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Research Article

Brandt Extensions and Primitive Topological Inverse Semigroups

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We study (countably) compact and (absolutely) H-closed primitive topological inverse semigroups. We describe the structure of compact and countably compact primitive topological inverse semigroups and show that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

In this paper all spaces are Hausdorff. A semigroup is a nonempty set with a binary associative operation. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element y in S is called *inverse* to x and denoted by x^{-1}. The map defined on an inverse semigroup S which maps to any element x of S its inverse x^{-1} is called the *inversion*.

A *topological semigroup* is a Hausdorff topological space with a jointly continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. A *topological group* is a topological space with a continuous group operation and an inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [1, Proposition II.1]). A Hausdorff topology τ on a (inverse) semigroup S is called (*inverse*) *semigroup* if (S, τ) is a topological (inverse) semigroup.

Further we shall follow the terminology of [2–8]. If S is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of S, and by S^1 $[S^0]$ we denote the semigroup S with the adjoined unit [zero] (see [7, page 2]). Also if a semigroup S has zero 0_S, then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\mathrm{cl}_Y(A)$ we denote the topological closure of A in Y. The set of positive integers is denoted by $\mathbb{N}$.

If E is a semilattice, then the semilattice operation on E determines the partial order $\leqslant$ on E:

$$e \leqslant f \quad \text{iff } ef = fe = e. \tag{1}$$

This order is called *natural*. An element e of a partially ordered set X is called *minimal* if $f \leqslant e$ implies $f = e$ for $f \in X$. An idempotent e of a semigroup S without zero (with zero) is called *primitive* if e is a minimal element in $E(S)$ (in $(E(S))^*$).

Let S be a semigroup with zero and let I_λ be a set of cardinality $\lambda \geqslant 1$. On the set $B_\lambda(S) = (I_\lambda \times S \times I_\lambda) \cup \{0\}$ we define the semigroup operation as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma, \\ 0, & \text{if } \beta \neq \gamma, \end{cases} \tag{2}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$ and $a, b \in S$. If $S = S^1$, then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup S* [9]. Obviously, $\mathcal{J} = \{0\} \cup \{(\alpha, \mathcal{O}, \beta) \mid \mathcal{O}$ is the zero of $S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$ and we shall call $B_\lambda^0(S)$ the *Brandt λ^0-extension of the semigroup S with zero* [10]. Further, if $A \subseteq S$, then we shall denote $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in I_\lambda$. If $\mathcal{I}$ is a trivial semigroup (i.e., $\mathcal{I}$ contains only one element), then by $\mathcal{I}^0$ we denote the semigroup $\mathcal{I}$ with the adjoined zero. Obviously, for any $\lambda \geqslant 2$ the Brandt λ^0-extension of the semigroup $\mathcal{I}^0$ is isomorphic to the semigroup of $I_\lambda \times I_\lambda$-matrix units and any Brandt λ^0-extension of a semigroup with zero contains the semigroup of $I_\lambda \times I_\lambda$-matrix units. Further by B_λ we shall denote the semigroup of $I_\lambda \times I_\lambda$-matrix units and by $B_\lambda^0(1)$ the subsemigroup of $I_\lambda \times I_\lambda$-matrix units of the Brandt λ^0-extension of a monoid S with zero. A completely 0-simple inverse semigroup is called a *Brandt semigroup* [8]. A semigroup S is a Brandt semigroup if and only if S is isomorphic to a Brandt λ-extension $B_\lambda(G)$ of some group G [8, Theorem II.3.5].

A nontrivial inverse semigroup is called a *primitive inverse semigroup* if all its nonzero idempotents are primitive [8]. A semigroup S is a primitive inverse semigroup if and only if S is an orthogonal sum of Brandt semigroups [8, Theorem II.4.3].

Green's relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ on a semigroup S are defined by

(i) $a\mathcal{L}b$ if and only if $a \cup Sa = b \cup Sb$;

(ii) $a\mathcal{R}b$ if and only if $a \cup aS = b \cup bS$;

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$

for $a, b \in S$. For details about Green's relations, see [4, Section 2.1] or [11]. We observe that two nonzero elements (α_1, s, β_1) and (α_2, t, β_2) of a Brandt semigroup $B_\lambda(G)$, $s, t \in G$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_\lambda$, are $\mathcal{H}$-equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ (see [8, page 93]).

By $\mathcal{S}$ we denote some class of topological semigroups.

Definition 1 (see [9, 12]). A semigroup $S \in \mathcal{S}$ is called *H-closed in $\mathcal{S}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as a subsemigroup.

If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*.

Definition 2 (see [13, 14]). A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class $\mathcal{S}$* if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

A semigroup S is called *algebraically closed in $\mathcal{S}$* if S with any semigroup topology τ is H-closed in $\mathcal{S}$ and $(S,\tau) \in \mathcal{S}$ [9]. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *algebraically closed*. A semigroup S is called *algebraically h-closed in $\mathcal{S}$* if S with the discrete topology $\mathfrak{d}$ is absolutely H-closed in $\mathcal{S}$ and $(S,\mathfrak{d}) \in \mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*.

Absolutely H-closed semigroups and algebraically h-closed semigroups were introduced by Stepp in [14]. There, they were called *absolutely maximal* and *algebraic maximal*, respectively.

Definition 3 (see [9]). Let λ be a cardinal $\geqslant 1$ and $(S,\tau) \in \mathcal{S}$. Let τ_B be a topology on $B_\lambda(S)$ such that

(i) $(B_\lambda(S),\tau_B) \in \mathcal{S}$;

(ii) $\tau_B|_{(\alpha,S^1,\alpha)} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_\lambda(S),\tau_B)$ is called a *topological Brandt λ-extension of (S,τ) in $\mathcal{S}$*. If $\mathcal{S}$ coincides with the class of all topological semigroups, then $(B_\lambda(S),\tau_B)$ is called a *topological Brandt λ-extension of (S,τ)*.

Definition 4 (see [10]). Let $\mathcal{S}_0$ be some class of topological semigroups with zero. Let λ be a cardinal $\geqslant 1$ and $(S,\tau) \in \mathcal{S}_0$. Let τ_B be a topology on $B^0_\lambda(S)$ such that

(a) $(B^0_\lambda(S),\tau_B) \in \mathcal{S}_0$;

(b) $\tau_B|_{(\alpha,S,\alpha)\cup\{0\}} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B^0_\lambda(S),\tau_B)$ is called a *topological Brandt λ^0-extension of (S,τ) in $\mathcal{S}_0$*. If $\mathcal{S}_0$ coincides with the class of all topological semigroups, then $(B^0_\lambda(S),\tau_B)$ is called a *topological Brandt λ^0-extension of (S,τ)*.

Gutik and Pavlyk in [9] proved that the following conditions for a topological semigroup S are equivalent:

(i) S is an H-closed semigroup in the class of topological inverse semigroups;

(ii) there exists a cardinal $\lambda \geqslant 1$ such that any topological Brandt λ-extension of S is H-closed in the class of topological inverse semigroups;

(iii) for any cardinal $\lambda \geqslant 1$ every topological Brandt λ-extension of S is H-closed in the class of topological inverse semigroups.

In [13] they showed that the similar statement holds for absolutely H-closed topological semigroups in the class of topological inverse semigroups.

In [10], Gutik and Pavlyk proved the following.

Theorem 5. *Let S be a topological inverse monoid with zero. Then the following conditions are equivalent:*

(i) *S is an (absolutely) H-closed semigroup in the class of topological inverse semigroups;*

(ii) *there exists a cardinal $\lambda \geqslant 1$ such that any topological Brandt λ^0-extension $B_\lambda^0(S)$ of the semigroup S is (absolutely) H-closed in the class of topological inverse semigroups;*

(iii) *for each cardinal $\lambda \geqslant 1$, every topological Brandt λ^0-extension $B_\lambda^0(S)$ of the semigroup S is (absolutely) H-closed in the class of topological inverse semigroups.*

Also, an example of an absolutely H-closed topological semilattice $\mathcal{N}$ with zero and a topological Brandt λ^0-extension $B_\lambda^0(\mathcal{N})$ of $\mathcal{N}$ with the following properties was constructed in [10]:

(i) $B_\lambda^0(\mathcal{N})$ is an absolutely H-closed semigroup for any infinite cardinal λ;

(ii) $B_\lambda^0(\mathcal{N})$ is a σ-compact inverse topological semigroup for any countable cardinal λ;

(iii) $B_\lambda^0(\mathcal{N})$ contains an absolutely H-closed ideal J such that the Rees quotient semigroup $B_\lambda^0(\mathcal{N})/J$ is not a topological semigroup.

We observe that for any topological Brandt λ-extension $B_\lambda(S)$ of a topological semigroup S there exist a topological monoid T with zero and a topological Brandt λ^0-extension $B_\lambda^0(T)$ of T, such that the semigroups $B_\lambda(S)$ and $B_\lambda^0(T)$ are topologically isomorphic. Algebraic properties of Brandt λ^0-extensions of monoids with zero and nontrivial homomorphisms between Brandt λ^0-extensions of monoids with zero and a category whose objects are ingredients of the construction of Brandt λ^0-extensions of monoids with zeros were described in [15]. Also, in [15, 16] was described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros.

In [9, 17] for every infinite cardinal λ, semigroup topologies on Brandt λ-extensions which preserve an H-closedness and an absolute H-closedness were constructed. An example of a non$H$-closed topological inverse semigroup S in the class of topological inverse semigroups such that for any cardinal $\lambda \geqslant 1$ there exists an absolute H-closed topological Brandt λ-extension of the semigroup S in the class of topological semigroups was constructed in [17].

In this paper we study (countably) compact and (absolutely) H-closed primitive topological inverse semigroups. We describe the structure of compact and countably compact primitive topological inverse semigroups and show that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

Lemma 6. *Let E be a topological semilattice with zero 0 such that every nonzero idempotent of E is primitive. Then every nonzero element of E is an isolated point in E.*

Proof. Let $x \in E^*$. Since E is a Hausdorff topological semilattice, for every open neighbourhood $U(x) \not\ni 0$ of the point x there exists an open neighbourhood $V(x)$ of x such that $V(x) \cdot V(x) \subseteq U(x)$. If x is not an isolated point of E, then $V(x) \cdot V(x) \ni 0$ which contradicts to the choice of $U(x)$. This implies the assertion of the lemma. □

Lemma 7. *Let S be a primitive inverse topological semigroup and let S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\}_{i\in\mathcal{A}}$ of topological Brandt semigroups with zeros, that is, $S = \sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$.*

Let $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(G_i)$ *be a nonzero element of* S. *Then*

(i) *there exists an open neighbourhood* U *of* (α_i, g_i, β_i) *such that* $U \subseteq S^*_{\alpha_i,\beta_i} \subseteq B_{\lambda_i}(G_i)$;

(ii) *every nonzero idempotent of* S *is an isolated point in* $E(S)$.

Proof. (i) Suppose to the contrary that $U \nsubseteq S^*_{\alpha_i,\beta_i} \subseteq B_{\lambda_i}(G_i)$ for any open neighbourhood U of the point (α_i, g_i, β_i). Since S is a Hausdorff space, there exists an open neighbourhood V of the point (α_i, g_i, β_i) such that $0 \notin V$. The continuity of the semigroup operation in S implies that there exists an open neighbourhood W of the point (α_i, g_i, β_i) such that $(\alpha_i, 1_i, \alpha_i) \cdot W \cdot (\beta_i, 1_i, \beta_i) \subseteq V$. Since $W \nsubseteq S^*_{\alpha_i,\beta_i}$, we have that $0 \in V$, a contradiction.

Statement (ii) follows from Lemma 6. □

Lemma 7 implies the following.

Corollary 8. *Every nonzero* $\mathscr{H}$*-class of a primitive inverse topological semigroup* S *is an open subset in* S.

Lemma 9. *If* S *is a primitive topological inverse semigroup, then every nonzero* $\mathscr{H}$*-class of* S *is a clopen subset in* S.

Proof. Let $H(e, f)$ be a nonzero $\mathscr{H}$-class in S for $e, f \in (E(S))^*$, that is,

$$H(e,f) = \left\{x \in S \mid x \cdot x^{-1} = e, x^{-1} \cdot x = f\right\}. \tag{3}$$

Since S is a topological inverse semigroup, the maps $\varphi : S \to E(S)$ and $\psi : S \to E(S)$ defined by the formulae $\varphi(x) = x \cdot x^{-1}$ and $\psi(x) = x^{-1} \cdot x$ are continuous. By Lemma 6, e and f are isolated points in $E(S)$. Then the continuity of the maps φ and ψ implies the statement of the lemma. □

The following example shows that the statement of Lemma 9 does not hold for primitive inverse locally compact H-closed topological semigroups.

Example 10. Let $\mathbb{Z}$ be the discrete additive group of integers. We extend the semigroup operation from $\mathbb{Z}$ onto $\mathbb{Z}^0 = \mathbb{Z} \cup \{\infty\}$ as follows:

$$x \cdot \infty = \infty \cdot x = \infty \cdot \infty = \infty, \quad \forall x \in \mathbb{Z}. \tag{4}$$

We observe that $\mathbb{Z}^0$ is the group with adjoined zero ∞. We determine a semigroup topology τ on $\mathbb{Z}^0$ as follows:

(i) every nonzero element of $\mathbb{Z}^0$ is an isolated point;

(ii) the family $\mathcal{B}(\infty) = \{U_n = \{\infty\} \cup \{x \in \mathbb{Z} \mid x \geqslant n\} \mid n$ is a positive integer$\}$ is a base of the topology τ at the point ∞.

A simple verification shows that $(\mathbb{Z}^0, \tau)$ is a primitive inverse locally compact topological semigroup.

Proposition 11. *$(\mathbb{Z}^0, \tau)$ is an H-closed topological semigroup.*

Proof. Suppose that $\mathbb{Z}^0$ is embedded into a topological semigroup T. If $\{n_\iota\}$ is a net in $\mathbb{N}$ for which $\{-n_\iota\}$ converges in T to $t \in T \setminus \mathbb{Z}^0$, then the equation $-n_\iota + (n_\iota + k) = k$ implies that $t \cdot \infty = k$ for every $k \in \mathbb{N}$—which is impossible. So $\mathbb{Z}^0$ is closed in T. □

Proposition 12. *Every completely 0-simple topological inverse semigroup S is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(G)$ of some topological group G and cardinal $\lambda \geqslant 1$ in the class of topological inverse semigroups. Furthermore one has the following:*

(i) *any nonzero subgroup of S is topologically isomorphic to G and every nonzero $\mathscr{H}$-class of S is homeomorphic to G and is a clopen subset in S;*

(ii) *the family $\mathcal{B}(\alpha, g, \beta) = \{(\alpha, g \cdot U, \beta) \mid U \in \mathcal{B}_G(e)\}$, where $\mathcal{B}_G(e)$ is a base of the topology at the unity e of G, is a base of the topology at the nonzero element $(\alpha, g, \beta) \in B_\lambda(G)$.*

Proof. Let G be a nonzero subgroup of S. Then by Theorem 3.9 of [4, 5] the semigroup S is isomorphic to the Brandt λ-extension of the subgroup G for some cardinal $\lambda \geqslant 1$. Since S is a topological inverse semigroup, we have that G is a topological group.

(i) Let e be the unity of G. We fix arbitrary $\alpha, \beta, \gamma, \delta \in I_\lambda$ and define the maps $\varphi_{\alpha\beta}^{\gamma\delta} : B_\lambda(G) \to B_\lambda(G)$ and $\varphi_{\gamma\delta}^{\alpha\beta} : B_\lambda(G) \to B_\lambda(G)$ by the formulae $\varphi_{\alpha\beta}^{\gamma\delta}(s) = (\gamma, e, \alpha) \cdot s \cdot (\beta, e, \delta)$ and $\varphi_{\gamma\delta}^{\alpha\beta}(s) = (\alpha, e, \gamma) \cdot s \cdot (\delta, e, \beta), s \in B_\lambda(G)$. We observe that $\varphi_{\gamma\delta}^{\alpha\beta}(\varphi_{\alpha\beta}^{\gamma\delta}((\alpha, x, \beta))) = (\alpha, x, \beta)$ and $\varphi_{\alpha\beta}^{\gamma\delta}(\varphi_{\gamma\delta}^{\alpha\beta}((\gamma, x, \delta))) = (\gamma, x, \delta)$ for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, $x \in G$, and hence the restrictions $\varphi_{\alpha\beta}^{\gamma\delta}|_{(\alpha,G,\beta)}$ and $\varphi_{\gamma\delta}^{\alpha\beta}|_{(\gamma,G,\delta)}$ are mutually invertible. Since the maps $\varphi_{\alpha\beta}^{\gamma\delta}$ and $\varphi_{\gamma\delta}^{\alpha\beta}$ are continuous on $B_\lambda(G)$, the map $\varphi_{\alpha\beta}^{\gamma\delta}|_{(\alpha,G,\beta)} : (\alpha, G, \beta) \to (\gamma, G, \delta)$ is a homeomorphism and the map $\varphi_{\alpha\alpha}^{\gamma\gamma}|_{(\alpha,G,\alpha)} : (\alpha, G, \alpha) \to (\gamma, G, \gamma)$ is a topological isomorphism. We observe that the subset (α, G, β) of $B_\lambda(G)$ is an $\mathscr{H}$-class of $B_\lambda(G)$ and (α, G, α) is a subgroup of $B_\lambda(G)$ for all $\alpha, \beta \in I_\lambda$. This completes the proof of assertion (i).

(ii) The statement follows from assertion (i) and Theorem 4.3 of [18]. □

We observe that Example 10 implies that the statements of Proposition 12 are not true for completely 0-simple inverse topological semigroups. Definition 3 implies that S is a topological Brandt λ-extension $B_\lambda(G)$ of the topological group G.

Gutik and Repovš, in [19], studied the structure of 0-simple countably compact topological inverse semigroups. They proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of a countably compact topological group H in the class of topological inverse semigroups for some finite cardinal $\lambda \geqslant 1$. This implies Pavlyk's Theorem (see [20]) on the structure of 0-simple compact topological inverse semigroups: *every 0-simple compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of a compact topological group H in the class of topological inverse semigroups for some finite cardinal $\lambda \geqslant 1$.*

The following theorem describes the structure of primitive countably compact topological inverse semigroups.

Theorem 13. *Every primitive countably compact topological inverse semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i \in \mathscr{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of* *countably compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geqslant 1$. Moreover the family*

$$\mathcal{B}(0) = \left\{ S \setminus \left(B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}) \right)^* \mid i_1, i_2, \ldots, i_n \in \mathcal{A}, n \in \mathbb{N} \right\} \tag{5}$$

determines a base of the topology at zero 0 *of* S.

Proof. By Theorem II.4.3 of [8] the semigroup S is an orthogonal sum of Brandt semigroups and hence S is an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of groups G_i. We fix any $i_0 \in \mathcal{A}$. Since S is a topological inverse semigroup, Proposition II.2 [1] implies that $B_{\lambda_{i_0}}(G_{i_0})$ is a topological inverse semigroup. By Proposition 12, $B_{\lambda_{i_0}}(G_{i_0})$ is a closed subsemigroup of S and hence by Theorem 3.10.4 [6], $B_{\lambda_{i_0}}(G_{i_0})$ is a countably compact 0-simple topological inverse semigroup. Then, by Theorem 2 of [19], the semigroup $B_{\lambda_{i_0}}(G_{i_0})$ is a topological Brandt λ_i-extension of countably compact topological group G_{i_0} in the class of topological inverse semigroups for some finite cardinal $\lambda_{i_0} \geqslant 1$. This completes the proof of the first assertion of the theorem.

Suppose on the contrary that $\mathcal{B}(0)$ is not a base at zero 0 of S. Then, there exists an open neighbourhood $U(0)$ of zero 0 such that $U(0) \bigcup (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}))^* \neq S$ for finitely many indexes $i_1, i_2, \ldots, i_n \in \mathcal{A}$. Therefore there exists an infinitely family $\mathcal{F}$ of nonzero disjoint $\mathcal{H}$-classes such that $H \not\subseteq U(0)$ for all $H \in \mathcal{F}$. Let $\mathcal{F}_0$ be an infinite countable subfamily of $\mathcal{F}$. We put $W = \bigcup\{H \mid H \in \mathcal{F} \setminus \mathcal{F}_0\}$. Lemma 9 implies that the family $\mathcal{C} = \{U(0), W\} \cup \mathcal{F}_0$ is an open countable cover of S. Simple observation shows that the cover $\mathcal{C}$ does not contain a finite subcover. This contradicts to the countable compactness of S. The obtained contradiction implies the last assertion of the theorem. □

Since any maximal subgroup of a compact topological semigroup T is a compact subset in T (see [2, Vol. 1, Theorem 1.11]), Theorem 13 implies the following.

Corollary 14. *Every primitive compact topological inverse semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geqslant 1$ and the family*

$$\mathcal{B}(0) = \left\{ S \setminus \left(B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}) \right)^* \mid i_1, i_2, \ldots, i_n \in \mathcal{A}, n \in \mathbb{N} \right\} \tag{6}$$

determines a base of the topology at zero 0 *of* S.

Theorem 15. *Every primitive countably compact topological inverse semigroup S is a dense subsemigroup of a primitive compact topological inverse semigroup.*

Proof. By Theorem 13 the topological semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geqslant 1$. Since any countably compact topological group G_i is pseudocompact, the Comfort-Ross Theorem (see [21, Theorem 4.1]) implies that the Stone-Čech compactification $\beta(G_i)$ is a compact topological group and the inclusion mapping f_i of G_i into $\beta(G_i)$ is a topological isomorphism for all $i \in \mathcal{A}$. On the orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ-extensions $B_{\lambda_i}(\beta(G_i))$, $i \in \mathcal{A}$, we determine a topology τ as follows:

(a) the family $\mathcal{B}(\alpha_i, g_i, \beta_i) = \{(\alpha_i, g_i \cdot U, \beta_i) \mid U \in \mathcal{B}_{\beta(G_i)}(e_i)\}$ is a base of the topology at the nonzero element $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(\beta(G_i))$, where $\mathcal{B}_{\beta(G_i)}(e_i)$ is a base of the topology at the unity e_i of the compact topological group $\beta(G_i)$;

(b) the family

$$\mathcal{B}(0) = \left\{ S \setminus \left(B_{\lambda_{i_1}}(\beta(G_{i_1})) \cup B_{\lambda_{i_2}}(\beta(G_{i_2})) \cup \cdots \cup B_{\lambda_{i_n}}(\beta(G_{i_n})) \right)^* \mid i_1, i_2, \ldots, i_n \in \mathcal{A}, n \in \mathbb{N} \right\} \quad (7)$$

determines a base of the topology at zero 0 of $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$.

By Theorem II.4.3 of [8], $\sum_{i\in\mathcal{A}} B_{\lambda_i}(\beta(G_i))$ is a primitive inverse semigroup and simple verifications show that $\sum_{i\in\mathcal{A}} B_{\lambda_i}(\beta(G_i))$ with the topology τ is a compact topological inverse semigroup.

We define a map $f : \sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i) \to \sum_{i\in\mathcal{A}} B_{\lambda_i}(\beta(G_i))$ as follows:

$$f(0) = 0, \qquad f((\alpha_i, g_i, \beta_i)) = (\alpha_i, f_i(g_i), \beta_i) \in B_{\lambda_i}(\beta(G_i)) \quad \text{for } (\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(G_i). \quad (8)$$

Simple verifications show that f is a continuous homomorphism. Since $f_i : G_i \to \beta(G_i)$ is a topological isomorphism, we have that $f : \sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i) \to \sum_{i\in\mathcal{A}} B_{\lambda_i}(\beta(G_i))$ is a topological isomorphism too. □

Gutik and Repovš in [19] showed that the Stone-Čech compactification $\beta(T)$ of a 0-simple countably compact topological inverse semigroup T is a 0-simple compact topological inverse semigroup. In this context the following question arises naturally.

Question 1. Is the Stone-Čech compactification $\beta(T)$ of a primitive countably compact topological inverse semigroup T a topological semigroup (a primitive topological inverse semigroup)?

Theorem 16. *Let $S = \bigcup_{\alpha\in\mathcal{A}} S_\alpha$ be a topological inverse semigroup such that*

(i) *S_α is an H-closed (resp., absolutely H-closed) semigroup in the class of topological inverse semigroups for any $\alpha \in \mathcal{A}$;*

(ii) *there exists an H-closed (resp., absolutely H-closed) subsemigroup T of S in the class of topological inverse semigroups such that $S_\alpha \cdot S_\beta \subseteq T$ for all $\alpha \neq \beta$, $\alpha, \beta \in \mathcal{A}$.*

Then S is an H-closed (resp., absolutely H-closed) semigroup in the class of topological inverse semigroups.

Proof. We consider the case of absolute H-closedness only.

Suppose on the contrary that there exist a topological inverse semigroup G and a continuous homomorphism $h : S \to G$ such that $h(S)$ is not closed subsemigroup in G. Without loss of generality we can assume that $\mathrm{cl}_G(h(S)) = G$. Thus, by Proposition II.2 of [1], G is a topological inverse semigroup.

Then, $G \setminus h(S) \neq \emptyset$. Let $x \in G \setminus h(S)$. Since S and G are topological inverse semigroups we have that $h(S)$ is an inverse subsemigroup in G and hence $x^{-1} \in G \setminus h(S)$. The semigroup

T which is an absolutely H-closed semigroup in the class of topological inverse semigroups implies that there exists an open neighbourhood $U(x)$ of the point x in T such that $U(x) \cap h(T) = \emptyset$. Since G is a topological inverse semigroup there exist open neighbourhoods $V(x)$ and $V(x^{-1})$ of the points x and x^{-1} in G, respectively, such that $V(x) \cdot V(x^{-1}) \cdot V(x) \subseteq U(x)$. But $x, x^{-1} \in \mathrm{cl}_G(h(S)) \setminus h(S)$ and since $\{S_\alpha \mid \alpha \in \mathcal{A}\}$ is the family of absolutely H-closed semigroups in the class of topological inverse semigroups, each of the neighbourhoods $V(x)$ and $V(x^{-1})$ intersects infinitely many subsemigroups $h(S_\beta)$ in G, $\beta \in \mathcal{A}$. Hence, $(V(x) \cdot V(x^{-1}) \cdot V(x)) \cap h(T) \neq \emptyset$. This contradicts the assumption that $U(x) \cap h(T) = \emptyset$. The obtained contradiction implies that S is an absolutely H-closed semigroup in the class of topological inverse semigroups.

The proof in the case of H-closeness is similar to the previous one. □

Theorem 16 implies the following.

Corollary 17. *Let $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ be an inverse semigroup such that*

(i) *S_α is an algebraically closed (resp., algebraically h-closed) semigroup in the class of topological inverse semigroups for any $\alpha \in \mathcal{A}$;*

(ii) *there exists an algebraically closed (resp., algebraically h-closed) sub-semigroup T of S in the class of topological inverse semigroups such that $S_\alpha \cdot S_\beta \subseteq T$ for all $\alpha \neq \beta$, $\alpha, \beta \in \mathcal{A}$.*

Then S is an algebraically closed (resp., algebraically h-closed) semigroup in the class of topological inverse semigroups.

Theorem 16 implies the following.

Theorem 18. *Let a topological inverse semigroup S be an orthogonal sum of the family $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ of H-closed (resp., absolutely H-closed) topological inverse semigroups with zeros in the class of topological inverse semigroups. Then S is an H-closed (resp., absolutely H-closed) topological inverse semigroup in the class of topological inverse semigroups.*

Corollary 17 implies the following.

Corollary 19. *Let an inverse semigroup S be an orthogonal sum of the family $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraically closed (resp., algebraically h-closed) inverse semigroups with zeros in the class of topological inverse semigroups. Then S is an algebraically closed (resp., algebraically h-closed) inverse semigroup in the class of topological inverse semigroups.*

Recall in [22], that a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H-closed in the class of topological groups. In [23] Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, that is, G is complete with respect to the two sided uniformity.

A topological group G is called *h-complete* if for every continuous homomorphism $f : G \to H$ into a topological group H the subgroup $f(G)$ of H is closed [24]. The h-completeness is preserved under taking products and closed central subgroups [24].

Gutik and Pavlyk in [13] showed that a topological group G is H-closed (resp., absolutely H-closed) in the class of topological inverse semigroups if and only if G is absolutely closed (resp., h-complete).

Theorem 20. *For a primitive topological inverse semigroup S the following assertions are equivalent:*

(i) *every maximal subgroup of S is absolutely closed;*

(ii) *the semigroup S with every inverse semigroup topology τ is H-closed in the class of topological inverse semigroups.*

Proof. (i)⇒(ii) Suppose that a primitive topological inverse semigroup S is an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of topological groups G_i in the class of topological inverse semigroups and every topological group G_i is absolutely closed. Then, by Theorem 3 of [9] any topological Brandt λ_i-extension $B_{\lambda_i}(G_i)$ of topological group G_i is H-closed in the class of topological inverse semigroups. Theorem 18 implies that S is an H-closed topological inverse semigroup in the class of topological inverse semigroups.

(ii)⇒(i) Let G be any maximal nonzero subgroup of S. Since S is a primitive topological inverse semigroup, we have that S is an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ-extensions $B_{\lambda_i}(G_i)$ of topological groups G_i and hence there exists a topological Brandt λ_{i_0}-extension $B_{\lambda_{i_0}}(G_{i_0})$, $i\in\mathcal{A}$, such that $B_{\lambda_{i_0}}(G_{i_0})$ contains the maximal subgroup G and $B_{\lambda_{i_0}}(G_{i_0})$ is a subsemigroup of S.

Suppose on the contrary that the topological group $G = G_{i_0}$ is not absolutely closed. Then there exists a topological group H which contains G as a dense proper subgroup. For every $i\in\mathcal{A}$ we put

$$H_i = \begin{cases} G_i, & \text{if } i \neq i_0, \\ H, & \text{if } i = i_0. \end{cases} \tag{9}$$

On the orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(H_i)$ of Brandt λ-extensions $B_{\lambda_i}(H_i)$, $i\in\mathcal{A}$, we determine a topology τ_0 as follows:

(a) the family $\mathcal{B}(\alpha_i, g_i, \beta_i) = \{(\alpha_i, g_i\cdot U, \beta_i) \mid U\in\mathcal{B}_{H_i}(e_i)\}$ is a base of the topology at the nonzero element $(\alpha_i, g_i, \beta_i)\in B_{\lambda_i}(H_i)$, where $\mathcal{B}_{H_i}(e_i)$ is a base of the topology at the unity e_i of the topological group H_i;

(b) the zero 0 is an isolated point in $(\sum_{i\in\mathcal{A}} B_{\lambda_i}(H_i), \tau_0)$.

By Theorem II.4.3 of [8], $\sum_{i\in\mathcal{A}} B_{\lambda_i}(H_i)$ is a primitive inverse semigroup and simple verifications show that $\sum_{i\in\mathcal{A}} B_{\lambda_i}(H_i)$ with the topology τ_0 is a topological inverse semigroup. Also we observe that the semigroup $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ which is induced from $(\sum_{i\in\mathcal{A}} B_{\lambda_i}(H_i), \tau_0)$ topology is a topological inverse semigroup which is a dense proper inverse sub-semigroup of $(\sum_{i\in\mathcal{A}} B_{\lambda_i}(H_i), \tau_0)$. The obtained contradiction completes the statement of the theorem. □

Theorem 20 implies the following.

Corollary 21. *For a primitive inverse semigroup S the following assertions are equivalent:*

(i) *every maximal subgroup of S is algebraically closed in the class of topological inverse semigroups;*

(ii) *the semigroup S is algebraically closed in the class of topological inverse semigroups.*

Theorem 22. *For a primitive topological inverse semigroup S the following assertions are equivalent:*

(i) *every maximal subgroup of S is h-complete;*

(ii) *the semigroup S with every inverse semigroup topology τ is absolutely H-closed in the class of topological inverse semigroups.*

Proof. (i)⇒(ii) Suppose that a primitive topological inverse semigroup S is an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of topological groups G_i in the class of topological inverse semigroups and every topological group G_i is h-complete. Then by Theorem 14 of [13] any topological Brandt λ_i-extension $B_{\lambda_i}(G_i)$ of topological group G_i is absolutely H-closed in the class of topological inverse semigroups. Theorem 18 implies that S is an absolutely H-closed topological inverse semigroup in the class of topological inverse semigroups.

(ii)⇒(i) Let G be any maximal nonzero subgroup of S. Since S is a primitive topological inverse semigroup, S is an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ-extensions $B_{\lambda_i}(G_i)$ of topological groups G_i. Hence there exists a topological Brandt λ_{i_0}-extension $B_{\lambda_{i_0}}(G_{i_0})$, $i \in \mathcal{A}$, such that $B_{\lambda_{i_0}}(G_{i_0})$ contains the maximal subgroup G and $B_{\lambda_{i_0}}(G_{i_0})$ is a subsemigroup of S.

Suppose on the contrary that the topological group $G = G_{i_0}$ is not h-completed. Then there exist a topological group H and continuous homomorphism $h : G \to H$ such that $h(G)$ is a dense proper subgroup of H. On the Brandt λ-extension $B_{\lambda_{i_0}}(H)$, we determine a topology τ_H as follows:

(a) the family $\mathcal{B}(\alpha_{i_0}, g_{i_0}, \beta_{i_0}) = \{(\alpha_{i_0}, g_i \cdot U, \beta_{i_0}) \mid U \in \mathcal{B}_H(e)\}$ is a base of the topology at the nonzero element $(\alpha_{i_0}, g_i, \beta_{i_0}) \in B_{\lambda_i}(H)$, where $\mathcal{B}_H(e)$ is a base of the topology at the unity e of the topological group H;

(b) the zero 0 is an isolated point in $(B_{\lambda_{i_0}}(H), \tau_H)$.

Then $B_{\lambda_{i_0}}(H)$ is an inverse semigroup and simple verifications show that $B_{\lambda_{i_0}}(H)$ with the topology τ_H is a topological inverse semigroup.

On the orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ-extensions $B_{\lambda_i}(G_i)$, $i \in \mathcal{A}$, we determine a topology $\tau_\star$ as follows:

(a) the family $\mathcal{B}(\alpha_i, g_i, \beta_i) = \{(\alpha_i, g_i \cdot U, \beta_i) \mid U \in \mathcal{B}_{G_i}(e_i)\}$ is a base of the topology at the nonzero element $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(G_i)$, where $\mathcal{B}_{G_i}(e_i)$ is a base of the topology at the unity e_i of the topological group G_i;

(b) the zero 0 is an isolated point in $(\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i), \tau_\star)$.

By Theorem II.4.3 of [8], $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ is a primitive inverse semigroup and simple verifications show that $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ with the topology $\tau_\star$ is a topological inverse semigroup.

We define the map $f : S \to B_{\lambda_{i_0}}(H)$ as follows:

$$f(x) = \begin{cases} h(x), & \text{if } x \in B_{\lambda_{i_0}}(G_{i_0}), \\ 0, & \text{if } x \notin B_{\lambda_{i_0}}(G_{i_0}), \end{cases} \tag{10}$$

where 0 is zero of S. Evidently the defined map f is a continuous homomorphism. Then $f(S) = B_{\lambda_{i_0}}(h(G_{i_0}))$ is a dense proper inverse subsemigroup of the topological inverse semigroup $(B_{\lambda_{i_0}}(H), \tau_H)$. The obtained contradiction completes the statement of the theorem. □

Theorem 22 implies the following.

Corollary 23. *For a primitive inverse semigroup S the following assertions are equivalent:*

(i) *every maximal subgroup of S is algebraically h-closed in the class of topological inverse semigroups;*

(ii) *the semigroup S is algebraically h-closed in the class of topological inverse semigroups.*

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