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Research Article

An Iterative Method for Solving the Generalized System of Relaxed Cocoercive Quasivariational Inclusions and Fixed Point Problems of an Infinite Family of Strictly Pseudocontractive Mappings

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We introduce an iterative scheme by the viscosity approximation to find the set of solutions of the generalized system of relaxed cocoercive quasivariational inclusions and the set of common fixed points of an infinite family of strictly pseudocontractive mappings problems in Hilbert spaces. We suggest and analyze an iterative scheme under some appropriate conditions imposed on the parameters; we prove that another strong convergence theorem for the above two sets is obtained. The results presented in this paper improve and extend the main results of Li and Wu (2010) and many others.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product and norm being denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let C be a nonempty closed convex subset of H. Recall that P_C is the *metric projection* of H onto C; that is, for each $x \in H$ there exists the unique point in $P_C x \in C$ such that

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{1.1}$$

A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C \tag{1.2}$$

and the mapping $f : C \to C$ is called a *contraction* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If $C \subset H$ is bounded, closed and convex and T is a nonexpansive mappings of C into itself, then $F(T)$ is nonempty (see [1]). Recall that a mapping $A : C \to H$ is said to be

(i) *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C, \tag{1.4}$$

(ii) *k-Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C, \tag{1.5}$$

if $k = 1$, then A is a nonexpansive,

(iii) *pseudocontractive* if

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C, \tag{1.6}$$

(iv) *k-strictly pseudocontractive* if there exists a constant $k \in [0,1)$ such that

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + k\|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C, \tag{1.7}$$

it is obvious that A is a nonexpansive if and only if A is a 0-strictly pseudocontractive,

(v) *α-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C, \tag{1.8}$$

(vi) *α-inverse-strongly monotone (or α-cocoercive)* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{1.9}$$

if $\alpha = 1$, then A is called that firmly nonexpansive; it is obvious that any α-inverse-strongly monotone mapping A is monotone and $(1/\alpha)$-Lipschitz continuous,

(vii) relaxed α-cocoercive if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{1.10}$$

(viii) relaxed (α, r)-cocoercive if there exists two constants $\alpha, r > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\alpha) \|Ax - Ay\|^2 + r\|x - y\|^2, \quad \forall x, y \in C, \tag{1.11}$$

it is obvious that any the r-strongly monotonicity implies to the relaxed (α, r)-cocoercivity.

Recall that a set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph of $G(M)$ of M is not properly contained in the graph of any other monotone mappings.

The existence common fixed points for a finite family of nonexpansive mappings has been considered by many authers (see [2–5] and the references therein).

In this paper, we study the mapping W_n defined by

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n S_n U_{n,n+1} + (1 - \mu_n) I, \\
U_{n,n-1} &= \mu_{n-1} S_{n-1} U_{n,n} + (1 - \mu_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \mu_k S_k U_{n,k+1} + (1 - \mu_k) I, \\
U_{n,k-1} &= \mu_{k-1} S_{k-1} U_{n,k} + (1 - \mu_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \mu_2 S_2 U_{n,3} + (1 - \mu_2) I, \\
W_n = U_{n,1} &= \mu_1 S_1 U_{n,2} + (1 - \mu_1) I,
\end{aligned} \tag{1.12}$$

where $\{\mu_i\}$ is nonnegative real sequence in $(0,1)$, for all $i \in \mathbb{N}$, $S_1, S_2, \ldots$ form a family of infinitely nonexpansive mappings of C into itself. It is obvious that W_n is nonexpansive from C into itself, such a mapping W_n is called a W-mapping generated by $S_1, S_2, \ldots, S_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping in a real Hilbert space H:

$$\min_{x \in C} \left\{ \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle \right\}, \tag{1.13}$$

where A is a bounded linear operator on H, C is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Recall that A be a strongly positive bounded linear operator on H if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.14}$$

Marino and Xu [6] introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [7]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad \forall n \in \mathbb{N}, \tag{1.15}$$

where A is a strongly positive bounded linear operator on H, f is a contraction on H and S is a nonexpansive on H. They proved that under those conditions are corrected, if $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.15) converges strongly to the unique solution z of the variational inequality

$$\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in F(S), \tag{1.16}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)} \left\{ \frac{1}{2}\langle Ax, x\rangle - h(x) \right\}, \tag{1.17}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

The so-called the system of generalized quasivariational inclusions problem is to find $(x^*, y^*) \in H \times H$ such that

$$\begin{aligned} &0 \in x^* - y^* + \lambda_1\big((B_1 + C_1)y^* + M_1x^*\big), \\ &0 \in y^* - x^* + \lambda_2\big((B_2 + C_2)x^* + M_2y^*\big), \end{aligned} \tag{1.18}$$

where $B_i, C_i : H \to H$, $M_i : H \to 2^H$ are nonlinear mappings and $\lambda_i > 0$ for each $i = 1, 2$. As special cases of problem (1.18), we have the following.

(1) If $B_1 = B_2 = B$, $C_1 = C_2 = C$ and $M_1 = M_2 = M$, then problem (1.18) is reduced to find $(x^*, y^*) \in H \times H$ such that

$$\begin{aligned} &0 \in x^* - y^* + \lambda_1\big((B + C)y^* + Mx^*\big), \\ &0 \in y^* - x^* + \lambda_2\big((B + C)x^* + My^*\big). \end{aligned} \tag{1.19}$$

(2) If $C_1 = C_2 = 0$, then problem (1.18) is reduced to find $(x^*, y^*) \in H \times H$ such that

$$\begin{aligned} &0 \in x^* - y^* + \lambda_1\big(B_1y^* + M_1x^*\big), \\ &0 \in y^* - x^* + \lambda_2\big(B_2x^* + M_2y^*\big), \end{aligned} \tag{1.20}$$

which called that the system of quasivariational inclusions problem.

(3) If $B_1 = B_2 = B$ and $M_1 = M_2 = M$, then problem (1.20) is reduced to find $(x^*, y^*) \in H \times H$ such that

$$0 \in x^* - y^* + \lambda_1 (By^* + Mx^*),$$
$$0 \in y^* - x^* + \lambda_2 (Bx^* + My^*). \tag{1.21}$$

(4) If $x^* = y^*$, then problem (1.21) is reduced to find $x^* \in H$ such that

$$0 \in Bx^* + Mx^*. \tag{1.22}$$

We denote by $\mathrm{VI}(H, B, M)$ the set of solutions of variational inclusion of the problem (1.22).

(5) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdifferential of ϕ, then problem (1.22) is equivalent to find $x^* \in H$ such that

$$\langle Bx^*, v - x^* \rangle + \phi(v) - \phi(x^*) \geq 0, \quad \forall v \in H, \tag{1.23}$$

which is said to be the *mixed quasivariational inequality* (see, e.g., [8, 9] for more details).

(6) If ϕ is the indicator function of C, then problem (1.23) is equivalent to the classical variational inequality problem, denoted by $\mathrm{VI}(C, B)$, to find $x^* \in C$ such that

$$\langle Bx^*, v - x^* \rangle \geq 0, \quad \forall v \in C. \tag{1.24}$$

Iiduka and Takahashi [10] introduced iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality (1.24) as the following theorem.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let B be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, B) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C (x_n - \lambda_n B x_n), \quad \forall n \in \mathbb{N}, \tag{1.25}$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ satisfying the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(C,B)} x$.

Definition 1.1 (see [11]). Let $M : H \to 2^H$ be a multivalued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u)$, for all $u \in H$, is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Recently, Zhang et al. [11] considered the problem (1.22). To be more precise, they proved the following theorem.

Theorem ZLC. *Let H be a real Hilbert space, $B : H \to H$ be an α-inverse-strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping, and $S : H \to H$ be a nonexpansive mapping. Suppose that the set $F(S) \cap VI(H,B,M) \neq \emptyset$, where $VI(H,B,M)$ is the set of solutions of variational inclusion* (1.22)*. Suppose that $x_1 = x \in H$ and $\{x_n\}$ is the sequence defined by*

$$y_n = J_{M,\lambda}(x_n - \lambda B x_n),$$
$$x_{n+1} = \alpha_n x + (1-\alpha_n) S y_n, \tag{1.26}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(H,B,M)} x$.

Very recently, Li and Wu [12] introduced an iterative scheme:

$$y_n = J_{M,\lambda}(x_n - \lambda B x_n),$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n) I - \alpha_n A\big)\big(\mu S_k x_n + (1-\mu) y_n\big), \tag{1.27}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, A is a strongly positive bounded linear operator on H, f is a contraction on H and S_k is a mapping on H defined by $S_k x = kx + (1-k)Sx$ such that S is a k-strictly pseudocontractive mapping on H with a fixed point. They proved that under missing condition of μ, it should be $0 < \mu < 1$ by those Lemma 1.6, others are corrected, if $\Omega = F(S) \cap \mathrm{VI}(H,B,M) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.27) converges strongly to $z = P_\Omega (I - A + \gamma f) z$ of the variational inequality

$$\big\langle (A - \gamma f) z, x - z \big\rangle \geq 0, \quad \forall x \in \Omega, \tag{1.28}$$

which is the optimality condition for the minimization problem:

$$\min_{x\in\Omega} \left\{ \frac{1}{2} \langle Ax, x\rangle - h(x) \right\}, \tag{1.29}$$

where h is a potential function for γf.

Inspired and motivated by the works mentioned above, in this paper, we introduce an iterative scheme (2.2) below by the viscosity approximation to find the set of solutions of the generalized system of relaxed cocoercive quasivariational inclusions and the set of common fixed points of an infinite family of strictly pseudocontractive mappings problems in Hilbert spaces. We suggest and analyze an iterative scheme under some appropriate conditions imposed on the parameters, we prove that another strong convergence theorem for the above two sets is obtained. The results presented in this paper improve and extend the main results of Li and Wu [12] and many others.

We collect the following lemmas which be used in the proof for the main results in the next section.

Lemma 1.2 (see [6]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then,*

(1) *if $0 < \gamma < \overline{\gamma}/\alpha$, then $\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2$, $x, y \in H$;*

(2) *if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 1.3 (see [13]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{1.30}$$

Suppose that

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n, \quad n \geq 1 \tag{1.31}$$

and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 1.4 (see [14]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \eta_n)a_n + \delta_n, \quad n \geq 1, \tag{1.32}$$

where $\{\eta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \eta_n = \infty$;

(2) $\limsup_{n\to\infty}(\delta_n/\eta_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5 (see [15]). *Let C be a nonempty closed convex subset of a Hilbert space H, define mapping W_n as* (1.12), *let $S_i : C \to C$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, and let $\{\mu_i\}$ be a sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{\infty} F(S_i)$ for each $n \geq 1$;*

(2) *for each $x \in C$ and for each positive integer k, $\lim_{n\to\infty} U_{n,k}x$ exists;*

(3) *the mapping $W : C \to C$ define by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{1.33}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\mu_1, \mu_2, \ldots$.

Lemma 1.6 (see [11]). *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$.*

Lemma 1.7 (see [11]). *$u \in H$ is a solution of variational inclusion* (1.22) *if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, for all $\lambda > 0$, that is,*

$$\mathrm{VI}(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{1.34}$$

Lemma 1.8. *For any $(x^*, y^*) \in H \times H$, where $y^* = J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))x^*$, we have (x^*, y^*) is a solution of problem* (1.18) *if and only if x^* is a fixed point of the mapping D defined by*

$$Dx = J_{M_1,\lambda_1}((I - \lambda_1(B_1 + C_1))J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))x). \tag{1.35}$$

Proof. Observe from (1.18) that

$$\begin{aligned} & 0 \in x^* - y^* + \lambda_1((B_1 + C_1)y^* + M_1x^*), \\ & 0 \in y^* - x^* + \lambda_2((B_2 + C_2)x^* + M_2y^*) \\ \Longleftrightarrow\ & \begin{aligned} & x^* = J_{M_1,\lambda_1}(I - \lambda_1(B_1 + C_1))y^*, \\ & y^* = J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))x^* \end{aligned} \\ \Longleftrightarrow\ & Dx^* = J_{M_1,\lambda_1}((I - \lambda_1(B_1 + C_1))J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))x^*) = x^*. \end{aligned} \tag{1.36}$$

□

Lemma 1.9 (see [16]). *Let C be a closed convex subset of a strictly convex Banach space E. Let S and T be two nonexpansive mappings on C. Suppose that $F(S) \cap F(T)$ is nonempty. Then a mapping R on C defined by $Rx = aSx + (1 - a)Tx$, where $a \in (0, 1)$, for $x \in C$ is well defined and nonexpansive and $F(R) = F(S) \cap F(T)$ holds.*

Lemma 1.10 (see [17]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

Lemma 1.11 (see [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudocontraction. Define $S : C \to H$ by $Sx = \alpha x + (1 - \alpha)Tx$ for each $x \in C$. Then, as $\alpha \in [k, 1)$, S is a nonexpansive such that $F(S) = F(T)$.*

2. Main Results

Let H be a real Hilbert space, $M_i : H \to 2^H$ be a maximal monotone mapping, $B_i : H \to H$ be a relaxed (α_i, r_i)-cocoercive and ξ_i-Lipschitz continuous mappings, respectively, $C_i : H \to H$ be a relaxed $(\overline{\alpha}_i, \overline{r}_i)$-cocoercive and $\overline{\xi}_i$-Lipschitz continuous mappings, respectively, for each $i = 1, 2$. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\|A\| \leq 1$ and $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\{T_n : H \to H\}$ be a family of k_n-strictly pseudocontractive mappings with a fixed point such that $k_n \in [0, 1)$ for all $n \in \mathbb{N}$. Define $S_nx = \delta_nx + (1 - \delta_n)T_nx$, where $\delta_n \in [k_n, 1)$, for all $n \in \mathbb{N}$, and let $W_n : H \to H$ be a W-mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$.

Define sequence of mappings $\{P_n : H \to H\}$ and mapping $Q : H \to H$ as follows:

$$P_n x = \alpha_n \gamma f(W_n x) + \beta_n x + \big((1-\beta_n)I - \alpha_n A\big)\big(\gamma_n W_n x + (1-\gamma_n)Qx\big),$$
$$Qx = J_{M_1,\lambda_1}((I - \lambda_1(B_1 + C_1))J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))x), \tag{2.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, \|A\|^{-1})$, $\lambda_i \in (0, \epsilon_i]$, $\epsilon_i = \min\{2(r_i - \alpha_i\xi_i^2)/\xi_i(\xi_i + \overline{\xi}_i), 2(\overline{r}_i - \overline{\alpha}_i\overline{\xi}_i^2)/\overline{\xi}_i(\xi_i + \overline{\xi}_i)\}$, $r_i > \alpha_i\xi_i^2$, $\overline{r}_i > \overline{\alpha}_i\overline{\xi}_i^2$, for each $i = 1, 2$ and $0 < \gamma < \overline{\delta}/\delta$.

Under some appropriate imposed on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$, we also know that $\|(1-\beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n\overline{\delta}$, and, we also have that W_n, J_{M_i,λ_i} and $I - \lambda_i(B_i + C_i)$ are nonexpansive for each $i = 1, 2$ (see argument in the proof of Theorem 2.1 below). Observe that Q is a nonexpansive, and so P_n is a contraction with coefficient $1 - (\overline{\delta} - \gamma\delta)\alpha_n$. Therefore, by Banach contraction principle guarantees that P_n has a unique fixed point in H.

By the idea above, we obtain an iteration scheme by the viscosity approximation for solving the generalized system of relaxed cocoercive quasivariational inclusions and fixed point problems of an infinite family of strictly pseudocontractive mappings as the following theorem.

Theorem 2.1. *Let H be a real Hilbert space, $M_i : H \to 2^H$ be a maximal monotone mapping, $B_i : H \to H$ be a relaxed (α_i, r_i)-cocoercive and ξ_i-Lipschitz continuous mappings, respectively, $C_i : H \to H$ be a relaxed $(\overline{\alpha}_i, \overline{r}_i)$-cocoercive and $\overline{\xi}_i$-Lipschitz continuous mappings, respectively, for each $i = 1, 2$. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\|A\| \leq 1$ and $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\{T_n : H \to H\}$ be a family of k_n-strictly pseudocontractive mappings with a fixed point such that $k_n \in [0, 1)$ for all $n \in \mathbb{N}$. Define $S_n x = \delta_n x + (1-\delta_n)T_n x$, where $\delta_n \in [k_n, 1)$, for all $n \in \mathbb{N}$, and let $W_n : H \to H$ be a W-mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(D) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$ where $D : H \to H$ defined by $Dx = J_{M_1,\lambda_1}((I - \lambda_1(B_1 + C_1))J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))x)$. For $x_1 \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$z_n = J_{M_2,\lambda_2}(x_n - \lambda_2(B_2 + C_2)x_n),$$
$$y_n = J_{M_1,\lambda_1}(z_n - \lambda_1(B_1 + C_1)z_n), \tag{2.2}$$
$$x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big)\big(\gamma_n W_n x_n + (1-\gamma_n)y_n\big),$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, $\lambda_i \in (0, \epsilon_i]$, $\epsilon_i = \min\{2(r_i - \alpha_i\xi_i^2)/\xi_i(\xi_i + \overline{\xi}_i), 2(\overline{r}_i - \overline{\alpha}_i\overline{\xi}_i^2)/\overline{\xi}_i(\xi_i + \overline{\xi}_i)\}$, $r_i > \alpha_i\xi_i^2$, $\overline{r}_i > \overline{\alpha}_i\overline{\xi}_i^2$, for each $i = 1, 2$, satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3) $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $0 < \lim_{n\to\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $w \in \Omega$ *where* $w = P_\Omega(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega, \tag{2.3}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \Omega} \left\{ \frac{1}{2} \langle Ax, x \rangle - h(x) \right\}, \tag{2.4}$$

where h *is a potential function for* γf *(i.e.,* $h'(x) = \gamma f(x)$ *for* $x \in H$*) and* $(w, J_{M_2,\lambda_2}(I - \lambda_2(B_2 + C_2))w)$ *is a solution of problem* (1.18).

Proof. From (C1) and (C2), we have $\alpha_n \to 0$ and $\limsup_{n\to\infty} \beta_n < 1$. Thus, we may assume without loss of generality that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. For each $i = 1, 2$, since

$$0 < \lambda_i \leq \epsilon_i \leq \frac{2(r_i - \alpha_i \xi_i^2)}{\xi_i \left(\xi_i + \overline{\xi}_i \right)}, \qquad 0 < \lambda_i \leq \epsilon_i \leq \frac{2\left(\overline{r}_i - \overline{\alpha}_i \overline{\xi}_i^2 \right)}{\overline{\xi}_i \left(\xi_i + \overline{\xi}_i \right)}, \tag{2.5}$$

where $r_i > \alpha_i \xi_i^2$ and $\overline{r}_i > \overline{\alpha}_i \overline{\xi}_i^2$, we have

$$\lambda_i^2 \xi_i \left(\xi_i + \overline{\xi}_i \right) + 2\lambda_i \alpha_i \xi_i^2 - 2\lambda_i r_i \leq 0, \qquad \lambda_i^2 \overline{\xi}_i \left(\xi_i + \overline{\xi}_i \right) + 2\lambda_i \overline{\alpha}_i \overline{\xi}_i^2 - 2\lambda_i \overline{r}_i \leq 0. \tag{2.6}$$

For any $x, y \in H$, it follows by the relaxed (α_i, r_i)-cocoercivity and ξ_i-Lipschitz continuity of B_i and the relaxed $(\overline{\alpha}_i, \overline{r}_i)$-cocoercivity and $\overline{\xi}_i$-Lipschitz continuity of C_i that

$$\begin{aligned}
&\left\| (I - \lambda_i(B_i + C_i))x - (I - \lambda_i(B_i + C_i))y \right\|^2 \\
&\quad = \left\| (x - y) - \lambda_i \left((B_i + C_i)x - (B_i + C_i)y \right) \right\|^2 \\
&\quad = \|x - y\|^2 - 2\lambda_i \langle x - y, (B_i + C_i)x - (B_i + C_i)y \rangle + \lambda_i^2 \left\| (B_i + C_i)x - (B_i + C_i)y \right\|^2 \\
&\quad = \|x - y\|^2 - 2\lambda_i \langle x - y, B_i x - B_i y \rangle - 2\lambda_i \langle x - y, C_i x - C_i y \rangle \\
&\qquad + \lambda_i^2 \left\| (B_i x - B_i y) + (C_i x - C_i y) \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \|x-y\|^2 - 2\lambda_i\left(-\alpha_i\|B_ix-B_iy\|^2 + r_i\|x-y\|^2\right) - 2\lambda_i\left(-\overline{\alpha}_i\|C_ix-C_iy\|^2 + \overline{r}_i\|x-y\|^2\right)\\
&\quad + \lambda_i^2(\|B_ix-B_iy\| + \|C_ix-C_iy\|)^2\\
&\le \|x-y\|^2 - 2\lambda_i\left(-\alpha_i\xi_i^2\|x-y\|^2 + r_i\|x-y\|^2\right) - 2\lambda_i\left(-\overline{\alpha}_i\overline{\xi}_i^2\|x-y\|^2 + \overline{r}_i\|x-y\|^2\right)\\
&\quad + \lambda_i^2\left(\xi_i\|x-y\| + \overline{\xi}_i\|x-y\|\right)^2\\
&= \left(1 + \left(\lambda_i^2\xi_i\left(\xi_i+\overline{\xi}_i\right) + 2\lambda_i\alpha_i\xi_i^2 - 2\lambda_i r_i\right) + \left(\lambda_i^2\overline{\xi}_i\left(\xi_i+\overline{\xi}_i\right) + 2\lambda_i\overline{\alpha}_i\overline{\xi}_i^2 - 2\lambda_i\overline{r}_i\right)\right)\|x-y\|^2\\
&\le \|x-y\|^2,
\end{aligned}
\tag{2.7}
$$

which implies that $I-\lambda_i(B_i+C_i)$ is a nonexpansive. Since, A is a linear bounded self-adjoint operator, we have

$$
\|A\| = \sup\{|\langle Ax,x\rangle| : x\in E,\ \|x\|=1\}. \tag{2.8}
$$

Observe that

$$
\begin{aligned}
\langle((1-\beta_n)I-\alpha_nA)x,x\rangle &= (1-\beta_n)\langle x,x\rangle - \alpha_n\langle Ax,x\rangle\\
&\ge (1-\beta_n) - \alpha_n\|A\|\\
&\ge 0.
\end{aligned}
\tag{2.9}
$$

Therefore, we obtain $(1-\beta_n)I-\alpha_nA$ is positive. Thus, by the strong positively of A, we get

$$
\begin{aligned}
\|(1-\beta_n)I-\alpha_nA\| &= \sup\{\langle((1-\beta_n)I-\alpha_nA)x,x\rangle : x\in E,\ \|x\|=1\}\\
&= \sup\{(1-\beta_n)\langle x,x\rangle - \alpha_n\langle Ax,x\rangle : x\in E,\ \|x\|=1\}\\
&\le 1-\beta_n-\alpha_n\overline{\delta}.
\end{aligned}
\tag{2.10}
$$

Pick $x^*\in\Omega$. Then, we have

$$
x^* = Dx^* = J_{M_1,\lambda_1}((I-\lambda_1(B_1+C_1))J_{M_2,\lambda_2}(I-\lambda_2(B_2+C_2))x^*). \tag{2.11}
$$

And, let $y^* = J_{M_2,\lambda_2}(x^* - \lambda_2(B_2 + C_2)x^*)$. Therefore, from (2.11), we have

$$x^* = J_{M_1,\lambda_1}\big(y^* - \lambda_1(B_1 + C_1)y^*\big). \tag{2.12}$$

By the nonexpansivity of J_{M_i,λ_i} and $I - \lambda_i(B_i + C_i)$, we have

$$\begin{aligned}\|z_n - y^*\| &= \|J_{M_2,\lambda_2}(x_n - \lambda_2(B_2 + C_2)x_n) - J_{M_2,\lambda_2}(x^* - \lambda_2(B_2 + C_2)x^*)\| \\ &\le \|(x_n - \lambda_2(B_2 + C_2)x_n) - (x^* - \lambda_2(B_2 + C_2)x^*)\| \\ &\le \|x_n - x^*\|.\end{aligned} \tag{2.13}$$

And, we have

$$\begin{aligned}\|y_n - x^*\| &= \big\|J_{M_1,\lambda_1}(z_n - \lambda_1(B_1 + C_1)z_n) - J_{M_1,\lambda_1}\big(y^* - \lambda_1(B_1 + C_1)y^*\big)\big\| \\ &\le \big\|(z_n - \lambda_1(B_1 + C_1)z_n) - \big(y^* - \lambda_1(B_1 + C_1)y^*\big)\big\| \\ &\le \|z_n - y^*\|.\end{aligned} \tag{2.14}$$

Therefore, by (2.13) and (2.14), we have

$$\|y_n - x^*\| \le \|z_n - y^*\| \le \|x_n - x^*\|. \tag{2.15}$$

Let $t_n = \gamma_n W_n x_n + (1 - \gamma_n) y_n$. Since $S_n x = \delta_n x + (1 - \delta_n) T_n x$, where $\delta_n \in [k_n, 1)$ and $\{T_n\}$ be a family of k_n-strict pseudocontraction. By Lemma 1.11, we have S_n is a nonexpansive and $F(S_n) = F(T_n)$. Therefore, by Lemma 1.5(1), we get $F(W_n) = \bigcap_{i=1}^{\infty} F(S_i) = \bigcap_{i=1}^{\infty} F(T_i)$, which implies that $W_n x^* = x^*$. It follows by (2.15) and the nonexpansivity of W_n that

$$\begin{aligned}\|t_n - x^*\| &= \|\gamma_n W_n x_n + (1 - \gamma_n) y_n - x^*\| \\ &= \|\gamma_n (W_n x_n - x^*) + (1 - \gamma_n)(y_n - x^*)\| \\ &\le \gamma_n \|x_n - x^*\| + (1 - \gamma_n)\|y_n - x^*\| \\ &\le \|x_n - x^*\|.\end{aligned} \tag{2.16}$$

From (2.16), by the contraction of f and the nonexpansivity of W_n, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|\alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)t_n - x^*\right\| \\
&= \left\|\alpha_n(\gamma f(W_n x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(t_n - x^*)\right\| \\
&\le \alpha_n \left\|\gamma f(W_n x_n) - Ax^*\right\| + \beta_n\|x_n - x^*\| + \left\|(1-\beta_n)I - \alpha_n A\right\| \|t_n - x^*\| \\
&\le \alpha_n \gamma \left\|f(W_n x_n) - f(x^*)\right\| + \alpha_n\left\|\gamma f(x^*) - Ax^*\right\| + \beta_n\|x_n - x^*\| \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\delta}\right)\|x_n - x^*\| \\
&\le \alpha_n\gamma\delta\|W_n x_n - x^*\| + \alpha_n\left\|\gamma f(x^*) - Ax^*\right\| + \left(1 - \alpha_n\overline{\delta}\right)\|x_n - x^*\| \\
&\le \left(1 - \left(\overline{\delta} - \gamma\delta\right)\alpha_n\right)\|x_n - x^*\| + \alpha_n\left\|\gamma f(x^*) - Ax^*\right\| \\
&\le \max\left\{\|x_n - x^*\|, \frac{\left\|\gamma f(x^*) - Ax^*\right\|}{\overline{\delta} - \gamma\delta}\right\}.
\end{aligned} \tag{2.17}$$

It follows from induction that

$$\|x_{n+1} - x^*\| \le \max\left\{\|x_1 - x^*\|, \frac{\left\|\gamma f(x^*) - Ax^*\right\|}{\overline{\delta} - \gamma\delta}\right\}, \tag{2.18}$$

for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{W_n x_n\}$, $\{f(W_n x_n)\}$ and $\{At_n\}$.

Next, we prove that $\|v_n - x_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|t_n - x_n\| \to 0$ as $n \to \infty$. By the nonexpansivity of J_{M_i,λ_i} and $I - \lambda_i(B_i + C_i)$, we have

$$\begin{aligned}
\left\|y_{n+1} - y_n\right\| &= \|J_{M_1,\lambda_1}(z_{n+1} - \lambda_1(B_1 + C_1)z_{n+1}) - J_{M_1,\lambda_1}(z_n - \lambda_1(B_1 + C_1)z_n)\| \\
&\le \|(z_{n+1} - \lambda_1(B_1 + C_1)z_{n+1}) - (z_n - \lambda_1(B_1 + C_1)z_n)\| \\
&\le \|z_{n+1} - z_n\|.
\end{aligned} \tag{2.19}$$

Similarly, we have

$$\|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\|. \tag{2.20}$$

Therefore, from (2.19) and (2.20), we have

$$\|y_{n+1} - y_n\| \le \|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\|. \tag{2.21}$$

By the nonexpansivity of S_i and $U_{n,i}$, we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &= \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
&= \left\|\mu_1 S_1 U_{n+1,2}x_n + (1-\mu_1)x_n - (\mu_1 S_1 U_{n,2}x_n + (1-\mu_1)x_n)\right\| \\
&\le \mu_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \mu_1\left\|\mu_2 S_2 U_{n+1,3}x_n + (1-\mu_2)x_n - (\mu_2 S_2 U_{n,3}x_n + (1-\mu_2)x_n)\right\| \\
&\le \mu_1\mu_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\
&\vdots \\
&\le \left(\prod_{i=1}^{n}\mu_i\right)\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&\le M\prod_{i=1}^{n}\mu_i,
\end{aligned} \tag{2.22}$$

for some constant M such that $M \ge \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \ge 0$. Therefore, from (2.22), by the nonexpansivity of W_{n+1}, we have

$$\begin{aligned}
\|W_{n+1}x_{n+1} - W_nx_n\| &\le \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\| \\
&\le \|x_{n+1} - x_n\| + M\prod_{i=1}^{n}\mu_i.
\end{aligned} \tag{2.23}$$

Since

$$\begin{aligned}
t_{n+1} - t_n &= \gamma_{n+1}W_{n+1}x_{n+1} + (1-\gamma_{n+1})y_{n+1} - (\gamma_nW_nx_n + (1-\gamma_n)y_n) \\
&= \gamma_{n+1}(W_{n+1}x_{n+1} - W_nx_n) + (\gamma_{n+1} - \gamma_n)(W_nx_n - y_n) \\
&\quad + (1-\gamma_{n+1})(y_{n+1} - y_n),
\end{aligned} \tag{2.24}$$

combining (2.21), (2.23), and (2.24), we have

$$\begin{aligned}
\|t_{n+1} - t_n\| &\le \gamma_{n+1}\|W_{n+1}x_{n+1} - W_nx_n\| + |\gamma_{n+1} - \gamma_n|\left\|W_nx_n - y_n\right\| + (1-\gamma_{n+1})\left\|y_{n+1} - y_n\right\| \\
&\le \gamma_{n+1}\left(\|x_{n+1} - x_n\| + M\prod_{i=1}^{n}\mu_i\right) + |\gamma_{n+1} - \gamma_n|\left\|W_nx_n - y_n\right\| + (1-\gamma_{n+1})\left\|y_{n+1} - y_n\right\| \\
&\le M\prod_{i=1}^{n}\mu_i + |\gamma_{n+1} - \gamma_n|\left\|W_nx_n - y_n\right\| + \|x_{n+1} - x_n\|.
\end{aligned} \tag{2.25}$$

Let $v_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$. Then we have

$$x_{n+1} = (1-\beta_n)v_n + \beta_n x_n. \tag{2.26}$$

Since

$$\begin{aligned} v_{n+1} - v_n &= \frac{\alpha_{n+1}\gamma f(W_{n+1}x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)t_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(W_n x_n) + ((1-\beta_n)I - \alpha_n A)t_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(W_{n+1}x_{n+1}) - At_{n+1}) - \frac{\alpha_n}{1-\beta_n}(\gamma f(W_n x_n) - At_n) + (t_{n+1} - t_n), \end{aligned} \tag{2.27}$$

combining (2.25) and (2.27), we have

$$\begin{aligned} \|v_{n+1} - v_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(W_{n+1}x_{n+1}) - At_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(W_n x_n) - At_n\| + \|t_{n+1} - t_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(W_{n+1}x_{n+1}) - At_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(W_n x_n) - At_n\| \\ &\quad + M\prod_{i=1}^{n}\mu_i + |\gamma_{n+1} - \gamma_n|\|W_n x_n - y_n\| + \|x_{n+1} - x_n\|. \end{aligned} \tag{2.28}$$

Therefore, by (C1), (C2) and $\lim_{n\to\infty}\prod_{i=1}^{n}\mu_i = 0$, we get

$$\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{2.29}$$

From (2.26) and (2.29), by (C2) and Lemma 1.3, we obtain

$$\|v_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.30}$$

From (2.26), by (2.30), we obtain

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|v_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.31}$$

Since

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n\gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)t_n - x_n \\ &= \alpha_n(\gamma f(W_n x_n) - At_n) + (1-\beta_n)(t_n - x_n), \end{aligned} \tag{2.32}$$

therefore,

$$(1-\beta_n)\|t_n - x_n\| \le \alpha_n \left\|\gamma f(W_n x_n) - At_n\right\| + \|x_{n+1} - x_n\|. \tag{2.33}$$

From (2.31), by (C1) and (C2), we obtain

$$\|t_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.34}$$

For all $x, y \in H$, by Lemma 1.2(2), the nonexpansivity of P_Ω and the contraction of f, we have

$$\begin{aligned}\left\|P_\Omega(I - A + \gamma f)x - P_\Omega(I - A + \gamma f)y\right\| &\le \left\|(I - A + \gamma f)x - (I - A + \gamma f)y\right\| \\ &\le \gamma\left\|f(x) - f(y)\right\| + \|I - A\|\left\|x - y\right\| \\ &\le \gamma\delta\left\|x - y\right\| + \left(1 - \overline{\delta}\right)\left\|x - y\right\| \\ &= \left(1 - \left(\overline{\delta} - \gamma\delta\right)\right)\left\|x - y\right\|.\end{aligned} \tag{2.35}$$

Therefore, $P_\Omega(I - A + \gamma f)$ is a contraction with coefficient $1 - (\overline{\delta} - \gamma\delta)$, by Banach contraction principle guarantees that $P_\Omega(I - A + \gamma f)$ has a unique fixed point, say $w \in H$, that is, $w = P_\Omega(I - A + \gamma f)w$.

Next, we claim that

$$\limsup_{n\to\infty}\langle \gamma f(w) - Aw, x_n - w\rangle \le 0. \tag{2.36}$$

To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(w) - Aw, x_n - w\rangle = \lim_{i\to\infty}\langle \gamma f(w) - Aw, x_{n_i} - w\rangle. \tag{2.37}$$

Since, $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $\overline{w}$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup \overline{w}$ as $i \to \infty$.

Next, we prove that $\overline{w} \in \Omega$. Define sequence of mappings $\{R_n : H \to \mathrm{H}\}$ and mapping $R : H \to H$ by

$$\begin{gathered} R_n x = \gamma_n W_n x + (1 - \gamma_n)Dx, \quad \forall n \in \mathbb{N}, \\ Rx = \lim_{n\to\infty} R_n x. \end{gathered} \tag{2.38}$$

Thus, by Lemma 1.5(3) and (C3), we have

$$Rx = aWx + (1 - a)Dx, \tag{2.39}$$

where $0 < a = \lim_{n\to\infty}\gamma_n < 1$. Since, W and D are nonexpansive and by Lemma 1.5(3), $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$. Therefore, by Lemma 1.9, we get R is a nonexpansive and $F(R) = F(W) \cap F(D) = \bigcap_{i=1}^{\infty} F(S_i) \cap F(D)$. Since, $S_n x = \delta_n x + (1-\delta_n)T_n x$, where $\delta_n \in [k_n, 1)$. Thus, by Lemma 1.11, we obtain

$$F(R) = \bigcap_{i=1}^{\infty} F(S_i) \cap F(D) = \bigcap_{i=1}^{\infty} F(T_i) \cap F(D) = \Omega. \tag{2.40}$$

From (2.34), we have $\|t_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$. Thus, from (2.38), we get $\|Rx_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$. It follows from $x_{n_i} \rightharpoonup \overline{w}$ and by Lemma 1.10 that $\overline{w} \in F(R)$, that is $\overline{w} \in \Omega$. Therefore, from (2.37), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(w) - Aw, x_n - w\rangle &= \lim_{i\to\infty}\langle \gamma f(w) - Aw, x_{n_i} - w\rangle \\ &= \langle (\gamma f - A)w, \overline{w} - w\rangle \le 0.\end{aligned} \tag{2.41}$$

Next, we prove that $x_n \to w$ as $n \to \infty$. Since $w \in \Omega$, we have $W_n w = w$, and the same as in (2.16), we have

$$\|t_n - w\| \le \|x_n - w\|. \tag{2.42}$$

It follows by the contraction of f and the nonexpansivity of W_n that

$$\begin{aligned}\|x_{n+1} - w\|^2 &= \left\|\alpha_n\gamma f(W_n x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)t_n - w\right\|^2 \\ &= \left\|\left((1-\beta_n)I - \alpha_n A\right)(t_n - w) + \beta_n(x_n - w) + \alpha_n\left(\gamma f(W_n x_n) - Aw\right)\right\| \\ &= \left\|\left((1-\beta_n)I - \alpha_n A\right)(t_n - w) + \beta_n(x_n - w)\right\|^2 + \alpha_n^2\left\|\gamma f(W_n x_n) - Aw\right\|^2 \\ &\quad + 2\alpha_n\beta_n\langle x_n - w, \gamma f(W_n x_n) - Aw\rangle \\ &\quad + 2\alpha_n\left\langle\left((1-\beta_n)I - \alpha_n A\right)(t_n - w), \gamma f(W_n x_n) - Aw\right\rangle \\ &\le \left(\left(1-\beta_n-\alpha_n\overline{\delta}\right)\|t_n - w\| + \beta_n\|x_n - w\|\right)^2 + \alpha_n^2\left\|\gamma f(W_n x_n) - Aw\right\|^2 \\ &\quad + 2\alpha_n\beta_n\gamma\langle x_n - w, f(W_n x_n) - f(w)\rangle + 2\alpha_n\beta_n\langle x_n - w, \gamma f(w) - Aw\rangle \\ &\quad + 2\alpha_n(1-\beta_n)\gamma\langle t_n - w, f(W_n x_n) - f(w)\rangle \\ &\quad + 2\alpha_n(1-\beta_n)\langle t_n - w, \gamma f(w) - Aw\rangle \\ &\quad - 2\alpha_n^2\langle A(t_n - w), \gamma f(W_n x_n) - Aw\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\alpha_n\overline{\delta}\right)^2\|x_n-w\|^2+2\alpha_n\beta_n\gamma\|x_n-w\|\left\|f(W_nx_n)-f(w)\right\| \\
&\quad +2\alpha_n(1-\beta_n)\gamma\|t_n-w\|\left\|f(W_nx_n)-f(w)\right\|+\theta_n \\
&\leq \left(1-\alpha_n\overline{\delta}\right)^2\|x_n-w\|^2+2\alpha_n\beta_n\gamma\delta\|x_n-w\|^2+2\alpha_n(1-\beta_n)\gamma\delta\|x_n-w\|^2+\theta_n \\
&= \left(1-\alpha_n\overline{\delta}\right)^2\|x_n-w\|^2+2\alpha_n\gamma\delta\|x_n-w\|^2+\theta_n \\
&= \left(1-2\left(\overline{\delta}-\gamma\delta\right)\alpha_n\right)\|x_n-w\|^2+\alpha_n^2\overline{\delta}^2\|x_n-w\|^2+\theta_n \\
&\leq (1-\eta_n)\|x_n-w\|^2+\delta_n',
\end{aligned}
\tag{2.43}
$$

where $\eta_n := (\overline{\delta}-\gamma\delta)\alpha_n \in (0,1)$, $\delta_n' := \alpha_n^2\overline{\delta}^2\|x_n-w\|^2+\theta_n$ and

$$
\begin{aligned}
\theta_n := {}&\alpha_n^2\left\|\gamma f(W_nx_n)-Aw\right\|^2+2\alpha_n\beta_n\langle x_n-w,\gamma f(w)-Aw\rangle \\
&+2\alpha_n(1-\beta_n)\langle t_n-w,\gamma f(w)-Aw\rangle-2\alpha_n^2\langle A(t_n-w),\gamma f(W_nx_n)-Aw\rangle.
\end{aligned}
\tag{2.44}
$$

By (2.34), (2.41), (C1) and (C3), we can found that $\sum_{n=1}^{\infty}\eta_n=\infty$ and $\limsup_{n\to\infty}(\delta_n'/\eta_n)\leq 0$. By Lemma 1.4, we obtain $\{x_n\}$ converges strongly to w. This proof is completed. □

Remarks 2.2. Theorem 2.1 improve and extend to the main results of Li and Wu [12] for solving the generalized system of relaxed cocoercive quasivariational inclusions and fixed points problems of an infinite family of strictly pseudocontractive mappings.

3. Applications

Theorem 3.1. *Let H be a real Hilbert space, $M : H \to 2^H$ be a maximal monotone mapping, $B : H \to H$ be a relaxed (α, r)-cocoercive and ξ-Lipschitz continuous mappings, respectively, $C : H \to H$ be a relaxed $(\overline{\alpha},\overline{r})$-cocoercive and $\overline{\xi}$-Lipschitz continuous mappings, respectively. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta}\in(0,1]$ such that $\|A\|\leq 1$ and $f : H \to H$ be a contraction mapping with coefficient $\delta\in(0,1)$. Let $\{T_n : H \to H\}$ be a family of k_n-strictly pseudocontractive mappings with a fixed point such that $k_n\in[0,1)$ for all $n\in\mathbb{N}$. Define $S_nx=\delta_nx+(1-\delta_n)T_nx$, where $\delta_n\in[k_n,1)$, for all $n\in\mathbb{N}$, and let $W_n : H \to H$ be a W-mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\}\subset(0,\mu]$, for some $\mu\in(0,1)$. Assume that $\Omega := \bigcap_{n=1}^{\infty}F(T_n)\cap F(D)\neq\emptyset$ and $0<\gamma<\overline{\delta}/\delta$ where $D : H \to H$ defined by $Dx = J_{M,\lambda_1}((I-\lambda_1(B+C))J_{M,\lambda_2}(I-\lambda_2(B+C))x)$. For $x_1\in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$
\begin{aligned}
z_n &= J_{M,\lambda_2}(x_n-\lambda_2(B+C)x_n), \\
y_n &= J_{M,\lambda_1}(z_n-\lambda_1(B+C)z_n), \\
x_{n+1} &= \alpha_n\gamma f(W_nx_n)+\beta_nx_n+\left((1-\beta_n)I-\alpha_nA\right)(\gamma_nW_nx_n+(1-\gamma_n)y_n),
\end{aligned}
\tag{3.1}
$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$, $\lambda_i \in (0,\epsilon]$, $\epsilon = \min\{2(r-\alpha\xi^2)/\xi(\xi+\overline{\xi}), 2(\overline{r}-\overline{\alpha}\overline{\xi}^2)/\overline{\xi}(\xi+\overline{\xi})\}$, $r > \alpha\xi^2$, $\overline{r} > \overline{\alpha}\overline{\xi}^2$, *for each* $i = 1,2$, *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}|\gamma_{n+1}-\gamma_n| = 0$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3) $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $0 < \lim_{n\to\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $w \in \Omega$ *where* $w = P_\Omega(I-A+\gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A-\gamma f)w, y-w\rangle \geq 0, \quad \forall y \in \Omega, \tag{3.2}$$

which is the optimality condition for the minimization problem

$$\min_{x\in\Omega}\left\{\frac{1}{2}\langle Ax,x\rangle - h(x)\right\}, \tag{3.3}$$

where h *is a potential function for* γf *(i.e.,* $h'(x) = \gamma f(x)$ *for* $x \in H$*) and* $(w, J_{M,\lambda_2}(I-\lambda_2(B+C))w)$ *is a solution of problem* (1.19).

Proof. It is concluded obviously, from Theorem 2.1 by putting $M_1 = M_2 = M$, $B_1 = B_2 = B$ and $C_1 = C_2 = C$. □

Theorem 3.2. *Let* H *be a real Hilbert space,* $M : H \to 2^H$ *be a maximal monotone mapping,* $B : H \to H$ *be a relaxed* (α, r)*-cocoercive and* ξ*-Lipschitz continuous mappings, respectively,* $C : H \to H$ *be a relaxed* $(\overline{\alpha},\overline{r})$*-cocoercive and* $\overline{\xi}$*-Lipschitz continuous mappings, respectively. Let* $A : H \to H$ *be a strongly positive linear bounded self-adjoint operator mapping with coefficient* $\overline{\delta} \in (0,1]$ *such that* $\|A\| \leq 1$ *and* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0,1)$*. Let* $\{T_n : H \to H\}$ *be a family of nonexpansive mappings. Define* $S_n x = \delta_n x + (1-\delta_n)T_n x$*, where* $\delta_n \in [0,1)$*, for all* $n \in \mathbb{N}$*, and let* $W_n : H \to H$ *be a* W*-mapping generated by* $\{S_n\}$ *and* $\{\mu_n\}$ *such that* $\{\mu_n\} \subset (0,\mu]$*, for some* $\mu \in (0,1)$*. Assume that* $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(D) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$ *where* $D : H \to H$ *defined by* $Dx = J_{M,\lambda_1}((I-\lambda_1(B+C))J_{M,\lambda_2}(I-\lambda_2(B+C))x)$*. For* $x_1 \in H$*, suppose that* $\{x_n\}$ *be generated iteratively by*

$$\begin{aligned} z_n &= J_{M,\lambda_2}(x_n - \lambda_2(B+C)x_n), \\ y_n &= J_{M,\lambda_1}(z_n - \lambda_1(B+C)z_n), \\ x_{n+1} &= \alpha_n\gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)(\gamma_n W_n x_n + (1-\gamma_n)y_n), \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$, $\lambda_i \in (0,\epsilon]$, $\epsilon = \min\{2(r-\alpha\xi^2)/\xi(\xi+\overline{\xi}), 2(\overline{r}-\overline{\alpha}\overline{\xi}^2)/\overline{\xi}(\xi+\overline{\xi})\}$, $r > \alpha\xi^2$, $\overline{r} > \overline{\alpha}\overline{\xi}^2$, *for each* $i = 1,2$, *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}|\gamma_{n+1}-\gamma_n| = 0$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3) $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $0 < \lim_{n\to\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega, \tag{3.5}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \Omega} \left\{ \frac{1}{2} \langle Ax, x \rangle - h(x) \right\}, \tag{3.6}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$) and $(w, J_{M,\lambda_2}(I - \lambda_2(B + C))w)$ is a solution of problem (1.19).

Proof. It is concluded obviously, from Theorem 3.1 by putting $k_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 3.3. *Let H be a real Hilbert space, $M_i : H \to 2^H$ be a maximal monotone mapping, $B_i : H \to H$ be a relaxed (α_i, r_i)-cocoercive and ξ_i-Lipschitz continuous mappings, respectively, for each $i = 1, 2$. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\|A\| \leq 1$ and $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\{T_n : H \to H\}$ be a family of k_n-strictly pseudocontractive mappings with a fixed point such that $k_n \in [0, 1)$ for all $n \in \mathbb{N}$. Define $S_n x = \delta_n x + (1 - \delta_n) T_n x$, where $\delta_n \in [k_n, 1)$, for all $n \in \mathbb{N}$, and let $W_n : H \to H$ be a W-mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(D) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$ where $D : H \to H$ defined by $Dx = J_{M_1,\lambda_1}((I - \rho_1 B_1) J_{M_2,\lambda_2}(I - \rho_2 B_2)x)$ such that $\lambda_i = \rho_i/2$ for each $i = 1, 2$. For $x_1 \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= J_{M_2,\lambda_2}(x_n - \rho_2 B_2 x_n), \\ y_n &= J_{M_1,\lambda_1}(z_n - \rho_1 B_1 z_n), \\ x_{n+1} &= \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A)(\gamma_n W_n x_n + (1 - \gamma_n) y_n), \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, $\rho_i \in (0, 2(r_i - \alpha_i \xi_i^2)/\xi_i^2]$, $r_i > \alpha_i \xi_i^2$, for each $i = 1, 2$, satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $0 < \lim_{n\to\infty} \gamma_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega, \tag{3.8}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \Omega} \left\{ \frac{1}{2} \langle Ax, x \rangle - h(x) \right\}, \tag{3.9}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$) and $(w, J_{M_2,\lambda_2}(I - \rho_2 B_2)w)$ is a solution of problem

$$\begin{aligned} &0 \in x^* - y^* + \lambda_1(2B_1 y^* + M_1 x^*), \\ &0 \in y^* - x^* + \lambda_2(2B_2 x^* + M_2 y^*). \end{aligned} \tag{3.10}$$

Proof. It is concluded obviously, from Theorem 2.1 by putting $B_i = C_i$ for each $i = 1, 2$. □

Theorem 3.4. *Let H be a real Hilbert space, $M : H \to 2^H$ be a maximal monotone mapping, $B : H \to H$ be a relaxed (α, r)-cocoercive and ξ-Lipschitz continuous mappings, respectively. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0,1]$ such that $\|A\| \leq 1$ and $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\{T_n : H \to H\}$ be a family of k_n-strictly pseudocontractive mappings with a fixed point such that $k_n \in [0,1)$ for all $n \in \mathbb{N}$. Define $S_n x = \delta_n x + (1 - \delta_n)T_n x$, where $\delta_n \in [k_n, 1)$, for all $n \in \mathbb{N}$, and let $W_n : H \to H$ be a W-mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0,1)$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(D) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$ where $D : H \to H$ defined by $Dx = J_{M,\lambda_1}((I - \rho_1 B)J_{M,\lambda_2}(I - \rho_2 B)x)$ such that $\lambda_i = \rho_i/2$ for each $i = 1, 2$. For $x_1 \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= J_{M,\lambda_2}(x_n - \rho_2 B x_n), \\ y_n &= J_{M,\lambda_1}(z_n - \rho_1 B z_n), \\ x_{n+1} &= \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)(\gamma_n W_n x_n + (1 - \gamma_n) y_n), \end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, $\rho_i \in (0, 2(r - \alpha\xi^2)/\xi^2]$, $r > \alpha\xi^2$, for each $i = 1, 2$, satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $0 < \lim_{n\to\infty} \gamma_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega, \tag{3.12}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \Omega} \left\{ \frac{1}{2}\langle Ax, x \rangle - h(x) \right\}, \tag{3.13}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$) and $(w, J_{M,\lambda_2}(I - \rho_2 B)w)$ is a solution of problem

$$\begin{aligned} &0 \in x^* - y^* + \lambda_1(2B y^* + M x^*), \\ &0 \in y^* - x^* + \lambda_2(2B x^* + M y^*). \end{aligned} \tag{3.14}$$

Proof. It is concluded obviously, from Theorem 3.3 by putting $B_1 = B_2 = B$ and $M_1 = M_2 = M$. □

Theorem 3.5. *Let H be a real Hilbert space, $M : H \to 2^H$ be a maximal monotone mapping, $B : H \to H$ be a relaxed (α, r)-cocoercive and ξ-Lipschitz continuous mappings, respectively. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0,1]$ such that $\|A\| \leq 1$ and $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\{T_n : H \to H\}$ be a family of nonexpansive mappings. Define $S_n x = \delta_n x + (1-\delta_n)T_n x$, where $\delta_n \in [0,1)$, for all $n \in \mathbb{N}$, and let $W_n : H \to H$ be a W-mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0,\mu]$, for some $\mu \in (0,1)$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(D) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$ where $D : H \to H$ defined by $Dx = J_{M,\lambda_1}((I-\rho_1 B)J_{M,\lambda_2}(I-\rho_2 B)x)$ such that $\lambda_i = \rho_i/2$ for each $i = 1,2$. For $x_1 \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= J_{M,\lambda_2}(x_n - \rho_2 B x_n), \\ y_n &= J_{M,\lambda_1}(z_n - \rho_1 B z_n), \\ x_{n+1} &= \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)(\gamma_n W_n x_n + (1-\gamma_n)y_n), \end{aligned} \tag{3.15}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$, $\rho_i \in (0, 2(r-\alpha\xi^2)/\xi^2]$, $r > \alpha\xi^2$, for each $i = 1,2$, satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $0 < \lim_{n\to\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_{\Omega}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega, \tag{3.16}$$

which is the optimality condition for the minimization problem

$$\min_{x\in\Omega}\left\{ \frac{1}{2}\langle Ax, x\rangle - h(x) \right\}, \tag{3.17}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$) and $(w, J_{M,\lambda_2}(I - \rho_2 B)w)$ is a solution of problem

$$\begin{aligned} 0 &\in x^* - y^* + \lambda_1(2By^* + Mx^*), \\ 0 &\in y^* - x^* + \lambda_2(2Bx^* + My^*). \end{aligned} \tag{3.18}$$

Proof. It is concluded obviously, from Theorem 3.4 by putting $k_n = 0$ for all $n \in \mathbb{N}$. □

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