

Research Article

Dynamics and Thresholds of a Simple Epidemiological Model: Example of HIV/AIDS in Mali

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The dynamics of many epidemiological models for infectious diseases that spread in the sexually active population presents a crucial period: the period of the influx or recruitment of susceptible. In this paper, we assume that the recruitment of susceptible is done among the juvenile group. We propose a dynamical system to modelize the disease spread, and we study the dynamical behavior of this system. Then, the controllability of the system is studied. We prove that the survival rate allows to control the dynamic of the system. Numerical simulations are given to illustrate the results.

1. Introduction

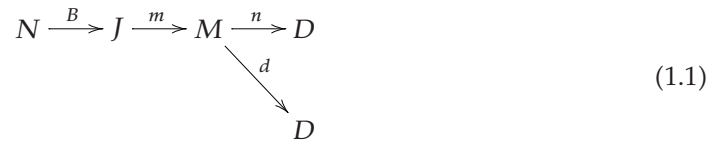
In recent years several authors have described interesting dynamical behavior of SIR epidemiological models in which the population can be portioned into two age structured classes: immature individuals and mature ones (see, e.g., [1, 2]). The HIV disease belongs to the class of diseases which spread essentially among sexually active individuals. Thus, it is meaningful to consider stage structure in epidemiological models. The population is initially divided into two compartments: those, who are mature individuals or adults and those who are in youthful age or immature individuals. All population groups are subject to the risk of dying from AIDS.

We denote by

- (i) B the birth density in the population;
- (ii) J the density of the immature individuals;

- (iii) M the density of the mature individuals;
- (iv) N the density of the population;
- (v) D the density of the dead individuals;
- (vi) d the probability of mature individuals to die of HIV;
- (vii) m the probability of immature individuals becoming mature individuals;
- (viii) n the probability of mature individuals to die of other causes.

Then a simple model with compartments and a single population with stage structure reads:



$$N = J + M \tag{1.2}$$

For describing the disease transmission, a dynamics between the compartments due to the disease has to be specified. A traditional SIR model is introduced. Each member of the population is considered to belong to one of the three classes: susceptible individuals (denoted by S), infected individuals (denoted by I) and removed individuals (denoted by R). Each individual begins in the class S , only to move to the class I after coming into contact with an infected person. Infected individuals eventually recover from the disease due to a medical treatment and then move to the class R and are unable to be infected one again. The disease is fueled by supply of susceptible issued from the compartment J . The size of the population is denoted by $N(t)$ and can be expressed as the following sum:

$$N(t) = S(t) + I(t) + R(t) + J(t). \tag{1.3}$$

The SIR model reads

$$\begin{aligned}
 \frac{dS}{dt} &= r_1 m (1 - \tau) J(t) - F_i(I, t) S + r_2 S, \\
 \frac{dI}{dt} &= F_i(I, t) S - r_3 (\sigma + \alpha) I, \\
 \frac{dR}{dt} &= r_3 \sigma I - \mu R,
 \end{aligned}
 \tag{1.4}$$

where $F_i(I, t)$ is the incidence function which may vary periodically because a part of the infected population represented by the truck drivers, for example, moves regularly. It is usual to take $F_i(I, t) = \Omega(t)I$ in which $\Omega(t)$ is the transmission rate; it is either constant, or a periodic modulation about a constant value, for example, $\Omega(t) = \Omega_0(1 + \Omega_1 \sin(\omega t))$; m is the rate of immature individuals becoming mature individuals; r_1 is the survival rate of the immature individuals; r_2 is the survival rate of the mature individuals; r_3 is the survival rate of the infected mature individuals and σ is the rate of the survivors subjected to the antiretroviral treatment. α is the rate of death due to the disease; τ is the fraction of infected immature from their mother; and μ is the rate of death due to other causes.

The aim of this work is to provide simple conditions for the parameters of the SIR model (1.4) that makes possible to control the infected individuals. By using the notion of the exterior contingent cone to a convex subset C of \mathbb{R}^2 , we prove that system (1.4) is controllable with three of its parameters. Whatever the initial conditions are, system (1.4) reaches the subset C and remains in C . The paper is organized as follows: the introduction ends with an existence and uniqueness result. In Section 2 the controllability of system (1.4) is studied and several numerical results are presented in connection with available data concerning Mali.

The dynamic behavior of (1.4) is determined by the variation of I and R . According to (1.3) the susceptible compartment is expressed as $S(t) = N(t) - I(t) - R(t) - J(t)$, thus (1.4) is reduced to

$$\begin{aligned} \frac{dI}{dt} &= F_i(I, t)[N(t) - J(t) - R(t) - I(t)] - r_3(\sigma + \alpha)I, \\ \frac{dR}{dt} &= r_3\sigma I - \mu R. \end{aligned} \tag{1.5}$$

Since $\mu > 0$, a new timescale $t' = \mu t$ is introduced. System (1.5) becomes:

$$\begin{aligned} \frac{dI}{dt'} &= \Omega(t')I(N - J - R - I) - r'_3(\sigma + \alpha)I, \\ \frac{dR}{dt'} &= r'_3\sigma I - R. \end{aligned} \tag{1.6}$$

We assume that $\Delta = N(t) - J(t) - R(t)$ is constant. Defining $\gamma = r'_3\sigma$, and omitting the prime notations, system (1.6) becomes:

$$\begin{aligned} \frac{dI}{dt} &= \Omega(t)I(\Delta - I) - \gamma I - \frac{\gamma\alpha}{\sigma}I, \\ \frac{dR}{dt} &= \gamma I - R. \end{aligned} \tag{1.7}$$

Theorem 1.1. *Assumes that Ω is a $C^1(\mathbb{R}_+; \mathbb{R})$ function with bounded primitive. For every initial condition $(I^*, R^*) \in \mathbb{R}_+^2$, the solution $(I(\cdot), R(\cdot)) : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ of (1.7) belongs to K where K is a compact subset of \mathbb{R}_+^2 .*

Proof. Set $\theta = \gamma(1 + \alpha/\sigma)$, by integrating the first equation of (1.7) we have

$$I(t) = \frac{I_0 e^{\int_0^t (\Omega(\tau)\Delta - \theta) d\tau}}{1 + \int_0^t I_0 e^{\int_0^s (\Omega(\tau)\Delta - \theta) d\tau} ds}. \tag{1.8}$$

Let M be a bound from below of a primitive of Ω , the we have

$$0 \leq I(t) < \frac{I_0 e^{M\Delta t}}{e^{\theta t} \left(1 + \int_0^t I_0 e^{\int_0^s (\Omega(\tau)\Delta - \theta) d\tau} ds \right)} = \bar{I}. \tag{1.9}$$

From $dR/dt = \gamma I - R$ we deduce

$$\begin{aligned} R(t) &= R_0 e^{-t} + \gamma e^{-t} \int_0^t e^s I(s) ds \\ &\leq R_0 e^{-t} + \gamma e^{-t} (e^t - 1) \bar{I} \\ &< R_0 e^{-t} + \gamma \bar{I} = \bar{R}. \end{aligned} \tag{1.10}$$

So the Poincaré-Bendixson's theorem [3] claims either the solution (I, R) of system (1.7) tends to a critical point when the time t goes to infinity, or it is a periodic solution.

A complete bifurcation analysis is beyond the objectives of this paper. For a precise study of the orbits the reader is referred to [4] or [5], for example. \square

2. Controllability of the Model with Its Coefficients

The question we address in this section reads: does there exist parameters which allow system (1.7) to evolve toward a fixed region C of the plane (I, R) , for any given initial condition? For $0 < \underline{x}_1$ fixed, we define the convex domain C of the plane and its associated truncated cylinder C_T by:

$$\begin{aligned} C &= \left\{ (x_1, x_2) \in \mathbb{R}_+^2; x_1 \leq \underline{x}_1; \text{ and } \frac{3}{4}x_1 \leq x_2 \right\}, \\ C_T &= \left\{ (t, x_1, x_2) \in \mathbb{R}_+^3; 0 \leq t \leq T; x_1 \leq \underline{x}_1; \text{ and } \frac{3}{4}x_1 \leq x_2 \right\}. \end{aligned} \tag{2.1}$$

Definition 2.1 (contingent and exterior contingent cone). The contingent cone to C_T at $x \in C_T$ is constituted by vectors $v \in \mathbb{R}^3$ verifying

$$\liminf_{h \rightarrow 0^+} \frac{d_{C_T}(x + hv, C_T)}{h} = 0, \tag{2.2}$$

where d_{C_T} denotes the distance to the subset C_T . The exterior contingent cone $T_{C_T}(x)$ is constituted by vectors $v \in \mathbb{R}^3$ verifying

$$\liminf_{h \rightarrow 0^+} \frac{d_{C_T}(x + hv, C_T) - d_{C_T}(x)}{h} \leq 0. \tag{2.3}$$

When a point x belongs to the boundary of C_T the definition of exterior contingent cone is equivalent to the definition of the contingent cone. We have the following result [6, Theorem 3.4.1 page 102].

Lemma 2.2. *The exterior contingent cone to C_T at point x is constituted by vectors $v \in \mathbb{R}^3$ satisfying:*

$$(x - P_{C_T}x, v) \leq 0; \tag{2.4}$$

where (\cdot, \cdot) denotes the Euclidean inner product, and P_{C_T} stands for the orthogonal projection on C_T .

Before stating the result of controllability, we give some technicalities. Setting

$$F(t, x_1, x_2) = \begin{pmatrix} 1 \\ \Omega(t)x_1(\Delta - x_1) - x_1\gamma\left(1 + \frac{\alpha}{\sigma}\right) \\ \gamma x_1 - x_2 \end{pmatrix}, \tag{2.5}$$

we have the following.

Lemma 2.3. *Let $X \in \{(t, x_1, x_2), 0 < t < T; 0 < x_1; 0 < x_2\} \cap C_T^c$ be fixed. Then $X - P_{C_T}X$ is the outward normal to C_T whenever it exists, and for $0 \leq s \leq 1$ is given by*

$$X - P_{C_T}X = \begin{pmatrix} 0 \\ 1 \\ \frac{4}{3}s \\ -\frac{1}{3}s \end{pmatrix}. \tag{2.6}$$

Furthermore, a sufficient condition for the vector $F(X)$ to belong to the exterior contingent cone T_{C_T} read as follows:

$$x_1 \left[\Omega(t)(\Delta - x_1) - \gamma \left(1 + \frac{\alpha}{\sigma} \right) + 1 \right] \leq 0. \tag{2.7}$$

Proof. From the definition of the exterior contingent cone (Figure 1) T_{C_T} we have:

$$\forall s \in [0, 1], \quad -\Omega(t)x_1^2 + x_1 \left(\Omega(t)\Delta - \gamma \left(1 + \frac{\alpha}{\sigma} \right) - \frac{4s}{3} \right) \leq \frac{4s}{3}x_2. \tag{2.8}$$

A sufficient condition independent of s for condition (2.8) to be satisfied is obtained when $x_2 \leq (3/4)x_1$ with $0 \leq x_1$ and read as follows:

$$-\Omega(t)x_1^2 + x_1 \left(\Omega(t)\Delta - \gamma \left(1 + \frac{\alpha}{\sigma} \right) \right) \leq -x_1. \tag{2.9}$$

□

Theorem 2.4. *Let $0 \leq \max_{0 \leq t} \Omega(t) = \bar{\Omega}$, and let parameters $\alpha, 0 < \underline{x}_1 < \Delta, \sigma$ be fixed. Whatever $(I_0, R_0) \in \mathbb{R}^2$ are, choose r'_3 in such a way that $\gamma = r'_3\sigma$ verifies:*

$$0 < \left(\gamma - \frac{3}{4} \right); \quad \bar{\Omega}(\Delta - \underline{x}_1) - \gamma \left(1 + \frac{\alpha}{\sigma} \right) + 1 \leq 0. \tag{2.10}$$

Then, there exists $0 \leq T_r$ such that for all time $t \geq T_r$, the solution $(I(t), R(t))$ of problem (1.7) belongs to the subset C .

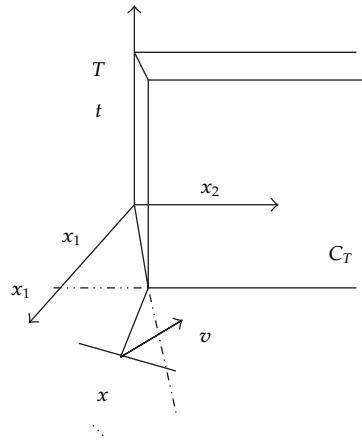


Figure 1: Exterior contingent cone.

Proof. Set $Y = (t, I(t), R(t))$, then problem (1.7) is expressed as the following autonomous system:

$$\begin{aligned} \frac{dY(t)}{dt} &= F(Y(t)); \quad 0 < t, \\ Y(0) &= (0, I_0, R_0). \end{aligned} \quad (2.11)$$

Define the function $G(t, I)$ by

$$G(t, I) = \Omega(t)(\Delta - I) - \gamma \left(1 + \frac{\alpha}{\sigma} \right) + 1. \quad (2.12)$$

Function G is a decreasing function with respect to I for all time. Thus if $G(t, x_1) \leq 0$, it will be negative for all $I > x_1$. Condition (2.10) implies that $F_2(t, I, R)$ is negative and $F_3(t, I, R)$ is positive for all $0 < t; x_1 < I; 0 \leq R$. Theorem 1.1 asserts the existence of $(I(t), R(t))$ for all time t . A simple continuity argument implies that the subset C defined in (2.1) is reached for a time T_r by the trajectory starting at the point (I_0, R_0) . Fix $T > T_r$, Lemma 2.2 claims that condition (2.10) is sufficient for $F(Y) \in T_{C_T}(Y)$ when Y belongs to the boundary of C_T . Nagumo's theorem applies for equation (2.11) with initial conditions $(T_r, I(t_r), R(T_r))$, and we get $(I(t), R(t)) \in C$ for $T_r \leq t$ [7, Theorem 1, page 27]. \square

As consequence of Theorem 2.4 the SIR models allow to improve the efficiency of medical policies. The sufficient condition (2.10) characterizes the treatment effort through the survival rate r_3 of the infected mature individuals recovered with the rate σ .

Let us end this section with numerical examples. The system (1.7) is discretized with a Runge-Kutta's method (RK4). By using available data from Mali in system (1.7) we have the following values for parameters: $\Delta = 6066573$; $\gamma = 0.25$; $\alpha = 0.01$; $x_1 = 1.5 \cdot 10^5$. The following graphs represent the phase portrait of system (1.7). When the time elapses, the values of the function I are along the x -axis and the values of the function R are along the y -axis. There is no limit cycle, and the last point of the simulation is represented with the green point.

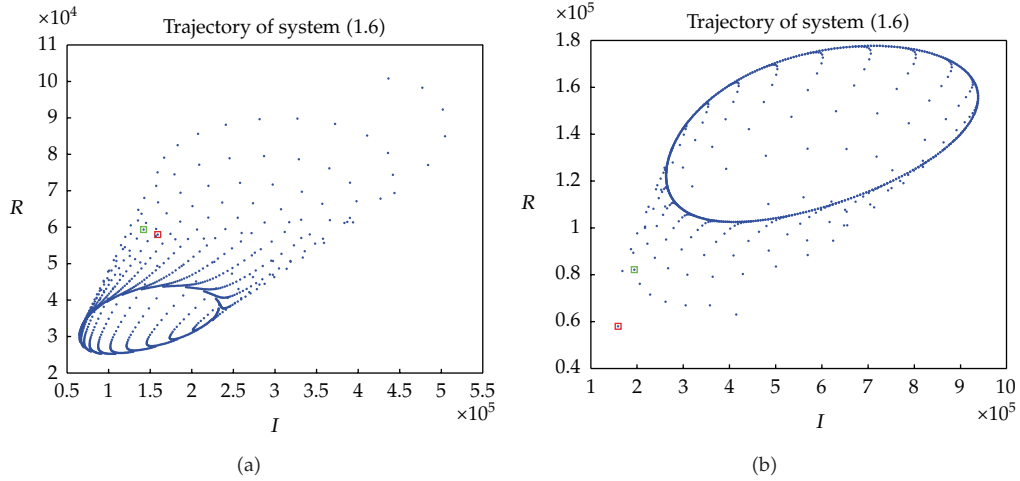


Figure 2: (a) $\bar{\Omega}(\Delta - \underline{x}_1) - \gamma(1 + \alpha/\sigma) \leq -1$, (b) $\bar{\Omega}(\Delta - \underline{x}_1) - \gamma(1 + \alpha/\sigma) + 1 > 0$.

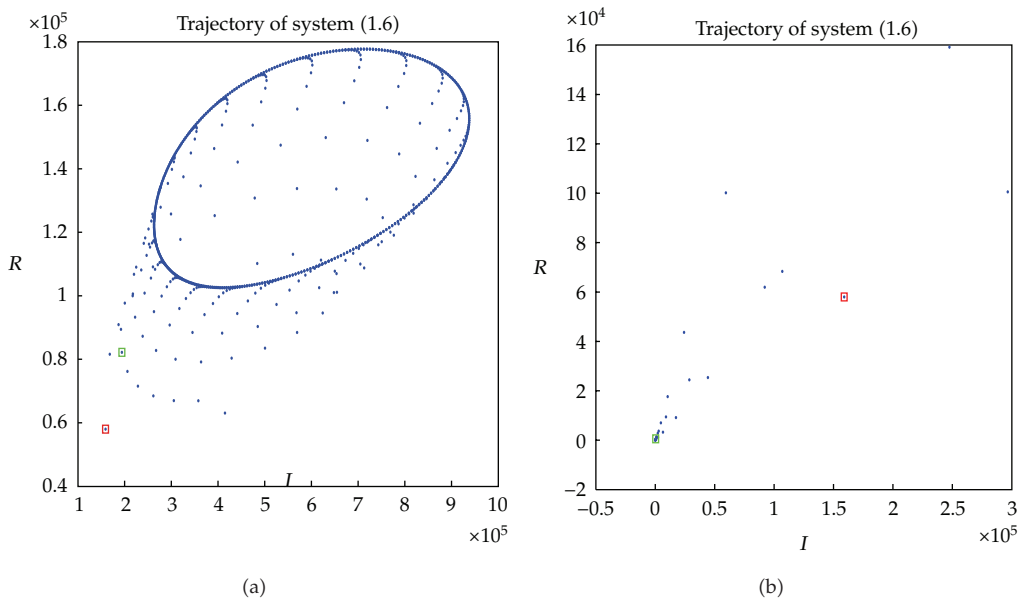


Figure 3: (a) $(3/4)x_1 \leq x_2$ is not satisfied with $r_3 = 0.85$; (b) $(3/4)x_1 \leq x_2$ is satisfied with $r_3 = 2$.

The initial conditions are $(I_0, R_0) = (1.59 \cdot 10^5, 5.8 \cdot 10^4)$ (denoted by the red point). In Figure 2 we have considered the case (a) with $\Omega = 4.3 \cdot 10^{-8}$; $\sigma = 0.5$. The sufficient condition (2.10) is not satisfied, nevertheless, it can be checked that after a long time the computed solution $(I(t), R(t))$ has a first component less than or equal to \bar{x}_1 . In case (b), we have $\Omega = 4.6 \cdot 10^{-8}$; $\sigma = 0.8$. The sufficient condition (2.10) is not satisfied. Here the trajectory is outside the cone C_T .

The cone C_T , roughly speaking, characterizes the treatment effort. The sufficient condition (2.10) is basically governed by two parameters: the transmission rate, and

the survival rate r_3 of the infected mature individuals recovered at a rate σ . In the following examples, keeping the same values for parameters as in case (a) except for r_3 . For $r_3 = 0.85$, The sufficient condition (2.10) is not satisfied, and we have in Figure 3(a) the trajectory outside C_T . For $r_3 = 2$, the sufficient condition (2.10) is satisfied, and the trajectory is concentrated in a neighborhood of the disease-free equilibrium $(0, 0)$ see Figure 3(b).

3. Conclusion

In this paper, it is shown by using the exterior contingent cone and a viability theorem, simple convex subsets are reachable with a SIR model by adjusting some coefficients. Thus, it will be possible to predict with a certain accuracy the evolution level of the disease by adjusting one or another of parameters. In our example, it is important to see that, if the survival rate r_3 attains 2%, the disease almost goes back at a level disease-free equilibrium. The controllability of the dynamical system has been established by using the exterior contingent cone technique. The mathematical model could be improved by introducing new compartments for describing, for example, the transmission of the disease between mothers and children. If the exterior contingent cone, for ordinary differential system of higher dimension, can be defined in the same way as before, it has to be calculable that which is an open question in general.

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