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Research Article

Some Properties of Fuzzy Quasimetric Spaces

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Some properties of fuzzy quasimetric spaces are studied. We prove that the topology induced by any $\widehat{M}$-complete fuzzy-quasi-space is a $\hat{d}$-complete quasimetric space. We also prove Baire's theorem, uniform limit theorem, and second countability result for fuzzy quasi-metric spaces.

1. Introduction and Preliminaries

Zadeh [1] introduced the concept of fuzzy sets as a new way to represent vagueness in our everyday life. Since then, many authors regarding the theory of fuzzy sets and its applications have developed a lot of literatures. Fuzzy Metric Spaces and existence of fixed points in fuzzy metric spaces have been emerged as two of the major of research activities. As natural, several mathematicians have introduced fuzzy metric spaces in different ways (Kramosil and Michálek [2], Erceg [3], Deng [4], Kaleva and Seikkla [5]). The definition proposed by Kramosil and Michalek in 1975 [2] is the most accepted one which is closely related to a class of probabilistic metric spaces [6]. Following this definition, a lot of research have been done on the existence of fixed points for the mappings under different conditions. Many authors have investigated and modified the definition of this concept and defined a Hausdorff topology on this fuzzy metric space. They showed that every metric induces a fuzzy metric, and, conversely, every fuzzy metric space generates a metrizable topology [2, 7–9].

Recently, many authors observed that various topological properties, fixed point theorems, and contraction mappings in metric spaces may be exactly translated into fuzzy metric spaces [10–21].

On the other hand, Künzi in [22] showed that the concepts from the theory of quasimetric spaces can be used as an efficient tool to solve several problems from theoretical computer science, approximation theory, topological algebra, and so forth.

In [23], Gregori and Romaguera introduced and studied the class of fuzzy quasimetric spaces as a natural generalization of the corresponding notion of fuzzy metric space to the quasimetric. Our paper continues to study such spaces and investigates some properties of this class. In particular, we address issues related to compactness, completeness, and Baire's theorem.

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm [6] if $*$ satisfies the following conditions:

(1) $*$ is associative and commutative.

(2) $*$ is continuous.

(3) $a * 1 = a$ for every $a \in [0,1]$.

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, with $a, b, c, d \in [0,1]$.

Example 1.1. The four basic t-norms are the following.

(1) The Lukasierviez t-norm: $*_{L:} I \times I \to I, a *_L b = \max\{a + b - 1, 0\}$.

(2) The product t-norm: $a *_P b = ab$.

(3) The minimum t-norm: $a *_M b = \min\{a, b\}$.

(4) The weakest t-norm, the drastic product:

$$*_D(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Using pointwise ordering, we have the inequalities $*_D < *_L < *_P < *_M$.

Let X be a nonempty set, $*$ a continuous t-norm, and M a fuzzy set in $X \times X \times [0, +\infty)$. For all $x, y, z \in X$ and $t, s > 0$, consider the following conditions.

(FM-1) $M(x, y, 0) = 0$

(FM-2) $M(x, x, t) = 1$ for all $t > 0$

(FM-3) $x = y$ if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$

(FM-3′) $x = y$ if $M(x, y, t) = 1$ for all $t > 0$

(FM-4) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$ for all $t, s \ge 0$

(FM-5) $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left continuous.

(FM-6) $M(x, y, t) = M(y, x, t)$ for all $t > 0$.

A fuzzy quasimetric on X [23] is a pair $(M, *)$ satisfying the conditions (FM-1), (FM-2), (FM-3), (FM-5), and (FM-6). If M satisfies conditions (FM-1), (FM-2), (FM-4), (FM-5), and (FM-6), then we call $(M, *)$ a T_1 fuzzy quasimetric on X. If M satisfies conditions (FM-1), (FM-2), (FM-3), (FM-5), (FM-6), and (FM-7), then we call $(M, *)$ a fuzzy metric space. A fuzzy quasimetric space is a triple $(X, M, *)$ such that X is a nonempty set and $(M, *)$ is a fuzzy quasimetric on X.

Remark 1.2. (1) It is clear that fuzzy metric ⇒ T_1 fuzzy quasimetric ⇒ fuzzy quasimetric.

(2) If $(M,*)$ is a fuzzy quasimetric on X, then M^{-1} is also fuzzy quasimetric on X, where M^{-1} is the fuzzy-set in $X \times X \times [0,+\infty)$ such that $M^{-1}(x,y,t) = M(y,x,t)$.

(3) If $(M,*)$ is a fuzzy quasimetric on X, then $\widehat{M}(x,y,t) = \min\{M(x,y,t), M^{-1}(x,y,t)\}$ is a fuzzy metric on X.

(4) If $(X,M,*)$ is a fuzzy quasimetric space, then for each $x,y,\in X$ the function $M(x,y,\cdot)$ is nondecreasing.

Proof of (4). Let $x,y \in X$ and $0 \le t < s$. By property (FM-4) $M(x,y,t) = M(x,x,s-t) * M(x,y,t) \le M(x,y,s)$. □

Example 1.3 (see [23]). *Let (X,d) be a quasimetric space. Define $a*b = ab$ for any $a,b,\in [0,1]$, and let M_d be the function defined on $X \times X \times [0,+\infty)$ by*

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}. \tag{1.2}$$

Then $(X,M_d,*)$ is a fuzzy quasimetric space and M_d is called the fuzzy quasimetric induced by d. Also the topology τ_d induced by the metric d and the topology τ_{M_d} induced by the fuzzy quasimetric $(M,*)$ are the same.

We call a fuzzy quasimetric $(M,*)$ on X a non-Archimedean if $M(x,z,t) \ge \min\{M(x,y,t), M(y,z,t)\}$ for all $x,y,z \in X, t > 0$. It is obvious to see that if M is a non-Archimedean fuzzy quasimetric on X, then $(M,*_M)$ is a fuzzy quasimetric on X.

The proof of the following theorem is straightforward. Therefore, it is omitted.

Theorem 1.4. *Let M_d be the standard fuzzy quasimetric of the quasimetric d on X. Then, M_d is non-Archimedean if and only if d is non-Archimedean.*

Definition 1.5. Let $(X,M,*)$ be a fuzzy quasimetric space and let $r \in (0,1), t > 0$, and $x \in X$.

The set $B_r^M(x,t) = \{y \in X : M(x,y,t) > 1-r\}$ is called the open ball with center x and radius r with respect to t. It is clear that $x \in B_r^M(x,t)$.

Lemma 1.6. *Let $(X,M,*)$ be a fuzzy quasimetric space. Than every open ball $B_r^M(x,t)$ is an open set.*

Proof. Let $y \in B_r^M(x,t)$. Then $M(x,y,t) > 1-r$. Set $r_0 = M(x,y,t_0)$. Since $r_0 > 1-r, \exists s \in (0,1)$ such that $r_0 > 1-s > 1-r$. Now given r_0 and s such that $r_0 > 1-s, \exists r_1 \in (0,1)$ such that $r_0 * r_1 > 1-s$. Consider the open ball $B_{1-r_1}^M(y,t-t_0)$. We claim $B_{1-r_1}^M(y,t-t_0) \subset B_r^M(x,t)$. Let $z \in B_{1-r_1}^M(y,t-t_0)$, then $M(y,z,t-t_0) > r_1$. So, $M(x,z,t) \ge M(x,y,t_0) * M(y,z,t-t_0) \ge r_0 * r_1 \ge 1-s > 1-r$. Thus $z \in B_r^M(x,t)$. □

2. Quasimetrization and Completeness Results

By Remark (2)–(4), if $x \in X, 0 < r_1 \le r_2 < 1$, and $0 < t_1 \le t_2$, then $B_{r_1}^M(x,t_1) \subseteq B_{r_2}^M(x,t_2)$. Hence the following theorem and lemma are easy to prove.

Theorem 2.1. *Let $(X, M, *)$ be a fuzzy (T_1) quasimetric space. Then $\tau_M = \{A \subset X :$ for each $x \in A, \exists t > 0$ and $r \in (0,1)$ such that $B_r^M(x,t) \subset A\}$ is a $T_0(T_1)$ topology on X.*

Lemma 2.2. *Let (X, M) be a fuzzy quasimetric space. Then for each $x \in X$, $\{B(x, 1/n, 1/n) : n \in \mathbb{N}\}$ is a neighborhood base at X for the topology τ_M. Moreover, the topology τ_M is first countable.*

Proposition 2.3 (A.H. Frink [24]). *T_1 space admits a quasiuniformity with a countable base if it is quasimetrizable.*

We can use a similar proof of [7, Theorem 1] to prove the following theorem.

Theorem 2.4. *Let $(X, M, *)$ be a fuzzy quasimetric space. Then (X, τ_M) is a quasimetric space.*

Proof. Let $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - (1/n), n \in \mathbb{N}\}$. We claim that $\{U_n : n \in \mathbb{N}\}$ is a base for quasiuniformity $\mathcal{U}$ on X whose induced topology coincides with τ_M. It is clear that $\{(x, x) : x \in X\} \subseteq U_n, U_{n+1} \subseteq U_n$. Also (by continuity of $*$) for each $n \in \mathbb{N}$, there is an m such that $m > 2n$ and $(1 - (1/m)) * (1 - (1/m)) > 1 - (1/n)$. Then $U_m \circ U_m \subseteq U_n$. Let $(x, y), (y, z) \in U_M$. From (Remark (2)–(4), $M(x, z, 1/n) \geq M(x, z, l/m)$. So, $M(x, z, 1/n) \geq M(x, y, 1/m) * M(y, z, 1/m) \geq (1 - (1/m)) * (1 - (-(1/m) > 1 - 1/n$. Therefore $(x, z) \in U_n$. Thus $\{U_n : n \in \mathbb{N}\}$ is a base for a quasiuniformity $\mathcal{U}$ on X. Since for each $x \in X$ and $n \in \mathbb{N}$, $U_n(x) = \{y \in X : M(x, y, 1/n) > 1 - (1/n)\} = B(x, y, 1/n)$, we deduce from Lemma 2.2 that the topology induced by $\mathcal{U}$ coincides with τ_M. By Proposition 2.3, (X, τ_M) is a quasimetrizable space. □

Corollary 2.5. *A topological space is quasimetrizable if and only if it admits a compatible fuzzy quasimetric.*

Definition 2.6. Let $(X, M, *)$ be a fuzzy quasimetric space. Then

(1) a sequence $\{x_n\}$ in X is said to be $\widehat{M}$-Cauchy if for each $\delta > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\widehat{M}(x_n, x_m, t) > 1 - \delta$, for all $n, m \geq n_0$;

(2) a sequence $\{x_n\}$ in XM-converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$;

(3) a sequence $\{x_n\}$ in $X\widehat{M}$-converges to x if and only if $\widehat{M}(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$;

(4) $(X, M, *)$ is $\widehat{M}$-complete if every $\widehat{M}$-Cauchy sequence is $\widehat{M}$-convergent with respect to $\tau_{\widehat{M}}$.

Remark 2.7. It is easy to prove that in a fuzzy quasimetric space $(X, M, *)$ if a sequence $\{x_n\}$ is $\widehat{M}$-Cauchy, then $\lim_{n\to\infty} \widehat{M}(x_n, x_{n+m}, t) = 1$ for each $m \in \mathbb{N}$ and $t > 0$.

Theorem 2.8. *Let $(X, M, *)$ be a fuzzy quasimetric space such that every $\widehat{M}$-Cauchy sequence in X has an $\widehat{M}$-convergent subsequence. Then $(X, M, *)$ is $\widehat{M}$-complete.*

Proof. Suppose $\{x_n\}$ is a $\widehat{M}$-Cauchy sequence and $\{x_{n_k}\}$ a subsequence of $\{x_n\}$ such that $\widehat{M}$-converges to x. To prove that $x_n\widehat{M}$-converges to x, let $t > 0$ and $\delta \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) \geq 1 - \delta$. Since $\{x_n\}$ is $\widehat{M}$-Cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $\widehat{M}(x_m, x_n, t/2) > 1 - r$ for all $m, n \geq n_0$. Since $x_{n_k}\widehat{M}$-converges to x, there is $k_1 \in \mathbb{N}$ such that

$k_1 > n_0, \widehat{M}(x_{k_1}, x, t/2) > 1 - r$. This implies that if $n \geq n_0$, then $\widehat{M}(x_n, x, t) \geq \widehat{M}(x_n, x_{k_1}, t/2) * \widehat{M}(x_{k_1}, x, t/2 > (1-r) * (1-r) \geq 1 - \delta$. Therefore, $x_n \widehat{M}$-converges to x and hence $(X, M, *)$ is $\widehat{M}$-complete. □

Let d be a quasimetric on X, M_d the corresponding fuzzy quasimetric, and $\widehat{d} = \max\{d(p,q), d(q,p)\}$ for each $p, q \in X$. Since a sequence $\{x_n\}$ is an $\widehat{M}_d$-Cauchy sequence in X if and only if $\{x_n\}$ is a $\widehat{d}$-Cauchy sequence in X, it is not difficult to prove the following lemma; hence we omit the proof.

Lemma 2.9. *Let (X, d) be a quasimetric space. Then (X, d) is $\widehat{d}$-complete if and only if (X, M_d) is $\widehat{M}_d$-complete.*

Definition 2.10. Let $(X, M, *)$ be a fuzzy quasimetric space. A collection $\{F_n\}_{n\in\mathbb{N}}$ is said to have fuzzy diameter zero if for each $r \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x, y, t) > 1 - r$ for all $x, y \in F_{n_0}$.

It is clear that a nonempty subset F of a fuzzy quasimetric space X has fuzzy diameter zero if and only if F is a singleton set.

For self-containment and clarity, we give the proofs of the following theorems even though they share similarities to those in [9].

Theorem 2.11. *A fuzzy quasimetric space $(X, M, *)$ is $\widehat{M}$-complete if and only if every nested sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty closed sets with fuzzy diameter zero has nonempty intersection.*

Proof. First, suppose that the given condition is satisfied. We claim that $(X, M, *)$ is $\widehat{M}$-complete. Let $\{x_n\}$ be a $\widehat{M}$-Cauchy sequence in X. Set $B_n = \{x_k : k \geqslant n\}$ and $F_n = \overline{B_n}$, then we claim that $\{F_n\}$ has a fuzzy diameter zero. For given $s \in (0,1)$ and $t > 0$, we choose $r \in (0,1)$ such that $(1-r) * (1-r) * (1-r) > 1 - s$. Since $\{x_n\}$ is $\widehat{M}$-Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $\widehat{M}(x_n, x_m, (1/3)t) > 1 - r$ for all $m, n \geqslant n_0$. Therefore, $\widehat{M}(x, y, (1/3)t) > 1 - r$ for all $x, y \in B_{n_0}$. Let $x, y \in F_{n_0}$. Then there exist sequences $\{x'_n\}$ and $\{y'_n\}$ in B_{n_0} such that $x'_n \to x$ and $y'_n \to y$. Hence $x'_n \in B(x, r, t/3)$ and $y'_n \in B(y, r, t/3)$ for sufficiently large n. Now we have $\widehat{M}(x, y, t) \geqslant \widehat{M}(x, x'_n, t/3) * \widehat{M}(x'_n, y'_n, t/3) * \widehat{M}(y'_n, y, t/3) > (1-r) * (1-r) * (1-r) > 1 - s$. Therefore, $\widehat{M}(x, y, t) > 1 - s$ for all $x, y \in F_{n_0}$. Thus $\{F_n\}$ has fuzzy diameter zero and hence by hypothesis $\bigcap_{n\in\mathbb{N}} F_n$ is nonempty.

Take $x \in \bigcap_{n\in\mathbb{N}} F_n$. We show that $x_n \to x$. Then, for $r \in (0,1)$ and $t > 0$, there exists $n_1 \in \mathbb{N}$ such that $\widehat{M}(x_n, x, t) > 1 - r$ for all $n \geqslant n_1$. Therefore, for each $t > 0$, $\widehat{M}(x_n, x, t) \to 1$ as $n \to \infty$ and hence $x_n \to x$. Therefore, $(X, M, *)$ is $\widehat{M}$-complete.

Conversely, suppose that $(X, M, *)$ is $\widehat{M}$-complete and $\{F_n\}_{n\in\mathbb{N}}$ is nested sequence of non empty closed sets with fuzzy diameter zero. For each $n \in \mathbb{N}$, choose a point $x_n \in F_n$. We claim that $\{x_n\}$ is a $\widehat{M}$-Cauchy sequence. Since $\{F_n\}$ has fuzzy diameter zero, for $t > 0$ and $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\widehat{M}(x, y, t) > 1 - r$ for all $x, y \in F_{n_0}$. Since $\{F_n\}$ is a nested sequence, $\widehat{M}(x_n, x_m, t) > l - r$ for all $n, m \geqslant n_0$. Hence $\{x_n\}$ is a $\widehat{M}$-Cauchy sequence. Since $(X, M, *)$ is $\widehat{M}$-complete, $x_n \to x$ for some $x \in X$. Therefore, $x \in \overline{F_n} = F_n$ for every n, and hence $x \in \bigcap_{n\in\mathbb{N}} F_n$. This completes the proof. □

Remark 2.12. The element $x \in \bigcap_{n\in\mathbb{N}} F_n$ is unique. For if there are two elements $x, y \in \bigcap_{n\in\mathbb{N}} F_n$ since $\{F_n\}$ has a fuzzy diameter zero, for each fixed $t > 0$, $M(x, y, t) > 1 - 1/n$ for each $n \in \mathbb{N}$. This implies that $M(x, y, t) = 1$ and hence $x = y$.

Note that the topologies induced by the standard fuzzy quasimetric and the corresponding quasimetric are the same. So we have the following.

Corollary 2.13. *Let (X,d) be a quasimetric space and $\widehat{d} = \max\{d(p,q),d(q,p)\}$ for each $p,q \in X$. Then (X,d) is $\widehat{d}$-complete if and only if every nested sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty closed sets with diameter tending to zero has a nonempty intersection.*

Theorem 2.14. *Every separable fuzzy quasimetric space is second countable.*

Proof. Let $(X,M,*)$ be the given separable fuzzy quasimetric space. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X. Consider the family $\mathcal{B} = \{B(x_j,1/k,1/k) : j,k \in \mathbb{N}\}$. Then $\mathcal{B}$ is countable. We claim that $\mathcal{B}$ is a base for the family of all open sets in X. Let U be any open set in X and let $x \in U$. Then there exist $t > 0$ and $r \in (0,I)$ such that $B(x,r,t) \subset U$. Since $r \in (0,1)$, we can choose a $s \in (0,1)$ such that $(1-s)*(l-s) > 1-r$. Take $m \in \mathbb{N}$ such that $1/m < min\{s,t/2\}$. Since A is dense in X, there exists $x_j \in A$ such that $x_j \in B(x,1/m,1/m)$. Now, if $y \in B(x_j,1/m,1/m)$, then $M(x,y,t) \geqslant M(x,x_j,t/2)*M(y,x_j,t/2) \geqslant M(x,x_j,1/m)*M(y,x_j,1/m) \geqslant (1-1/m)*(1-1/m) \geqslant (1-s)*(1-s) > 1-r$. Thus, $y \in B(x,r,t) \subset U$ and hence $\mathcal{B}$ is a base. □

Remark 2.15. Since second countability is a hereditary property and second countability implies separability, we obtain the following: every subspace of a separable fuzzy quasimetric space is separable.

A fuzzy quasimetric $(X,M,*)$ is said to be totally bounded if for all $\epsilon > 0, x \in X$, there exist $x_1,x_2,x_3,\ldots,x_n \in X$ and $i \in \{1,2,3,\ldots,n\}$ such that $\widehat{M}(x,x_i,t) > 1-\epsilon$ for all $t > 0$.

We can use proofs similar to that in [24, Proposition 7.2, Proposition 7.5, and Corollary 7.6] to prove the following theorems.

Theorem 2.16. *Let $(X,M,*)$ be a fuzzy quasimetric space. Then (X,τ_M) is a second countable if and only if it is totally bounded.*

A fuzzy quasimetric M on a set X is continuous provided that for each $x \in X$ the function $M(x,\cdot,t) : X \to [0,1]$ defined by $M(x,\cdot,t)(p) = M(x,p,t)$ is a continuous function.

Theorem 2.17. *Let $(X,M,*)$ be a fuzzy quasimetric space and let $x \in X, t > 0$. Then $M(x,\cdot,t)$ is a continuous function if and only if for each $\delta > 0$, $\overline{B_\delta^M(x,t)} \subset \{y \in X : M(x,y,t) \geq 1-\delta\}$.*

Theorem 2.18. *Let $(X,M,*)$ be a fuzzy quasimetric space. If $M(\cdot,x,t)$ is an upper semicontinuous function for each $x \in X, t > 0$, then (X,τ_M) is metrizable.*

Definition 2.19. Let X be any nonempty set and $(Y,M,*)$ a fuzzy quasimetric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to converge uniformly to a function f from X to Y if given $t > 0$ and $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n(x),f(x),t) > 1-r$ for all $n \geqslant n_0$ and for all $x \in X$.

Theorem 2.20 (Uniform Limit Theorem). *Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a fuzzy quasimetric space $(Y,M,*)$. If $\{f_n\}$ converges uniformly to f: $X \to Y$, then f is continuous.*

Proof. Let V be on open set of Y and let $x_0 \in f^{-1}(V)$. We want to find a neighborhood U of x_0 such that $f(U) \subset V$. Since V is open, there exist $t > 0$ and $r \in (0,1)$ such that we choose a $s \in (0,1)$ such that $(1-s) * (1-s) * (1-s) > 1-r$. Since $\{f_n\}$ converges uniformly to f, given $t > 0$ and $s \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n(x), f(x), t/3) > 1-s$ for all $n \geqslant n_0$ and for all $x \in X$. Since f_n is continuous for all $n \in \mathbb{N}$, there exists a neighborhood U of x_0 such that $f_n(U) \subset B(f_n(x_0), s, t/3)$. Hence $M(f_n(x), f_n(x_0), t/3) > 1-s$ for all $x \in U$. Now $M(f(x), f(x_0), t) \geqslant M(f(x), f_n(x), t/3) * M(f_n(x), f_n(x_0), t/3) * M(f_n(x_0), f(x_0), t/3) \geqslant (1-s) * (1-s) * (1-s) > 1-r$. Thus, $f(x) \in B(f(x_0), r, t) \subset V$ for all $x \in U$. Hence $f(U) \subset V$ and so f is continuous. □

Remark 2.21. Let $(X, M, *)$ be a fuzzy quasimetric space. It is easy to prove that if $t > 0$ and $r, s \in (0,1)$ such that $(1-s) * (1-s) \geq (1-r)$, then $\overline{B(x, s, t/2)} \subset B(x, r, t)$.

Lemma 2.22. *A subset A of a fuzzy quasimetric space $(X, M, *)$ is nowhere dense if and only if every nonempty open set in X contains an open ball whose closure is disjoint from A.*

Proof. Let U be a nonempty open subset of X. Then there exists a nonempty open set V such that $V \subset U$ and $V \cap \overline{A} \neq \phi$. Let $x \in V$. Then there exist $r \in (0,1)$ and $t > 0$ such that $B(x, r, t) \subset V$. Choose $s \in (0,1)$ such that $(1-s) * (1-s) \geqslant 1-r$. By Remark 2.21, $\overline{B(x, s, t/2)} \subset B(x, r, t)$. Thus $B(x, s, t/2) \subset U$ and $\overline{B(x, s, t/2)} \cap A = \phi$.

Conversely, suppose A is not nowhere dense. Then $\operatorname{int}(\overline{A}) \neq \phi$, so there exists a nonempty open set U such that $U \subset \overline{A}$. Let $B(x, r, t)$ be an open ball such that $B(x, r, t) \subset U$. Then $\overline{B(x, r, t)} \cap A \neq \phi$. This is a contradiction. □

Theorem 2.23 (Baire's Theorem). *Suppose $\{U_n\}_{n \in \mathbb{N}}$ is a sequence of dense open subsets of a $\widehat{M}$-complete fuzzy quasimetric space $(X, M, *)$. Then $\bigcap_{n \in \mathbb{N}} U_n$ is also dense in X.*

Proof. Let V be a nonempty open set of X. Since U_1 is dense in X, $V \cap U_1 \neq \phi$. Let $x_1 \in V \cap U_1$. Since $V \cap U_1$ is open, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that $\overline{B(x_1, r_1, t_1)} \subset V \cap U_1$. Choose $r'_1 < r_1$ and $t'_1 = \min(t_1, 1)$ such that $B(x_1, r'_1, t'_1) \subset V \cap U_1$. Since U_2 is dense in X, $B(x_1, r'_1, t'_1) \cap U_2 \neq \phi$. Let $x_2 \in B(x_1, r'_1, t'_1) \cap U_2$. Since $B(x_1, r'_1, t'_1) \cap U_2$ is open, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that $\overline{B(x_2, r_2, t_2)} \subset B(x_1, r'_1, t'_1) \cap U_2$. Choose $r'_2 < r_2$ and $t'_2 = \min\{t_2, 1/2\}$ such that $B(x_2, r'_2, t'_2) \subset B(x_1, r'_1, t'_1) \cap U_2$. Continuing in this manner, we obtain a sequence $\{x_n\}$ in X and a sequence $\{t'_n\}$ such that $0 < t'_n < 1/n$ and $\overline{B(x_n, r'_n, t'_n)} \subset B(x_{n-1}, r'_{n-1}, t'_{n-1}) \cap U_n$.

Now we claim that $\{x_n\}$ is a $\widehat{M}$-Cauchy sequence. For a given $t > 0$ and $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $1/n_0 - t$ and $1/n_0 < \epsilon$. Then for $n \geqslant n_0$ and $m \geqslant n$, $\widehat{M}(x_n, x_m, t) \geqslant \widehat{M}(x_n, x_m, 1/n) \geqslant 1 - 1/n > 1 - \epsilon$.

Therefore, $\{x_n\}$ is a $\widehat{M}$-Cauchy sequence. Since X is $\widehat{M}$-complete, there exists $x \in X$ such that $x_n \to x$. Since $x_k \in B(x_n, r'_n, t'_n)$ for $k \geqslant n$, we obtain $x \in \overline{B(x_n, r'_n, t'_n)}$. Hence $x \in \overline{B(x_n, r'_n, t'_n)} \subset B(x_{n-1}, r'_{n-1}, t'_{n-1}) \cap U_n$ for all n. Therefore, $V \cap (\bigcap_{n \in \mathbb{N}} U_n) \neq \phi$.

Hence $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X. □

3. Compactness Results

Theorem 3.1. *Every T_1 fuzzy quasimetric space is Hausdorff.*

Proof. Let $(X, M, *)$ be a fuzzy quasimetric space. Let x and y be two distinct points in X. Then $0 < M(x, y, t) < 1$. Put $r_1 = M(x, y, t)$ and $r = \max\{r_1, 1 - r_1\}$. For each $r_0 \in (r, 1)$, there exists r_2 such that $r_2 * r_2 \geqslant r_0$. Put $r_3 = \max\{r_2, 1 - r_2\}$ and consider the open balls $B(x, 1 - r_3, t/2)$ and $B(y, 1 - r_3, t/2)$. Then clearly $B(x, 1 - r_3, t/2) \cap B(y, 1 - r_3, t/2) = \phi$. For if there exists $z \in B(x, 1 - r_3, t/2) \cap B(y, 1 - r_3, t/2)$, then $r_1 = M(x, y, t) \geqslant M(x, z, t/2) * M(z, y, t/2) \geqslant r_3 * r_3 \geqslant r_2 * r_2 \geqslant r_0 > r_1$, which is a contradiction. Hence $(X, M, *)$ is Hausdorff. □

Definition 3.2. Let $(X, M, *)$ be a T_1 fuzzy quasimetric space. A subset A of X is said to be IF-bounded if there exist $t > 0$ and $r \in (0, 1)$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Remark 3.3. Let $(X, M, *)$ be a T_1 fuzzy quasimetric space induced by a quasimetric d on X. Then $A \subset X$ is IF-bounded if and only if it is bounded.

Theorem 3.4. *Every compact subset A of a T_1 fuzzy quasimetric space $(X, M, *)$ is IF-bounded.*

Proof. Let A be a compact subset of a fuzzy quasimetric space X. Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B(x, r, t) : x \in A\}$ of A. Since A is compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r, t)$. Let $x, y \in A$.

Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some i, j. Thus we have $M(x, x_i, t) > 1 - r, M(y, x_j, t) > 1 - r$. Now, let $\alpha = \min\{M(x_i, x_j, t) : 1 \leqslant i, j \leqslant n\}$. Then $\alpha > 0$. Now, we have $M(x, y, 3t) \geqslant M(x, x_j, t) * M(x_i, x_j, t) * M(x_j, y, t) \geqslant (1 - r) * (1 - r) * \alpha > 1 - s_1$ for some $0 < s_1 < 1$. Taking $s = \max\{s_1, 1 - s_1\}$ and $t' = 3t$, we have $M(x, y, t') > 1 - s$ for all $x, y \in A$. Hence A is IF-bounded. □

From Remark 3.3 and Theorems 3.1 and 3.4, we have the following corollary.

Corollary 3.5. *In a T_1 fuzzy quasimetric space, every compact set is closed and bounded.*

4. Conclusion

This work studies the concept of fuzzy quasimetric spaces which was introduced by Gregori and Romaguera in 2004 [23] as a natural generalization of quasimetric spaces. A topological space is quasimetrizable if and only if it admits a compatible fuzzy–quasimetric. This has been expressed in our first main result, Corollary 2.5. Moreover, in Corollary 2.13, we have proved that the $\widehat{M}$-completeness of quasimetric space can be characterized in terms of a nested sequence of nonempty closed sets with diameter tending to zero which have nonempty intersection in a natural way. Following the proofs of Fletcher and Lindgren [24] of a metrization theorem of quasimetrizable spaces, we can prove a similar result as in Theorem 2.18. We also obtained Baire's Theorem, Uniform Limit Theorem, and a second countability result for fuzzy quasimetric spaces.

One important point which has been left for further study is the behaviour of fuzzy quasimetric spaces under mappings.

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