$ can be taken as initial data yielding the other t-structures as, so to speak, secondary data obtained by gluing and ungluing. Lifting this game to the level of Bridgeland stability conditions, we can then raise the question of what (categorical) structure may be imparted to these maps. It is at this stage, then, that we will return to the matter of the fine structure of these admittedly bizarre topological spaces of Bridgeland stability conditions we have evolved, in other words, the matter of the appearance and structure of points in this final *Übermannigfaltigkeit*; recall, after all, that, generally, $\mathfrak{D}_X = D^+(\mathfrak{Sh}/X)$ has complexes of sheaves on X as its objects.

As already indicated, beyond the present task of exhibiting the geometric composition of our pending *Übermannigfaltigkeit*, and critically dependent on this determination, we are called to make choices regarding the type or kind of singularity analysis we should train on this highest tier of our architecture. The goal is to demonstrate that for $\nu \neq 0$ our structure is degenerate at $\overline{Y}_{\zeta^\nu}$ (seeing that, as per Proposition 2.1, we need that $Y_{\zeta^\nu} = \emptyset$, if $\nu \neq 0$). For this purpose it is enough to show, of course, that there is, as it were, "nothing above" these $n - 1$ vertices. See Proposition 6.4, below.

This geometric pathology accordingly redounds to the classes of Bridgeland stability conditions we develop in what follows, and it is there that our final battles will eventually be fought.

Given the condition that we are imparting point status to collections of sheaf complexes, in this quasidualized formalism aimed at getting at n-Hilbert reciprocity along the lines sketched by Kubota in [1], we might project that the singularity analysis that will get the nod, down the line, to bring the aforementioned $n - 1$-fold degeneracy out into the open, will include Fourier analysis in the setting of derived categories as developed by Deligne and Laumon (cf. [17, 18]).

Furthermore, given the comparative arithmetical paucity of (2.1), on which the present geometrical constructs are to be built, we must look toward bringing in the effects of various objects occurring in superdiagrams of (2.1) to carry out these final manoeuvres. Since these superdiagrams, such as those in [2, (4.20)], are only future players and are cumbersome entities requiring explication that would take us far afield regarding what the present paper is concerned with, we omit them at this point in the proceedings. The present order of business is to adapt Bridgeland's results to our needs, to develop the *analysis situs*, to use an outdated phrase, underlying any upcoming singularity analysis, and to delineate some of the fine structure of the ensuing architecture in view of future needs.

4. Bridgeland Stability Conditions: The Relevant Results

Apparently the definition of a stability condition in the sense of [4–6] has its immediate antecedents in an investigation by Douglas [19] in the area of D-branes and mirror symmetry situated at the intersection of physics and mathematics. However, for our purposes we focus exclusively on the mathematics in question, that is, stability conditions as part of the theory of triangulated categories (of which derived categories comprise the most important example) and Bridgeland's remarkable characterization of classes of stability conditions admitting the structure of a metric topological space.

Moreover, we will see that a Bridgeland stability condition σ is not just a pair, $(z;\mathfrak{P})$, where z is a homomorphism from the Grothendieck group of the underlying triangulated category to $\mathbf{C}$ and $\mathfrak{P}$ is a certain mapping from $\mathbf{R}$ to the collection of full subcategories of this category (subject to four axioms); *qua* data, it is also a bounded t-structure equipped with a Harder-Narasimhan filtration on its central charge function, which can in fact be identified with z. If we have an exact triple of triangulated (or derived) categories to deal with (or four of these, as in (2.2)), and once a suitable pair of stability conditions is assigned to the extremes of the triple, we can glue these extreme t-structures to get a t-structure on the middle, or mean, category. It then falls to us to determine how to extend this to the indicated metric spaces or f.d. **C**-manifold, in such a way as to open the door for singularity analysis.

Despite the fact that derived categories, and rather special ones at that, will exclusively be dealt with in this projected singularity analysis, we follow Bridgeland in presenting the fundamentals of his stability conditions in the most general context of triangulated categories. But the reader should bear in mind that, soon, derived categories will take over for the upcoming triangulated categories, truncation functors will take on their prosaic meaning engendering actual "physical" truncations of chain complexes of sheaves of Abelian groups over topological spaces, and the ensuing cohomology will display familiar connections.

Given a triangulated category, then, whose definitions and main properties we present at the outset, we proceed in what follows by recalling the formalism of attendant t-structures on such a category, of *recollement* of t-structures on an exact triple of triangulated categories, and the alternative formulation of some of these entities favored by Bridgeland. Subsequently we present Bridgeland's notions of stability conditions, slicings (*sub rosa*), and central changes with a Harder-Narasimhan (HN) condition on them, and his results regarding metric topological structure (or better) on classes of stability conditions.

Before getting down to business, however, we should make two observations. First, our presentation of the background material on t-structures on triangulated or even derived categories is not in any sense exhaustive. The standard sources in this regard include Gelfand and Manin [16], Kashiwara and Schapira [15], Dimca [20], and of course Beĭlinson et al. [14], and we have opted to be somewhat liberal as regards specific attributions. Additionally, a good deal of the theory of t-structures as such, in the form given in the aforementioned sources, is present in our earlier papers in this series; see especially [3, Section 7]. Furthermore, with our objective being the application of Bridgeland's "technology" to our architecture in order to get at a question in analytic number theory, we quickly adopt the abbreviated notation Bridgeland favors for t-structures so that, as a result, our ensuing discussion approaches self-containment.

Second, we stipulate at this early point in the development that the object classes of the triangulated categories we deal with below are sets, or that the indicated categories can be replaced by equivalent categories with this property. In other words, our categories are either small or essentially small. The categories arising in direct connection with our number theoretic applications meet these requirements for, generally speaking, very straightforward reasons.

This having been said, then, in [4, 5] Bridgeland presents the following compact definition of a t-structure.

Definition 4.1. If $\mathfrak{D}$ is a triangulated category and $\mathfrak{f} \subset \mathfrak{D}$ is a full subcategory, then $\mathfrak{f}$ (itself) is said to be a t-structure on $\mathfrak{D}$ if, first, $\mathfrak{f}[1] \subset \mathfrak{f}$, and, second, if, by definition,

$$\mathfrak{f}^{\perp} := \{Y \in \mathfrak{D} \mid \operatorname{Hom}_{\mathfrak{D}}(X, Y) = 0 \ \forall Y \in \mathfrak{f}\}, \tag{4.1}$$

then for every $Z \in \mathfrak{D}$ there exists a distinguished triangle $Z_0 \to Z \to Z_1 \to Z_0[1]$ with $Z_0 \in \mathfrak{f}$ and $Z_1 \in \mathfrak{f}^{\perp}$.

Evidently this characterization of a t-structure varies from the standard one (cf., [3, page 18]), to wit: a t-structure on $\mathfrak{D}$ is the pair $t(\mathfrak{D}) := (\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$ of full subcategories such that, with $\mathfrak{D}^{\leq n} := \mathfrak{D}^{\leq 0}[-n]$ and $\mathfrak{D}^{\geq n} := \mathfrak{D}^{\geq 0}[-n]$, we have that $\mathfrak{D}^{\leq 0} \subset \mathfrak{D}^{\leq 1}$ (there is a misprint in loc. cit., where it reads $\mathfrak{D}^{\leq 0} \subset \mathfrak{D}^{\geq 1}$) and $\mathfrak{D}^{\geq 0} \supset \mathfrak{D}^{\geq 1}$; that if $A \in \mathfrak{D}^{\leq 0}$ and $B \in \mathfrak{D}^{\geq 1}$ then

$\mathrm{Hom}_{\mathfrak{D}}(A, B) = 0$; and, finally, that if $A \in \mathfrak{D}$ then there exist objects $\tau^{\leq 0}A \in \mathfrak{D}^{\leq 0}$, $\tau^{\geq 1}A \in \mathfrak{D}^{\geq 1}$, functorially, such that $\tau^{\leq 0}A \to A \to \tau^{\geq 1}A \xrightarrow{+1}$ is distinguished. However, the connection between these two definitions is in essence that $\mathfrak{f} = \mathfrak{D}^{\leq 0}$ and $\mathfrak{f}^{\perp} = \mathfrak{D}^{\geq 1}$, whence $\mathfrak{D}^{\geq 0} = \mathfrak{f}^{\perp}[1]$.

With these agreements in place, we are now in a position to confuse these two conventions at will, or, rather, as a function of convenience and clarity. *À propos*, we obviously play no favorites either between the renderings $A \to B \to C \to A[1]$ and $A \to B \to C \xrightarrow{+1}$ for a distinguished triangle; after all, both are entirely standard in the literature.

We stipulate, too, with Bridgeland, that the t-structures we are dealing with are bounded.

Definition 4.2. A t-structure $\mathfrak{f}$ on a triangulated category $\mathfrak{D}$ is bounded if

$$\mathfrak{D} = \bigcup_{i,j \in \mathbf{Z}} \left(\mathfrak{f}[i] \cup \mathfrak{f}^{\perp}[j] \right). \tag{4.2}$$

Proceeding along, then, the heart (or core) of a t-structure $t(\mathfrak{D})$, on $\mathfrak{D}$, being the Abelian category $\mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$, is given as $\mathfrak{f} \cap \mathfrak{f}^{\perp}[1]$ in Bridgeland's notation, and, for future reference, the standard cohomological functor $H^{\circ} := \tau^{\geq 0}\tau^{\leq 0} = \tau^{\leq 0}\tau^{\geq 0}$ (see [8]) maps into the core:

$$H^{\circ} : \mathfrak{D} \longrightarrow \mathfrak{f} \cap \mathfrak{f}^{\perp}[1]. \tag{4.3}$$

Next, recall that if $\mathfrak{A}$ is any Abelian category, its Grothendieck group, $K(\mathfrak{A})$, is the quotient of the free Abelian group on $\mathfrak{A}$ by the relation that $X = X' + X''$ in $K(\mathfrak{A})$ if and only if there is a short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathfrak{A}$. In the *leitmotiv* case of a derived category, say $\mathfrak{D} = D(\mathfrak{Sh}/_T)$ for a topological space T, it is a standard fact [21] that in the presence of the standard t-structure on $\mathfrak{D}$ short exact sequences of chain complexes of sheaves on T correspond to distinguished triangles in $D(\mathfrak{Sh}/_T)$; note also that the Abelian category $\mathfrak{Sh}/_T$ arises here as the core of the aforementioned standard t-structure $\mathfrak{D}$ (loc. cit.). It follows from these observations that $K(\mathfrak{Sh}/_T) \cong K(\mathfrak{D})$, where $K(\mathfrak{D})$ is defined as the free Abelian group of $\mathfrak{D}$ divided out by the relation that $F^{\cdot} = F'^{\cdot} + F''^{\cdot}$ if and only if we have a distinguished triangle $F'^{\cdot} \to F^{\cdot} \to F''^{\cdot} \xrightarrow{+1}$. Moreover, it turns out that this is in fact true for triangulated categories [5, page 15]: if $\mathfrak{D}$ is a triangulated category equipped with a t-structure whose core is the Abelian category $\mathfrak{A}$, then $K(\mathfrak{D}) \cong K(\mathfrak{A})$.

With these notions and facts in place we come to the main player in the game.

Definition 4.3. A Bridgeland stability condition, or just a stability condition, on a triangulated category $\mathfrak{D}$ is the data $\sigma = (z_{\sigma}; \mathfrak{P}_{\sigma})$, where, first,

$$z_{\sigma} : K(\mathfrak{D}) \longrightarrow \mathbf{C} \tag{4.4}$$

is a group homomorphism called the central charge of σ, and where, second,

$$\mathfrak{P}_{\sigma} : \mathbf{R} \longrightarrow \{\text{full additive subcategories of } \mathfrak{D}\} \tag{4.5}$$

is a so-called slicing of $\mathfrak{D}$, with this data being by definition subject to the following four axioms.

(i) If $E \in \mathfrak{P}_\sigma(\varphi)$, $\varphi \in \mathbf{R}$, then $\arg z_\sigma(E) = \pi\varphi$; that is to say, $z_\sigma(E) = |z_\sigma(E)| \cdot e^{i\pi\varphi}$.

(ii) For all $\varphi \in \mathbf{R}$, $\mathfrak{P}_\sigma(\varphi + 1) = \mathfrak{P}_\sigma(\varphi)[1]$.

(iii) If $A_1 \in \mathfrak{P}_\sigma(\varphi_1)$, $A_2 \in \mathfrak{P}_\sigma(\varphi_2)$, and $\varphi_1 > \varphi_2$, then $\operatorname{Hom}_{\mathfrak{D}}(A_1, A_2) = 0$.

(iv) If E is a nonzero object in $\mathfrak{D}$ (written somewhat abusively as $E \neq 0$), there is a finite sequence of real numbers,

$$\varphi_\sigma^+(E) := \varphi_1 > \varphi_2 > \cdots > \varphi_{i-1} > \varphi_i > \cdots > \varphi_{n-1} > \varphi_n =: \varphi_\sigma^-(E), \tag{4.6}$$

and a corresponding collection of distinguished triangles,

$$E_{i-1} \longrightarrow E_i \longrightarrow A_i \longrightarrow E_{i-1}[1], \tag{4.7}$$

rendered $E_{i-1} \to E_i$ (with Bridgeland), such that $A_i \in \mathfrak{P}_\sigma(\varphi_i)$ for every $1 \leq i \leq n$, and A_i we have (uniquely up to isomorphism)

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{i-1} \arrow[rr] & & E_i \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_i \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd} \tag{4.8}$$

Given this decomposition of $E \in \mathfrak{A}$, we say that E has mass

$$m_\sigma(E) := \sum_{i=1}^{n} (z_\sigma(A_i)). \tag{4.9}$$

It turns out that there is an equivalent way of characterizing stability conditions which is better suited to our near-future needs. First of all, for any Abelian category $\mathfrak{A}$ we get the following definition.

Definition 4.4. A stability function on $\mathfrak{A}$ is a group homomorphism

$$z : K(\mathfrak{A}) \longrightarrow \mathbf{C} \tag{4.10}$$

with the property that if $0 \neq E \in \mathfrak{A}$ then $z(E) \in \mathfrak{H}$, the (usual) complex upper half-plane. And then the phase of $E \in \mathfrak{A}$ is the real number

$$\varphi(E) := \frac{1}{\pi} \arg z(E) \tag{4.11}$$

in $(0, 1]$.

Definition 4.5. One says that $0 \neq E \subset \mathfrak{A}$ is semistable if one has that, for all $0 \neq E' \subset E$, $\varphi(E') \leq \varphi(E)$.

With these definitions in hand we obtain the notion of a H(arder-) N(arasimhan) stability function as follows.

Definition 4.6. A stability function z, as per (4.10), satisfies a Harder-Narasimhan (HN) condition if every $0 \neq E \in \mathfrak{A}$ admits a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{i-1} \subset E_i \subset \cdots \subset E_{n-1} \subset E_n = E \tag{4.12}$$

such that, for each i, the quotient object E_i/E_{i-1} is semistable in $\mathfrak{A}$ and the inequality $\varphi(E_i/E_{i-1}) > \varphi(E_{i+1}/E_i)$ is satisfied.

Accordingly, on one hand, we have the data afforded by a stability condition $\sigma = (z_\sigma; \mathfrak{P}_\sigma)$ in keeping with (4.4), (4.5), and (i)–(iv), above, while, on the other hand, we have the notion of an HN stability function, $z : K(\mathfrak{A}) \to \mathbf{C}$, on the Grothendieck group of an Abelian category, together with the notion of a t-structure on $\mathfrak{D}$. However, we know, too, from Proposition 4.7, that if $\mathfrak{A}$ is the core of a (suitable) t-structure on $\mathfrak{D}$ then $K(\mathfrak{A})$ and $K(\mathfrak{D})$ can be identified, and, in view of the proposition that (iv), specifically (4.5), can evidently be regarded as Harder-Narasimhan data, this suggests that there should be an identification possible between certain central charges z_σ and HN stability functions z. Indeed, starting with z_σ, that is, with $\sigma = (z_\sigma; \mathfrak{P}_\sigma)$, define, for any interval $I \subset \mathbf{R}$, the set

$$\mathfrak{P}_\sigma(I) := \{0 - \text{objects of } \mathfrak{D}\} \coprod \{E \in \mathfrak{D} \mid \varphi_\sigma^-(E), \varphi_\sigma^+(E) \in I\}, \tag{4.13}$$

and write, for convenience (and with Bridgeland), $\mathfrak{P}_\sigma((\varphi,\infty)) = \mathfrak{P}_\sigma(>\varphi)$, $\mathfrak{P}_\sigma([\varphi,\infty)) = \mathfrak{P}_\sigma(\geq \varphi)$, $\mathfrak{P}_\sigma((-\infty,\varphi)) = \mathfrak{P}_\sigma(<\varphi)$, and $\mathfrak{P}_\sigma((-\infty,\varphi]) = \mathfrak{P}_\sigma(\leq\varphi)$. It is easy to see that $\mathfrak{P}_\sigma(>\varphi)[1] = \mathfrak{P}_\sigma((\varphi,\infty))[1] = \mathfrak{P}_\sigma((\varphi+1,\infty)) \subset \mathfrak{P}((\varphi,\infty)) = \mathfrak{P}_\sigma(>\varphi)$, and that the same sort of thing happens for $\mathfrak{P}_\sigma(\geq\varphi)$. It accordingly stands to reason that each of these subcategories of $\mathfrak{D}$ should qualify as a t-structure (in Bridgeland's sense) on $\mathfrak{D}$; indeed, if, for example, $\mathfrak{P}_\sigma(>\varphi) =: \mathfrak{f}_{\sigma,\varphi}$ is such, then $\mathfrak{P}_\sigma(\leq\varphi) = \mathfrak{f}_{\sigma,\varphi}^\perp$. Naturally, the t-structure of choice (to correspond to a given $\mathfrak{P}_\sigma$) is $\mathfrak{f}_{\sigma,0}$, which we just term $\mathfrak{f}_\sigma$ from now on; in other words, $t_{\mathfrak{P}_\sigma}(\mathfrak{D}) = (\mathfrak{f}_\sigma, \mathfrak{f}_\sigma^\perp[1]) = (\mathfrak{P}_\sigma(>0), \mathfrak{P}_\sigma(\leq 1))$ and core $\mathfrak{f}_\sigma = \operatorname{core} t_{\mathfrak{P}_\sigma}(\mathfrak{D}) = \mathfrak{f}_\sigma \cap \mathfrak{f}_\sigma^\perp[1] = \mathfrak{P}_\sigma((0,1])$, an Abelian category. Under these circumstances, then, we identify the Grothendieck groups $K(\mathfrak{D})$ and $K(\operatorname{core}\mathfrak{f}_\sigma)$. (Given that our focus is not on t-structures and stability conditions for their own sake, we do not pursue the details of these arguments here. Again, the interested reader is referred to the literature mentioned earlier.)

Next, having indicated a means whereby to go from $\mathfrak{P}_\sigma \in \sigma = (z_\sigma; \mathfrak{P}_\sigma)$ to a t-structure, $\mathfrak{f}_\sigma$, we note that the fact that the central charge z_σ satisfies condition (iv), above, and the easily verified proposition that the aforementioned t-structure has core $\mathfrak{f}_\sigma \cap \mathfrak{f}_\sigma^\perp[1]$ implies the following conclusion.

Proposition 4.7. *As a function on this core, z_σ is in fact a Harder-Narasimhan function.*

Putting these things together we obtain that σ determines the data $(z_\sigma, \mathfrak{f}_\sigma)$, of an HN-stability function and a t-structure on $\mathfrak{D}$. Furthermore, the opposite implication is true, too, so that we obtain (*verbatim* Bridgeland) the following.

Proposition 4.8. *"To give a stability condition on $\mathfrak{D}$ is equivalent to giving a bounded t-structure on $\mathfrak{D}$ and a stability function on its heart with the Harder-Narasimhan property."*

Proof. See [4, page 10] or [5, page 15]. □

In light of this characterization we take the liberty of identifying any data $\sigma = (z_\sigma; \mathfrak{P}_\sigma)$ with the data $(z_\sigma|_{\mathrm{core}\,\mathfrak{f}_\sigma}; \mathfrak{f}_\sigma)$ where $\mathfrak{f}_\sigma := \mathfrak{P}_\sigma(> 0)$, in accord with our earlier remarks.

5. Equivalence Classes of Stability Conditions

Returning to our construct (2.8), which is the blueprint, as it were, for the *Übermannigfaltigkeit* on which we propose to carry out singularity analysis, the seven indicated t-structures (arising from three given ones) need to be "inflated" to Bridgeland stability conditions if we propose to use Bridgeland's topological results (loc. cit.). The obvious first requirement we face, however, is the imperative that the rather ramified *recollement* interplay depicted in (2.8) be carried over to these stability conditions. In other words, if we want (2.8) to evolve into a proper diagram with each of the seven t-structures in question replaced by a stability condition (i.e., a point on our expected *Übermannigfaltigkeit*), then the aforementioned move of "inflation" must commute with *recollement*. This requirement would make it incumbent on us to pick very special HN-functions on the cores of the seven given t-structures whereby to effect this inflation. Indeed, we would have to address the autonomous problem of extending the process of *recollement* to Bridgeland stability conditions in a well-defined and systematic fashion. Thus, given, for example, an arrangement of triangulated categories

$$\mathfrak{C} \xrightarrow{P} \mathfrak{D} \xrightarrow{Q} \mathfrak{E} \tag{5.1}$$

making up an exact triple [7, 15], and given a Bridgeland stability condition $\sigma = (z_\sigma; \mathfrak{P}_\sigma) = (z_\sigma; \mathfrak{f}_\sigma)$ on $\mathfrak{D}$, we have *à priori* that $\mathfrak{f}_\sigma$ yields t-structures ${}^{\lambda}\mathfrak{f}_\sigma$ and ${}^{\varrho}\mathfrak{f}_\sigma$ on $\mathfrak{C}$ and $\mathfrak{E}$, respectively; here we have taken the obvious luxury of writing ${}^{\lambda}\mathfrak{f}_\sigma$ and ${}^{\varrho}\mathfrak{f}_\sigma$ instead of ${}^{\lambda}t_{\mathfrak{P}_\sigma}(\mathfrak{D})$ and ${}^{\varrho}t_{\mathfrak{P}_\sigma}(\mathfrak{D})$, where $\mathfrak{P}_\sigma(> 0) = \mathfrak{f}_\sigma$, $\mathfrak{P}_\sigma(\leq 1) = \mathfrak{f}_\sigma^{\perp}[1]$. But we still need to address the issue of attendant central charges: we are given that $z_\sigma \in \mathrm{Hom}(K(\mathrm{core}\,\mathfrak{f}_\sigma); \mathbf{C}) \cong \mathrm{Hom}(K(\mathfrak{D}); \mathbf{C})$, and we need suitable ${}^{\lambda}z_\sigma \in \mathrm{Hom}(K(\mathrm{core}\,{}^{\lambda}\mathfrak{f}_\sigma); \mathbf{C}) \cong \mathrm{Hom}(K(\mathfrak{C}); \mathbf{C})$ and ${}^{\varrho}z_\sigma \in \mathrm{Hom}(K(\mathrm{core}\,{}^{\varrho}\mathfrak{f}_\sigma); \mathbf{C}) \cong \mathrm{Hom}(K(\mathfrak{E}); \mathbf{C})$, making for stability conditions ${}^{\lambda}\sigma$, ${}^{\varrho}\sigma$, on $\mathfrak{C}$, $\mathfrak{E}$, respectively, such that $z_{\lambda_\sigma} = {}^{\lambda}z_\sigma$ and $z_{\varrho_\sigma} = {}^{\varrho}z_\sigma$. Additionally, we have to arrange that the fact that *recollement* engenders that gluing and ungluing undo each other carries over to the level of stability conditions.

On the other hand, if we look ahead to our goal of carrying out a special kind of singularity analysis on the *Übermannigfaltigkeit* we seek to manufacture, it is clearly possible to do an end run, and avoid the difficulties raised above by introducing what we might call a "fat" equivalence relation on the set of stability conditions, placing the full burden of commuting with *recollement* on the t-structures occupying the stability conditions' second coordinates. Specifically, we have the following.

Definition 5.1. If $\sigma = (z_\sigma; \mathfrak{P}_\sigma)$ and $\tau = (z_\tau; \mathfrak{P}_\tau)$ are both stability conditions being defined on the same underlying derived category $\mathfrak{D}$, then $\sigma \sim \tau$ if and only if $\mathfrak{P}_\sigma = \mathfrak{P}_\tau$, or, equivalently, $\mathfrak{f}_\sigma = \mathfrak{f}_\tau$, using our earlier nomenclature conventions.

The effect of this equivalence is to attach to each t-structure $\mathfrak{f}$ in the game, specifically to each of the seven t-structures appearing in (2.8), a "fat" equivalence class $^{\lceil}\mathfrak{f}^{\rceil}$ of Bridgeland stability conditions. Thus, each $t(\mathfrak{D}) = (\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$ in (2.8), which can be rendered as $\mathfrak{f} = \mathfrak{D}^{\leq 0}$ (so that $\mathfrak{D}^{\geq 0} = \mathfrak{f}^{\perp}[1]$: see Section 3, above), is effectively inflated into the class $^{\lceil}\mathfrak{f}^{\rceil}$ simply by attaching to $\mathfrak{f}$, now identified with an appropriate $\mathfrak{P}$, all suitable central charges z, taking into account that the data $(z; \mathfrak{P}) = (z; \mathfrak{f})$ (*via* $\mathfrak{f} = \mathfrak{P}(> 0)$) is equivalent to the data provided by a stability condition.

Furthermore, isolating $\mathfrak{P}$ in this way is in fact tantamount to restricting our attention to slicings of $\mathfrak{D}$, as opposed to the obviously more restrictive stability conditions. Indeed in [5] we find the following.

Definition 5.2. A slicing of a given triangulated category $\mathfrak{D}$ is the data $\mathfrak{P}(\varphi)$, $\varphi \in \mathbf{R}$, cut out by the counterparts to (ii), (iii), and (iv) in the earlier definitions of a stability condition: $\mathfrak{P}(\varphi + 1) = \mathfrak{P}(\varphi)[1]$; $A_1 \in \mathfrak{P}(\varphi_1)$; $A_2 \in \mathfrak{P}(\varphi_2)$, $\varphi_1 > \varphi_2 \Rightarrow \mathrm{Hom}_{\mathfrak{D}}(A_1, A_2) = 0$; and each nonzero $E \in \mathfrak{D}$ associates to a sequence $\varphi_1 > \varphi_2 > \cdots > \varphi_n$ (for some n) such that

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{i-1} \arrow[r] & E_i \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & A_2 \arrow[ul, dotted] & & & A_i \arrow[ul, dotted] & & A_n \arrow[ul, dotted] &
\end{tikzcd} \tag{5.2}$$

with the triangles distinguished and $A_i \in \mathfrak{P}(\varphi_i)$ for $1 \leq i \leq n$.

With Bridgeland, write $\mathrm{Slice}(\mathfrak{D})$ for the set of all slicings of $\mathfrak{D}$, and, provisionally, $\mathrm{Stab}(\mathfrak{D})$ for the set of stability conditions on $\mathfrak{D}$ (with another defining condition to be discussed presently: see Section 6). Then, evidently, $\mathrm{Stab}(\mathfrak{D}) \subseteq \mathrm{Slice}(\mathfrak{D}) \times \mathrm{Hom}_{\mathbf{Z}}(K(\mathfrak{D}), \mathbf{C})$ (loc. cit., pages 17-18). For our purposes, however, instead of working with the structurally sparser $\mathrm{Slice}(\mathfrak{D})$, we focus on $\mathrm{Stab}(\mathfrak{D})$, which, courtesy of Bridgeland's metric, provides the topological structures holding the most promise.

Parenthetically, it is without question fascinating in its own right to pursue the question of extending *recollement* from t-structures to stability conditions in the narrow and exacting sense discussed above, and we propose to look into this matter in a separate investigation [22]. But for what we have in mind here, that is, our projected singularity analysis, that much fine structure is evidently not needed.

6. Bridgeland's Metric and Topological Spaces of Stability Conditions

We now head for the remarkable result Bridgeland presented in [4–6] to the effect that collections of stability conditions can be endowed with the structure of a metric space and, under the right circumstances, even that of a f(inite-) d(imensional) $\mathbf{C}$-manifold. This material is provided in complete detail in Bridgeland's papers so we present it here without proofs, soon to tailor these results to our needs in Sections 6 and 7. Before any of this, however, we need to say something about the matter of the proper characterization of $\mathrm{Stab}(\mathfrak{D})$, that is to say, the question of local finiteness of Bridgeland stability conditions.

Definition 6.1 (see [5, page 17]). A stability condition $\sigma = (z_\sigma; \mathfrak{P}_\sigma)$ is locally finite if there exists an $\epsilon > 0$ such that, for all $\varphi \in \mathbf{R}$, $\mathfrak{P}_\sigma(\varphi - \epsilon, \varphi + \epsilon)$ is both Artinian and Noetherian, that is, finite as a category.

$\mathrm{Stab}(\mathfrak{D})$ is the set of locally finite stability conditions on the triangulated category, $\mathfrak{D}$.

Thus, our earlier fat equivalence classes certainly induce a natural partitioning of $\mathrm{Stab}(\mathfrak{D})$, as it stands. However, Bridgeland also notes that the indicated constructions of metrics on sets of stability conditions, or even on slicings (see immediately below) of $\mathfrak{D}$, go through unchanged without the condition of local finiteness, so we postpone judgment for now regarding whether to include this requirement as part of the characterization of our $\mathrm{Stab}(\mathfrak{D})$'s, with the obvious abuse of language in place. Regardless, $\mathrm{Stab}(\mathfrak{D})$ splits up, or partitions, into fat equivalence classes as defined in Section 4.

Next, regarding the aforementioned metric, or distance function between stability conditions, first Bridgeland proves the following.

Proposition 6.2. *The assignment*

$$(\mathfrak{P}_1, \mathfrak{P}_2) \longmapsto \sup_{0 \neq E \in \mathfrak{D}} \left\{ \left| \varphi^-_{\mathfrak{P}_2}(E) - \varphi^-_{\mathfrak{P}_1}(E) \right|, \left| \varphi^+_{\mathfrak{P}_2}(E) - \varphi^+_{\mathfrak{P}_1}(E) \right| \right\} \tag{6.1}$$

defines a metric on $\mathrm{Slice}(\mathfrak{D})$; *to be proper, this rule actually defines or generalized metric in the sense that the range is the set of extended nonnegative real numbers,* $[0, \infty]$.

An equivalent way of presenting this metric is *via* the rule

$$(\mathfrak{P}_1, \mathfrak{P}_2) \longmapsto \inf\{\epsilon \geq 0 \mid \mathfrak{P}_2(\varphi) \subset \mathfrak{P}_1(\varphi - \epsilon, \varphi + \epsilon), \ \forall \varphi \in \mathbf{R}\}. \tag{6.2}$$

For proofs, the reader is referred (again) to [5, page 17]. Then, recalling that generally $\sigma = (z_\sigma; \mathfrak{P}_\sigma)$, Bridgeland obtains the following.

Proposition 6.3. *The mapping*

$$\begin{aligned} &d : \mathrm{Stab}(\mathfrak{D}) \times \mathrm{Stab}(\mathfrak{D}) \longrightarrow [0, \infty] \\ &(\sigma_1, \sigma_2) \longmapsto \sup_{0 \neq E \in \mathfrak{D}} \left\{ \left| \varphi^-_{\sigma_2}(E) - \varphi^-_{\sigma_1}(E) \right|, \left| \varphi^+_{\sigma_2}(E) - \varphi^+_{\sigma_1}(E) \right|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \end{aligned} \tag{6.3}$$

provides a metric on $\mathrm{Stab}(\mathfrak{D})$.

For the proof, consult [5, pages 24–26].

Returning to the construct (2.8), which is, the figure that needs to be replicated n-fold (indexed on $0 \leq \nu \leq n-1$) in order to manufacture the framework for our *Übermannigfaltigkeit* by means of defining a hub at the $\widetilde{X}^2_{\mathbf{A}}$ locale, that is, in the common derived category $\mathfrak{D}_{\widetilde{X}^2_{\mathbf{A}}}$, it is clearly notationally unwieldy to situate seven $\mathrm{Stab}(\mathfrak{D}_-)$'s in the indicated places. More importantly, in view of our future singularity analysis, in which metric space topology or even $\mathbf{C}$-manifold structure is to be exploited, a more evocative notation is desirable. So, we will systematically write

$$\mathcal{M}_{\mathfrak{D}} = \operatorname{Stab}(\mathfrak{D}) \tag{6.4}$$

for the indicated derived categories, $\mathfrak{D}$.

Moreover, in order to mitigate the cumbersome notation that results from the subscripted categories occurring in (2.8), we also write, systematically,

$$\mathcal{M}_X = \operatorname{Stab}(\mathfrak{D}_X) \tag{6.5}$$

and carry this convenience over to the other players in the game.

Accordingly, systematically writing Bridgeland's $\mathfrak{f}$'s in place of the more traditional t's currently found in (2.8), then writing $\mathfrak{f}_X$ to signal the t-structure data $t(\mathfrak{D}_X) = (\mathfrak{D}_X^{\leq 0}, \mathfrak{D}_X^{\geq 0})$, with $\mathfrak{D}_X^{\leq 0} = \mathfrak{f}_X$, $\mathfrak{D}_X^{\geq 0} = \mathfrak{f}_X^{\perp}[1]$ as before, and then denoting the according fat equivalence class by ${}^{\lceil}\mathfrak{f}_X^{\rceil} \subset \mathcal{M}_X$, we can now recast (2.8) as follows, highlighting not the individual t-structures but the fat classes and the $\mathcal{M}_{-}$:

$$\begin{tikzcd}
\mathcal{M}_{\overline{Y}_{\zeta^\nu}} \supset {}^{\lceil\lambda}\mathfrak{f}^{\rceil}_{\breve{X}_{\zeta^\nu}} \arrow[r, dotted] & \mathfrak{D}_{\overline{Y}_{\zeta^\nu}} \arrow[d, "\hat{i}_{\zeta^\nu,*}"'] \arrow[dr, "i_{\zeta^\nu,*}"] & & \begin{array}{c} {}^{\lceil\lambda}\mathfrak{f}_{\breve{X}_{\zeta^\nu}} \wedge (\mathfrak{f}_{Z_{\zeta^\nu}} \wedge \mathfrak{f}_{\breve{U}_{\zeta^\nu}})^{\rceil} \\ = {}^{\lceil}\mathfrak{f}_{\breve{X}_{\zeta^\nu}} \wedge \mathfrak{f}^{\rceil}_{\breve{U}_{\zeta^\nu}} \subset \mathcal{M}_{\tilde{X}^2_{\mathbf{A}}} \end{array} \arrow[dl, dotted] & \\
\mathcal{M}_{\breve{X}_{\zeta^\nu}} \supset {}^{\lceil}\mathfrak{f}^{\rceil}_{\breve{X}_{\zeta^\nu}} \arrow[r] & \mathfrak{D}_{\breve{X}_{\zeta^\nu}} \arrow[r, "\breve{i}_{\zeta^\nu,*}"] \arrow[d, "j^*"'] & \mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}} \arrow[r, "\breve{j}^*_{\zeta^\nu}"] \arrow[dr, "j^*_{\zeta^\nu}"'] & \mathfrak{D}_{\breve{U}_{\zeta^\nu}} & {}^{\lceil}\mathfrak{f}^{\rceil}_{\breve{U}_{\zeta^\nu}} \subset \mathcal{M}_{\breve{U}_{\zeta^\nu}} \arrow[l] \\
\mathcal{M}_{W_{\zeta^\nu}} \supset {}^{\lceil\varrho}\mathfrak{f}^{\rceil}_{\breve{X}_{\zeta^\nu}} \arrow[r, dotted] & \mathfrak{D}_{W_{\zeta^\nu}} & & \mathfrak{D}_{U_{\zeta^\nu}} \arrow[u, "\hat{j}^*_{\zeta^\nu}"'] & {}^{\lceil}\mathfrak{f}_{Z_{\zeta^\nu}} \wedge \mathfrak{f}^{\rceil}_{\breve{U}_{\zeta^\nu}} \subset \mathcal{M}_{U_{\zeta^\nu}} \arrow[l, dotted] \\
& & & \mathfrak{D}_{Z_{\zeta^\nu}} \arrow[u, "i_*"'] & {}^{\lceil}\mathfrak{f}^{\rceil}_{Z_{\zeta^\nu}} \subset \mathcal{M}_{Z_{\zeta^\nu}} \arrow[l]
\end{tikzcd} \tag{6.6}$$

With $\mu = \langle \zeta \rangle$, we have in (6.6), that is to say, in the (full) data provided by these diagrams, all the ingredients needed to define our *Ubermannigfaltigkeit*, which we will denote as $\mathbf{\Omega}_n$, placing us in the position to launch the singularity analysis alluded to above.

It behooves us at this stage to note that we are indeed closing in on our objective.

Proposition 6.4. *In order to obtain n-Hilbert reciprocity for the global field k, it suffices to show that the $n-1$ locales $\mathfrak{D}_{\overline{Y}_{\zeta^\nu}}$, $\nu = 1, 2, \ldots, n-1$, are void; equivalently, it suffices to show that all the action takes place above $\mathfrak{D}_{\overline{Y}_1}$.*

Proof. This follows from Proposition 2.1. ☐

By way of anticipation of Section 8, coming up, and, more importantly, the projected fourth and last paper in this series, we note two obvious but exceedingly important facts at this stage of the proceedings. First, the guiding idea is that $\mathbf{\Omega}_n$ should exhibit $n-1$ "fissures," so to speak, coming from the nullity of the aforementioned $n-1$ locales, so that our ultimate task will be along the lines of proving that these fissures are present by means of proving that a particular pathological situation arises at the level of function spaces (of a type to be determined) on indicated subsets of $\mathbf{\Omega}_n$. Evidently this bears a similarity to what occurs in regards to homology as a measure of the shape of a geometric object, in the presence of, say, a duality with suitably defined cohomology.

Second, the structure of $\mathbf{\Omega}_n$ as a geometrical object, which on a more local level involves the geometric structure of the $\mathcal{M}_{-}$, immediately takes us in the direction of geometrical and topological questions arranged in a natural sequence in such a way that resolving the later questions or problems would translate to hypotheses whose impositions on players in (6.6) would yield more structure for $\mathbf{\Omega}_n$. We say more about this in Section 8 below.

7. The "Points" of $\mathcal{M}_{\mathfrak{D}}$

All the $\mathcal{M}_{-}$ of (6.6) live above topological spaces supporting derived categories of sheaves (which, in due course, we take to be of a conveniently special sort, i.e., coherent sheaves), and these $\mathcal{M}_{-}$, which are also denoted as $\mathcal{M}_{\mathfrak{D}}$ for $\mathfrak{D}$ being any such derived category, are partitioned into fat classes of stability conditions. So it is important to address the question of the appearance of the points that make up $\mathcal{M}_{\mathfrak{D}}$ as a metric space *via* (6.3). Employing the notation of Section 5, this means that we have to explicate the inclusions

$$\mathcal{M}_{\mathfrak{D}} \supset \ulcorner\sigma\urcorner \ni \sigma \ni \mathfrak{f}_\sigma, \tag{7.1}$$

where $\ulcorner\sigma\urcorner$ is a fat equivalence class of Bridgeland stability conditions; for example, $\sigma = (z_\sigma, \mathfrak{f}_\sigma)$ is an individual Bridgeland stability condition with central charge z_σ and t-structure $\mathfrak{f}_\sigma$; it is $\mathfrak{f}_\sigma$ which, identified with a suitable $t(\mathfrak{D})$, takes us back to the players in the initial diagram (2.8). So, properly speaking, a point of $\mathcal{M}_{\mathfrak{D}}$ is a σ, so we start by briefly revisiting the definitions of z_σ and $\mathfrak{f}_\sigma$ as given above.

We have, accordingly, that $\sigma = (z_\sigma; \mathfrak{f}_\sigma) = (z_\sigma; \mathfrak{P}_\sigma)$, where z_σ is a central charge, that is, an HN stability function, a group homomorphism mapping the Grothendieck group of $\mathfrak{D}$ into $\mathbf{C}$ (cf., (4.4)), and $\mathfrak{P}_\sigma$ is a slicing of $\mathfrak{D}$ (as per (4.5)); then the relationship between $\mathfrak{P}_\sigma$ and the Bridgeland stability conditions $\mathfrak{f}_\sigma$ is given by the stipulation that $\mathfrak{P}_\sigma(>0) = \mathfrak{P}_\sigma((0,\infty)) = \mathfrak{f}_\sigma$, so that *qua* t-structure we have $t(\mathfrak{D})(=:t_\sigma(\mathfrak{D})) = (\mathfrak{P}_\sigma(>0), \mathfrak{P}_\sigma(\leq 1)) = (\mathfrak{f}_\sigma, \mathfrak{f}_\sigma^{\perp}[1])$. So, in relation to the nomenclature originating with [14] if we also write $t_\sigma(\mathfrak{D}) = (\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$, then $\mathfrak{D}^{\leq 0} = \mathfrak{P}_\sigma(>0) = \mathfrak{f}_\sigma$ and $\mathfrak{D}^{\geq 0} = \mathfrak{P}_\sigma(\leq 1) = \mathfrak{f}_\sigma^{\perp}[1]$.

Parenthetically, the cumbersome quality of the preceding identifications can possibly be somewhat mitigated by employing the fact that t-structures are self-dual [15, page 412]; however, that would engender yet more notational variations because of the fact that this self-duality of t-structures involves opposite triangulated categories. Seeing that from now on we work primarily with Bridgeland's t-structures (that is to say, $\mathfrak{f}$'s "by themselves"), this turns out not to be an issue.

Going on, then, if $\sigma = (z_\sigma; f_\sigma)$ is a typical point of $\mathcal{M}_{\mathfrak{D}} = \mathrm{Stab}(\mathfrak{D})$, writing also $\mathcal{M}_X$ for $\mathcal{M}_{\mathfrak{D}}$ when $\mathfrak{D} = \mathfrak{D}_X$ in accord with (2.8) and (6.6), then we have $\mathfrak{f}_\sigma = \mathfrak{f}_\tau$. In the presence of our earlier fat equivalence relation, $\mathcal{M}_{\mathfrak{D}}$ is partitioned into a disjoint union of such $\ulcorner\sigma\urcorner$.

Finally, seeing that the triangulated categories appearing in (6.6) are in fact derived categories of sheaf complexes, a point $\sigma = (z_\sigma; \mathfrak{f}_\sigma)$ engenders in $\mathfrak{f}_\sigma$ a full subcategory of an underlying $\mathfrak{D}_X = D^b(\mathfrak{Sh}/_X)$, or even $D^+({}^{\mathrm{coh}}\mathfrak{Sh}/_X)$, where ${}^{\mathrm{coh}}\mathfrak{Sh}\,/_X$ stands for the category of coherent sheaves on X (which is often rendered more compactly as Coh(X), e.g., by Bridgeland in [23]). So, *qua* data, the points of the various $\mathcal{M}_X$ in the game, and, as we will soon see, of our construct $\mathbf{\Omega}_n$, are innately tied to chain complexes of sheaves on the topological space X, with the reason being that $D(\mathfrak{Sh}/X) = \mathbf{K}(X)_{\mathrm{Qis}}$, the localization of the category $\mathbf{K}(X) = \mathrm{Kom}(X)/\simeq$ (chain complexes of sheaves on X modulo chain homotopy) at the localizing class of quasi-isomorphisms [15, 16, 21].

This augers for unusual characterizations of functions on (or along) paths on $\mathbf{\Omega}_n$ and thus for interesting opportunities in singularity analysis. But we are being premature: before anything else we have to deal with the matter of the global structure of $\mathbf{\Omega}_n$.

8. The Large-Scale Structure of Ω_n: Toward Singularity Analysis

The building blocks for $\mathbf{\Omega}_n$ are of course the n diagrams (6.6) with ν running through $\{0, 1, 2, 3, \ldots, n-1\}$. With the seven structures $\mathcal{M}_{-}$ in (6.6) being topological spaces, we can certainly form, first, the n spaces

$$\mathbf{\Omega}_{\zeta^n} := \mathcal{M}_{W_{\zeta^\nu}} \times \mathcal{M}_{\check{X}_{\zeta^\nu}} \times \mathcal{M}_{\overline{Y}_{\zeta^\nu}} \times \mathcal{M}_{\widetilde{X}^2_{\mathbf{A}}} \times \mathcal{M}_{\check{U}_{\zeta^\nu}} \times \mathcal{M}_{U_{\zeta^\nu}} \times \mathcal{M}_{Z_{\zeta^\nu}} \tag{8.1a}$$

$$\approx \mathcal{M}_{\widetilde{X}^2_{\mathbf{A}}} \times \mathcal{M}_{\overline{Y}_{\zeta^\nu}} \times \mathcal{M}_{W_{\zeta^\nu}} \times \mathcal{M}_{\check{X}_{\zeta^\nu}} \times \mathcal{M}_{\check{U}_{\zeta^\nu}} \times \mathcal{M}_{U_{\zeta^\nu}} \times \mathcal{M}_{Z_{\zeta^\nu}} \tag{8.1b}$$

$$\approx \mathcal{M}_{\widetilde{X}^2_{\mathbf{A}}} \times \mathcal{M}_{\overline{Y}_{\zeta^\nu}} \times \Theta_{\zeta^\nu} \tag{8.1c}$$

for each $0 \leq \nu \leq n-1$; here we have defined, *en passant*,

$$\Theta_{\zeta^\nu} := \mathcal{M}_{W_{\zeta^\nu}} \times \mathcal{M}_{\check{X}_{\zeta^\nu}} \times \mathcal{M}_{\check{U}_{\zeta^\nu}} \times \mathcal{M}_{U_{\zeta^\nu}} \times \mathcal{M}_{Z_{\zeta^\nu}}. \tag{8.2}$$

Obviously, the prevailing topology is the product topology.

The reason for our rendering Ω_{ζ^ν} as (8.1c) is that, first, $\widetilde{X}^2_{\mathbf{A}}$ is the shared locale (in $\mathfrak{Top}$) underlying all n of our diagrams (6.6), and, second, that, as we established already in Propositions 2.1 and 6.4, meeting Hecke's challenge depends on having $\overline{Y}_{\zeta^\nu} = \emptyset$ if $\nu \neq 0$, which is of course quite the same as having $Y_{\zeta^\nu} = \emptyset$ for $\nu \neq 0$, that is, $1 \leq \nu \leq n-1$, or, equivalently, having the corresponding $\mathfrak{D}_{\overline{Y}_{\zeta^\nu}}$ degenerate for $1 \leq \nu \leq n-1$. Thus, the objects in our structure above these empty locales are themselves null, or degenerate, too, meaning that we have, at last, the following.

Proposition 8.1. *n-Hilbert reciprocity for the number field k will follow if the $n-1$ topological spaces $\mathcal{M}_{\overline{Y}_{\zeta^\nu}}$, $1 \leq \nu \leq n-1$, are degenerate (i.e., zero).*

Proof. If $\overline{Y}_{\zeta^\nu} = \emptyset$, then any Abelian sheaf $\mathfrak{F}$ on $\overline{Y}_{\zeta^\nu}$ is evidently just the constant sheaf 0. So, $\mathfrak{Sh}/\overline{Y}_{\zeta^\nu} = \{0\}$, or, more precisely, $Ob(\mathfrak{Sh}/\overline{Y}_{\zeta^\nu}) = 0$. Immediately, therefore, $D(\mathfrak{Sh}/\overline{Y}_{\zeta^\nu}) = 0$, too. Realizing a Bridgeland stability condition on $\mathfrak{D}_{\overline{Y}_{\zeta^\nu}} \subset D(\mathfrak{Sh}/\overline{Y}_{\zeta^\nu})$ as $\sigma = (z; \mathfrak{P})$, obtain that $z : K(D(\mathfrak{Sh}/\overline{Y}_{\zeta^\nu})) = (0) \to \mathbf{C}$, that is, $z \equiv 0$, and for all $\varphi \in \mathbf{R}$, $\mathfrak{P}(\varphi) \subset D(\mathfrak{Sh}/\overline{Y}_{\zeta_\nu}) = \{0\}$, that is, $\mathfrak{P} \equiv 0$ too. Thus $\sigma = (0, 0)$. □

So the handwriting is on the wall: with $\widetilde{X}^2_{\mathbf{A}}$ as the single locale shared between the n seven-vertex diagrams in $\mathfrak{Top}$ underlying everything we have above along these lines, we define

$$\mathbf{\Omega}_n := \mathcal{M}_{\widetilde{X}^2_{\mathbf{A}}} \times \prod_{\nu=0}^{n-1}\left(\mathcal{M}_{\overline{Y}_{\zeta^\nu}} \times \Theta_{\zeta^\nu}\right) \tag{8.3}$$

in $\mathfrak{Top}$, still exploiting the product topology. We get, as an immediate consequence of Proposition 8.1, the following critical fact.

Corollary 8.2. *n-Hilbert reciprocity for k follows if* $\mathbf{\Omega}_n \approx \mathcal{M}_{\widetilde{X}^2_{\mathbf{A}}} \times \mathcal{M}_{\overline{Y}_1} \times \prod_{\nu=0}^{n-1}\Theta_{\zeta^\nu}$.

Proof. Clear from the foregoing.

And this brings us to the endgame. The quasidualization of Kubota's formalism for n-Hilbert reciprocity for the number field k by sheaf complex theoretic methods developed in [2, 3] has finally reached the stage where the game will be won if the geometrical (or topological) construct $\mathbf{\Omega}_n$, as above, is revealed to be singular in the sense presented by Corollary 8.2. Thus, to be sure, if [2] dealt with laying out the topological foundation of our strategy, and [3] subsequently focused on the ensuing homological algebra, then the present considerations can be rightly termed geometrical in the particular sense that we now have a construct, at worst a metric space, at best an f.d. $\mathbf{C}$-manifold (and the latter structure may only appear at certain factors of $\mathbf{\Omega}_n$), where singularity analysis would bring the matter to resolution.

The apparent best-case scenario for singularity analysis on $\mathbf{\Omega}_n$ would be if it were amenable to being dealt a finite-dimensional complex manifold structure. Following Bridgeland [4, 23], this would mean requiring the sheaves in our construction to be coherent, which is not a problem, of course, and, more problematically, having certain rather stringent conditions in place on the underlying topological spaces. Specifically, Bridgeland's hypotheses include that these spaces should be complex projective manifolds; admittedly these entail sufficient conditions, not necessary ones, but it is already evident that this much structure comes at a high price, and it is not yet clear how important finite dimensionality should be, given what we have in mind.

On the other hand, it is certain that, as a Cartesian product of metric spaces, $\mathbf{\Omega}_n$ is itself a metric space and this affords us the luxury of a handful of preliminary observations, along the following lines. Evidently the first possibility *vis-à-vis* revealing degeneracy at the aforementioned $n-1$ locales is to carry out a Morse-theoretic analysis of the situation, using particularly elementary Morse functions in the process. The main extrinsic objection to this consists in recalling that Hecke's original challenge asks for an analytic resolution of the problem, so the function-theoretic element in such a Morse-theoretic approach would have to be introduced in what might be a somewhat unusual fashion.

A more promising and not altogether disjoint approach, from the outset algebraic-topological in flavor, is to go at $\mathbf{\Omega}_n$ with the machinery of intersection homology and cohomology (cf., [8–10]). This line is particularly attractive because of the earlier observation (Section 6) to the effect that $\mathbf{\Omega}_n$'s points involve in some innate sense chain complexes with the prospect of using the formalism of functorial integral transforms as per Grothendieck, Deligne, Laumon (cf., [17, 18]), and so on. This route would be more likely to lead to a final singularity analysis on $\mathbf{\Omega}_n$ centered on the Fourier transform's relatively recent incarnation as a functor between derived categories [15].

Finally, there is some hope of going with a combination of the preceding approaches, together with functional analysis (of a very abstract sort), so as to couch the indicated singularity analysis in terms of spectral theory. This is again very appealing on the obvious grounds that such a resolution of Hecke's challenge would be genuinely analytic. These are matters for the next (and hopefully final) paper in this sequence. □

Appendix

Hecke's original proof of relative quadratic reciprocity (for an arbitrary algebraic number field, k, rather than the base field, $\mathbf{Q}$: the absolute case) is effected by means of passing from a functional equation for what are now called the Hecke ϑ-functions to a reciprocity between Gauss sums from which a generalized Gauss-Euler reciprocity law is easily derived. Our point of departure, however, is an equivalent formulation of this proof due to Weil [12] and rephrased by Kubota [13].

In Weil's treatment, Hecke's Fourier analytic deduction of the centrally important ϑ-functional equation is replaced by an application of the Stone-Von Neumann theorem to produce a projective unitary representation of the symplectic group. This projective representation is now called the Weil representation (or the oscillator representation), which, in the setting of the adèlization of the symplectic group, allows a natural group action on a certain functional having a Fourier analytic character. This functional, now called the Weil Θ-functional, is in fact invariant under the induced action of the subgroup of rational points, and so, given that the 2-cocycle produced by the (adèlic) Weil representation is built up from 2-Hilbert symbols, this invariance translates to the product formula giving 2-Hilbert reciprocity which, of course, is equivalent to Gauss-Euler reciprocity. It is not too difficult an exercise to get Hecke's ϑ-functional equations from the indicated invariance of the Weil Θ-functional (cf., [12]).

In Weil's presentation of this argument, Hilbert-Hasse reciprocity is addressed, and the aforementioned 2-cocycle, making for a double cover of the symplectic group, both 3-adically and adèlically, is not explicitly given in terms of the 2-Hilbert symbol. This transition was effected by Kubota in [13], and unitary representations of symplectic groups were simultaneously replaced by (second) cohomology of special linear groups (again 3-adically and adèlically, of course). To wit, it is a standard fact of representation theory [24], or algebraic topology, that the behavior outlined above, that is to say, the splitting of the according adèlic 2-cocycle on the rational points with the 2-cocycle and its constituent 2-Hilbert symbols taking values in $\{1,-1\} = \mu_2$, engenders that the accompanying double cover of $SL_2(k)_{\mathbf{A}}$, written, $\widetilde{SL}_2(k)_{\mathbf{A}}^{(2)} = SL_2(k)_{\mathbf{A}} \times_{c_{\mathbf{A}}^{(2)}} \mu_2$, is also split on $SL_2(k)$ (the all-important rational points). As already suggested, it is this splitting, following from the invariance of the adèlic Weil Θ-functional under the action of $SL_2(k)$ facilitated by the (projective) Weil representation, that permits us to characterize this line of argument as a genuine Fourier-analytic derivation of quadratic reciprocity.

Getting back to Hecke, at the end of [11] he had asked for a generalization of this analytic proof of quadratic reciprocity to arbitrary degrees $n \geq 2$, and Kubota went on to this matter in [1], in the indicated context of unitary group representations and low-dimensional group cohomology. Although not laying down a proof of higher reciprocity (because of his need to presuppose it in his discussion, as we will see momentarily), Kubota presented a promising formalism, dealing with n-fold covers of $SL_2(k)_{\mathbf{A}}$, through which to approach Hecke's challenge with various new tools. Specifically, he defined an n-fold (metaplectic) cover $\widetilde{SL}_2(k)_{\mathbf{A}}^{(n)} = SL_2(k)_{\mathbf{A}} \times_{c_{\mathbf{A}}^{(n)}} \mu_n$, where μ_n ($\subset k$, by assumption) is the group of nth roots of unity, and $c_{\mathbf{A}}^{(n)} \in H^2(SL_2(k)_{\mathbf{A}}, \mu_n)$, Kubota's (2-)cocycle, is built up from n-Hilbert symbols; see [2, Section 3] for specifics. As we indicated in [2, Section 3], if one demonstrates that $c_{\mathbf{A}}^{(n)}$ splits on $SL_2(k)$ by analytic means, Hecke's challenge is met. Kubota did not give this derivation in [1] where only the stated equivalence was set forth, and the problem remains open. In this connection see also [25, page 51].

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