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Research Article

The Diagrammatic Soergel Category and $sl(N)$-Foams, for $N \geq 4$

Marco Mackaay[1,2] and Pedro Vaz[2,3]

[1] *Departamento de Matemática, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal*
[2] *CAMGSD, Instituto Superior Técnico, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal*
[3] *Institut de Mathématiques de Jussieu, Université Paris 7, 175 Rue du Chevaleret, 75013 Paris, France*

Correspondence should be addressed to Marco Mackaay, mmackaay@ualg.pt

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For each $N \geq 4$, we define a monoidal functor from Elias and Khovanov's diagrammatic version of Soergel's category of bimodules to the category of $sl(N)$ foams defined by Mackaay, Stošić, and Vaz. We show that through these functors Soergel's category can be obtained from the $sl(N)$ foams.

1. Introduction

In [1] Soergel categorified the Hecke algebra using bimodules. Just as the Hecke algebra is important for the construction of the HOMFLY-PT link polynomial, so is Soergel's category for the construction of Khovanov and Rozansky's HOMFLY-PT link homology [2], as explained by Khovanov in [3]. Elias and Khovanov [4] constructed a diagrammatic version of the Soergel category with generators and relations, which Elias and Krasner [5] used for a diagrammatic construction of Rouquier's complexes associated to braids.

In [6] Bar-Natan gave a new version of Khovanov's [7] original link homology, also called the $sl(2)$ link homology, using 2d-cobordisms modulo certain relations, which we will call $sl(2)$ foams. Using 2d-cobordisms with a particular sort of singularity modulo certain relations, which we will call $sl(3)$ foams, Khovanov constructed the $sl(3)$ link homology [8]. Khovanov and Rozansky [9] then constructed the $sl(N)$ link homologies, for any $N \geq 1$, using matrix factorizations. These link homologies are closely related to the HOMFLY-PT link homology by Rasmussen's spectral sequences [10], with E_1-page isomorphic to the HOMFLY-PT homology and converging to the $sl(N)$ homology, for any $N \geq 1$. In [11] Mackaay et al. gave an alternative construction of these $sl(N)$ link homologies, for $N \geq 4$, using $sl(N)$ foams, which are 2d-cobordisms with two types of singularities satisfying relations determined by a formula from quantum field theory, originally obtained by Kapustin and Li [12] and later adapted by Khovanov and Rozansky [13].

Khovanov and Rozansky in [2, 9] and Rasmussen in [10] used matrix factorizations for their constructions. Therefore, the question arises whether their results can be understood in diagrammatic terms and what could be learned from that. In [14] Vaz constructed functors from Elias and Khovanov's diagrammatic version of Soergel's category to the categories of $sl(2)$ and $sl(3)$ foams. In this paper we construct the analogous functors from the same version of Soergel's category to the category of $sl(N)$ foams for $N \geq 4$. To complete the picture, one would like to construct the analogues of Rasmussen's spectral sequences in this setting. However for this, one would first have to understand the Hochschild homology of bimodules in diagrammatic terms, which has not been accomplished yet. Hochschild homology plays an integral part of the construction. Nevertheless, there is an interesting result which can already be shown using the functors in this paper. In a certain technical sense, which we will make precise in Proposition 4.2, Soergel's category can be obtained from the $sl(N)$ foams, and therefore from the Kapustin-Li formula, using our functors. This result should be compared to Rasmussen's Theorem 1 in [10].

We thank Catharina Stroppel for pointing out the connection of our work to results in [15]. We quote her directly: In [15] a categorification of "special trivalent" graphs modulo the MOY relations was constructed by exact functors acting between certain blocks of parabolic category $\mathcal{O}$. Using Soergel's functor passing from Lie theory to the combinatorial bimodule category the construction in [15] in fact produces an action of the diagrammatic Soergel category on these various category $\mathcal{O}$s.

We have tried to make the paper as self-contained as possible, but the reader should definitely leaf through [4, 5, 11, 14] before reading the rest of this paper.

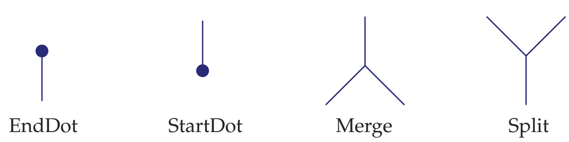
In Section 2 we recall Elias and Khovanov's version of Soergel's category. In Section 3 we review $sl(N)$ foams, as defined by Mackaay, Stošic', and Vaz. Section 4 contains the new results: the definition of our functors, the proof that they are indeed monoidal, and a statement on faithfulness in Proposition 4.2.

2. Elias and Khovanov's Version of Soergel's Category

This section is a reminder of the diagrammatics for Soergel categories introduced by Elias and Khovanov in [4]. Actually we give the version which they explained in [4, Section 4.5] and which can be found in detail in [5].

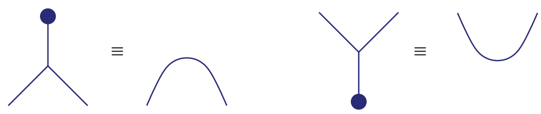
Fix a positive integer n. The category $\mathcal{SC}_1$ is the category whose objects are finite length sequences of points on the real line, where each point is colored by an integer between 1 and n. We read sequences of points from left to right. Two colors i and j are called adjacent if $|i-j| = 1$ and distant if $|i-j| > 1$. The morphisms of $\mathcal{SC}_1$ are given by generators modulo relations. A morphism of $\mathcal{SC}_1$ is a $\mathbb{C}$-linear combination of planar diagrams constructed by horizontal and vertical gluings of the following generators (by convention no label means a generic color j)

(i) Generators involving only one color are



(2.1)

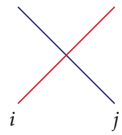
It is useful to define the cap and cup as



(2.2)

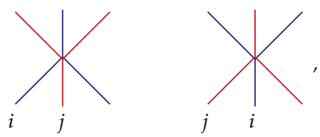
(ii) Generators involving two colors are

–The 4-valent vertex, with distant colors,



(2.3)

and the 6-valent vertex, with adjacent colors i and j,

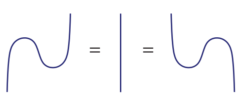


(2.4)

read from bottom to top. In this setting a diagram represents a morphism from the bottom boundary to the top. We can add a new colored point to a sequence and this endows $\mathcal{SC}_1$ with a monoidal structure on objects, which is extended to morphisms in the obvious way. Composition of morphisms consists of stacking one diagram on top of the other.

We consider our diagrams modulo the following relations.

"Isotopy" relations are



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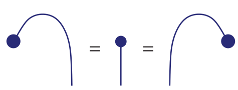

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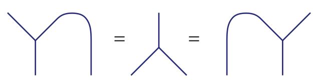

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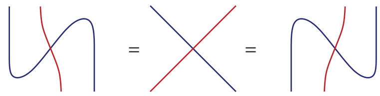

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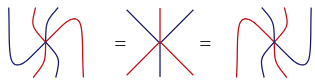
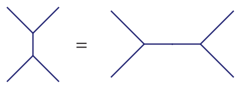


(2.9)

The relations are presented in terms of diagrams with generic colorings. Because of isotopy invariance, one may draw a diagram with a boundary on the side, and view it as a morphism in $\mathcal{SC}_1$ by either bending the line up or down. By the same reasoning, a horizontal line corresponds to a sequence of cups and caps.

One color relations are



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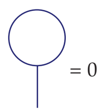
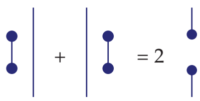
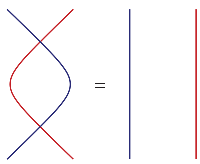


(2.11)



(2.12)

Relations involvingtwo distant colors are



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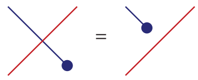

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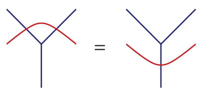

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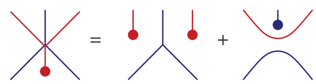
Relations involving two adjacent colors are



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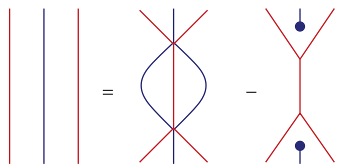
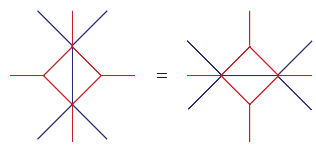


(2.17)



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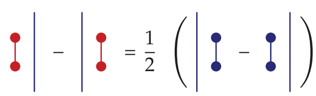
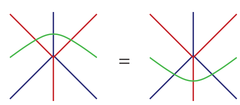
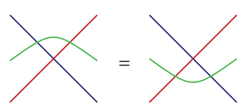


(2.19)

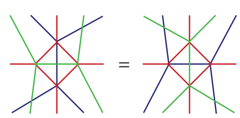
Relations involving three colors are (adjacency is determined by the vertices which appear)



(2.20)

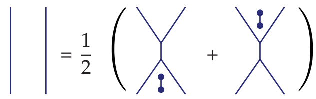


(2.21)



(2.22)

Furthermore, we also have a useful implication of relation (2.12) as follows:



(2.23)

Introduce a q-grading on $\mathcal{SC}_1$ declaring that dots have degree 1, trivalent vertices have degree −1, and 4- and 6-valent vertices have degree 0.

Definition 2.1. The category $\mathcal{SC}_2$ is the category containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$ and whose morphisms are the grading-preserving morphisms from $\mathcal{SC}_1$.

Definition 2.2. The category $\mathcal{SC}$ is the Karoubi envelope of the category $\mathcal{SC}_2$.

Elias and Khovanov's main result in [4] is the following theorem.

Theorem 2.3 (Elias-Khovanov). *The category $\mathcal{SC}$ is equivalent to the Soergel category in [1].*

From Soergel's results from [1] we have the following corollary.

Corollary 2.4. *The Grothendieck algebra of $\mathcal{SC}$ is isomorphic to the Hecke algebra.*

Notice that $\mathcal{SC}$ is an additive category but not abelian and we are using the (additive) split Grothendieck algebra.

In Section 4 we will define a a family of functors from $\mathcal{SC}_{1,n}$ to the category of $sl(N)$ foams, one for each $N \geq 4$. These functors are grading preserving, so they obviously extend uniquely to $\mathcal{SC}_{2,n}$. By the universality of the Karoubi envelope, they also extend uniquely to functors between the respective Karoubi envelopes.

3. Foams

3.1. Prefoams

In this section we recall the basic facts about foams. For the definition of the Kapustin-Li formula, for proofs of the relations between foams, and for other details, see [11, 16]. The foams in this paper are composed of three types of facets: simple, double, and triple facets. The double facets are coloured and the triple facets are marked to show the difference. Intersecting such a foam with a generic plane results in a web, as long as the plane avoids the singularities where six facets meet, such as on the right in Figure 1.

Definition 3.1. Let s_γ be a finite oriented closed 4-valent graph, which may contain disjoint circles and loose endpoints. We assume that all edges of s_γ are oriented. A cycle in s_γ is defined to be a circle or a closed sequence of edges which form a piecewise linear circle. Let Σ be a compact orientable possibly disconnected surface, whose connected components are simple, double, or triple, denoted by white, coloured, or marked. Each component can have a boundary consisting of several disjoint circles and can have additional decorations which we discuss below. A closed *prefoam* u is the identification space Σ/s_γ obtained by gluing boundary circles of Σ to cycles in s_γ such that every edge and circle in s_γ are glued to exactly three boundary circles of Σ and such that for any point $p \in s_\gamma$,

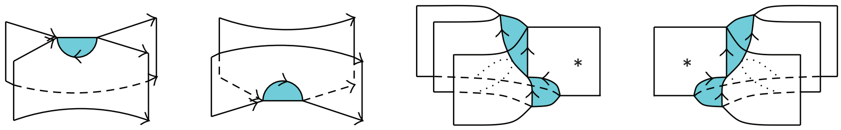


Figure 1: Some elementary prefoams.

(1) if p is an interior point of an edge, then p has a neighborhood homeomorphic to the letter Y times an interval with exactly one of the facets being double, and at most one of them being triple; for an example see Figure 1,

(2) if p is a vertex of s_γ, then it has a neighborhood as shown in Figure 1.

We call s_γ the *singular graph*, its edges and vertices *singular arcs* and *singular vertices*, and the connected components of $u - s_\gamma$ the *facets*.

Furthermore the facets can be decorated with dots. A simple facet can only have black dots (•), a double facet can also have white dots (◦), and a triple facet besides black and white dots can have double dots (⊙). Dots can move freely on a facet but are not allowed to cross singular arcs.

Note that the cycles to which the boundaries of the simple and the triple facets are glued are always oriented, whereas the ones to which the boundaries of the double facets are glued are not, as can be seen in Figure 1. Note also that there are two types of singular vertices. Given a singular vertex v, there are precisely two singular edges which meet at v and bound a triple facet: one oriented toward v, denoted as e_1, and one oriented away from v, denoted as e_2. If we use the “left-hand rule”, then the cyclic ordering of the facets incident to e_1 and e_2 is either $(3,2,1)$ or $(3,1,2)$, respectively, or the other way around. We say that v is of type I in the first case and of type II in the second case. When we go around a triple facet, we see that there have to be as many singular vertices of type I as there are of type II for the cyclic orderings of the facets to match up. This shows that for a closed prefoam the number of singular vertices of type I is equal to the number of singular vertices of type II.

We can intersect a prefoam u generically by a plane W in order to get a closed web, as long as the plane avoids the vertices of s_γ. The orientation of s_γ determines the orientation of the simple edges of the web according to the convention in Figure 2.

Suppose that, for all but a finite number of values $i \in]0,1[$, the plane $W \times i$ intersects u generically. Suppose also that $W \times 0$ and $W \times 1$ intersect u generically and outside the vertices of s_γ. Furthermore, suppose that $D \subset W$ is a disc in W and $C \subset D$ its boundary circle such that $C \times [0,1] \cap u$ is a disjoint union of vertical line segments. This means that we are assuming that s_γ does not intersect $C \times [0,1]$. We call $D \times [0,1] \cap u$ an *open* prefoam between the *open* webs $D \times \{0\} \cap u$ and $D \times \{1\} \cap u$. Interpreted as morphisms, we read open prefoams from bottom to top, and their composition consists of placing one prefoam on top of the other, as long as their boundaries are isotopic and the orientations of the simple edges coincide.

Definition 3.2. Let **Pfoam** be the category whose objects are webs and whose morphisms are $\mathbb{Q}$-linear combinations of isotopy classes of prefoams with the obvious identity prefoams and composition rule.

We now define the q-degree of a prefoam. Let u be a prefoam, u_1, u_2, and u_3 the disjoint union of its simple, double, and marked facets, respectively, and $s_\gamma(u)$ its singular

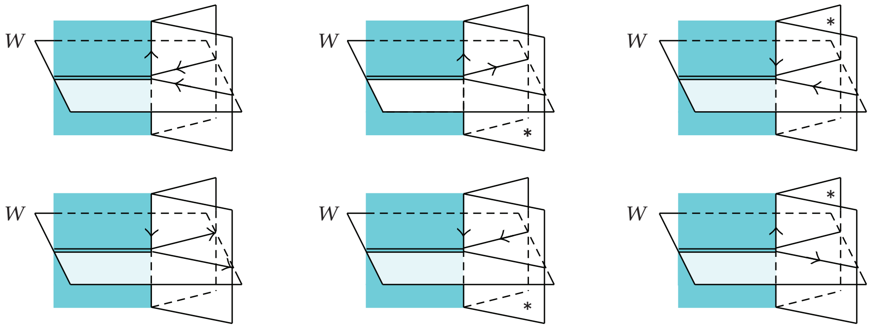


Figure 2: Orientations near a singular arc.

graph. Furthermore, let b_1, b_2, and b_3 be the number of simple, double, and marked vertical boundary edges of u, respectively. Define the partial q-gradings of u as

$$q_i(u) = \chi(u_i) - \frac{1}{2}\chi(\partial u_i \cap \partial u) - \frac{1}{2}b_i, \quad i = 1, 2, 3,$$
$$q_{s_\gamma}(u) = \chi\left(s_\gamma(u)\right) - \frac{1}{2}\chi\left(\partial s_\gamma(u)\right), \tag{3.1}$$

where χ is the Euler characteristic and ∂ denotes the boundary.

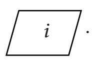
Definition 3.3. Let u be a prefoam with $d_\bullet$ dots of type $\bullet$, $d_\circ$ dots of type $\circ$, and $d_\odot$ dots of type $\odot$. The q-grading of u is given by

$$q(u) = -\sum_{i=1}^{3} i(N-i)q_i(u) - 2(N-2)q_{s_\gamma}(u) + 2d_\bullet + 4d_\circ + 6d_\odot. \tag{3.2}$$

The following result is a direct consequence of the definitions.

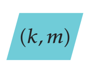
Lemma 3.4. *$q(u)$ is additive under the gluing of prefoams.*

We denote a simple facet with i dots by



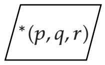
. (3.3)

Recall that the two-variable Schur polynomial $\pi_{k,m}$ can be expressed in terms of the elementary symmetric polynomials $\pi_{1,0}$ and $\pi_{1,1}$. By convention, the latter correspond to $\bullet$ and $\circ$ on a double facet, respectively, so that



(3.4)

is defined to be the linear combination of dotted double facets corresponding to the expression of $\pi_{k,m}$ in terms of $\pi_{1,0}$ and $\pi_{1,1}$. Analogously we can express the three-variable Schur polynomial $\pi_{p,q,r}$ in terms of the elementary symmetric polynomials $\pi_{1,0,0}$, $\pi_{1,1,0}$, and $\pi_{1,1,1}$. By convention, the latter correspond to $\bullet$, $\circ$, and $\odot$ on a triple facet, respectively, so we can make sense of



. (3.5)

3.2. Foams

In [11, 16] we gave a precise definition of the Kapustin-Li formula, following Khovanov and Rozansky's work [13]. We will not repeat that definition here, since it is complicated and unnecessary for our purposes in this paper. The only thing one needs to remember is that the Kapustin-Li formula associates a number to any closed prefoam and that those numbers have very special properties, some of which we will recall below. By $\langle u \rangle_{KL}$, we denote the Kapustin-Li evaluation of a closed prefoam u.

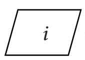
Definition 3.5. The category $\mathbf{Foam}_N$ is the quotient of the category **Pfoam** by the kernel of $\langle\,\rangle_{KL}$, that is, by the following identifications: for any webs Γ, Γ' and finite sets $f_i \in \mathrm{Hom}_{\mathbf{Pfoam}}(\Gamma, \Gamma')$ and $c_i \in \mathbb{Q}$ we impose the relations

$$\sum_i c_i f_i = 0 \Longleftrightarrow \sum_i c_i \langle \overline{f_i} \rangle_{KL} = 0, \tag{3.6}$$

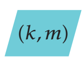
for any fixed way of closing the f_i, denoted by $\overline{f_i}$. By "fixed" we mean that all the f_i are closed in the same way. The morphisms of $\mathbf{Foam}_N$ are called *foams*.

In the next proposition we recall those relations in $\mathbf{Foam}_N$ that we need in the sequel. For their proofs and other relations we refer to [11].

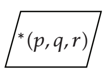
Proposition 3.6. *The following identities hold in* $\mathbf{Foam}_N$
The dot conversion relations are



$= 0$ if $i \geq N$, (3.7)

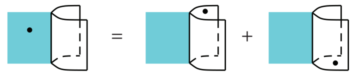


$= 0$ if $k \geq N-1$, (3.8)

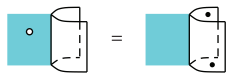


$= 0$ if $p \geq N-2$. (3.9)

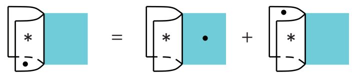
The dot migration relations are



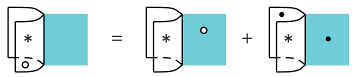
(3.10)



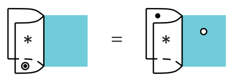
(3.11)



(3.12)

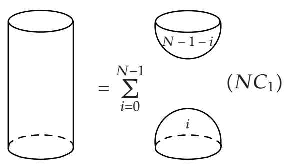


(3.13)

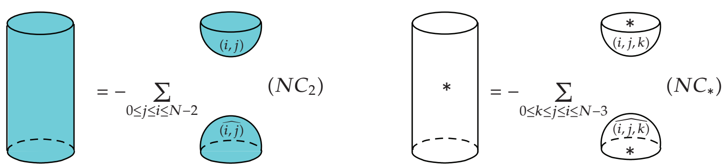


(3.14)

The neck cutting relations are (these were called cutting neck relations in [11, 16])

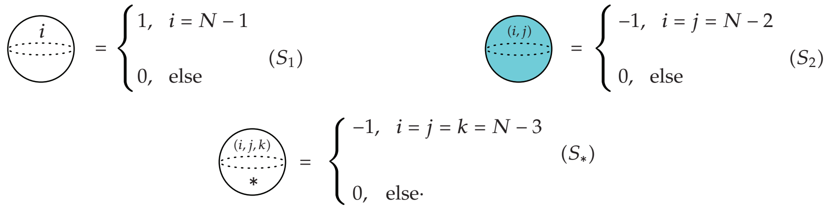


(3.15)



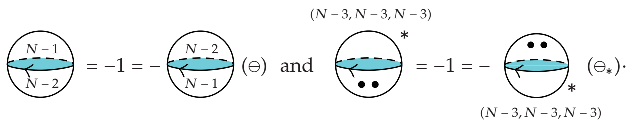
(3.16)

The sphere relations are



(3.17)

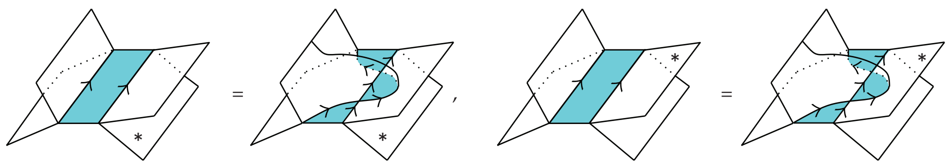
The Θ-foam relations are



(3.18)

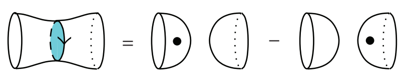
Inverting the orientation of the singular circle of $(\ominus_*)$ *inverts the sign of the corresponding foam. A theta-foam with dots on the double facet can be transformed into a theta-foam with dots only on the other two facets, using the dot migration relations.*

The Matveev-Piergalini relation is

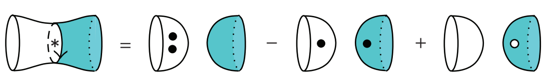


(MP)

The disc removal relations are

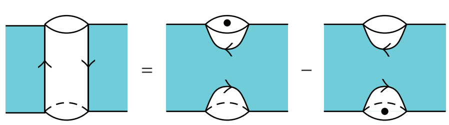


(RD_1)

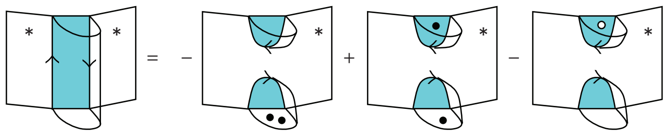


(RD_2)

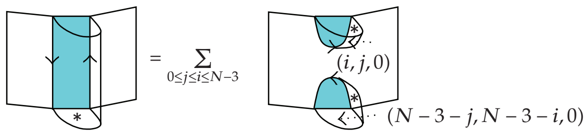
The digon removal relations are



(DR_1)

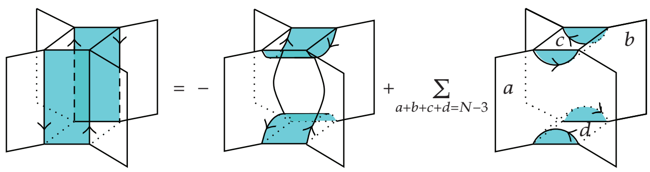


(DR_{3_1})

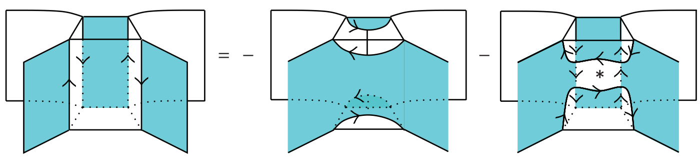


(DR_{3_2})

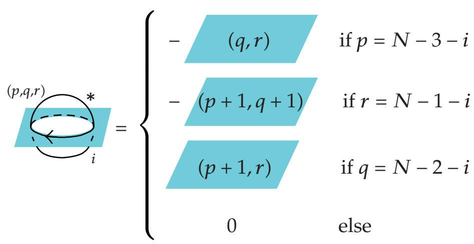
The square removal relations are



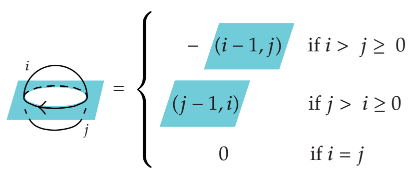
(SqR_1)



(SqR_2)



(3.19)

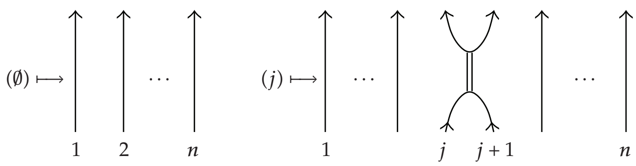


(3.20)

4. The Functors $\mathcal{F}_{N,n}$

Let $n \geq 1$ and $N \geq 4$ be arbitrary but fixed. In this section we define a monoidal functor $\mathcal{F}_{N,n}$ between the categories $\mathcal{SC}_{1,n}$ and $\mathbf{Foam}_N$.

On Objects. $\mathcal{F}_{N,n}$ sends the empty sequence to 1_n and the one-term sequence (j) to w_j:

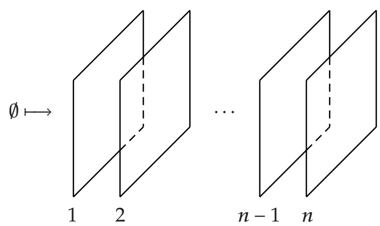


(4.1)

with $\mathcal{F}_{N,n}(jk)$ given by the vertical composite $w_j w_k$.

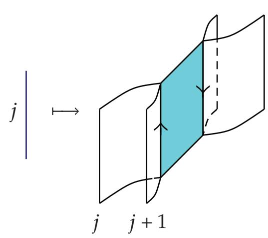
On Morphisms.

(i) The empty diagram is sent to n parallel vertical sheets:



(4.2)

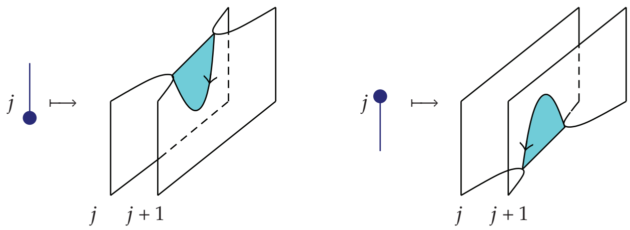
(ii) The vertical line coloured j is sent to the identity cobordism of w_j:



(4.3)

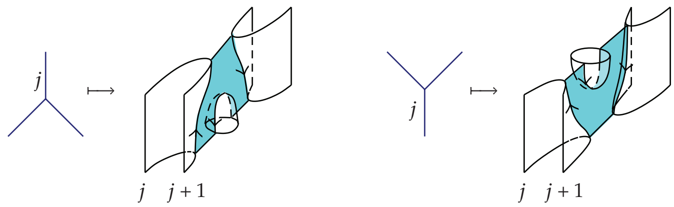
The remaining $n-2$ vertical parallel sheets on the r.h.s. are not shown for simplicity, a convention that we will follow from now on.

(iii) The *StartDot* and *EndDot* morphisms are sent to the zip and the unzip, respectively:



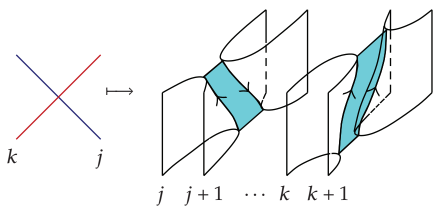
(4.4)

(iv) *Merge* and *Split* are sent to cup and cap cobordisms:



(4.5)

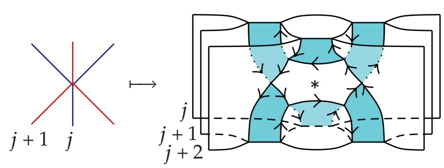
(v) The *4-valent vertex* with distant colors. For $j + 1 < k$ we have



(4.6)

The case $j > k + 1$ is given by reflexion around a horizontal plane.

(vi) For the *6-valent vertices* we have



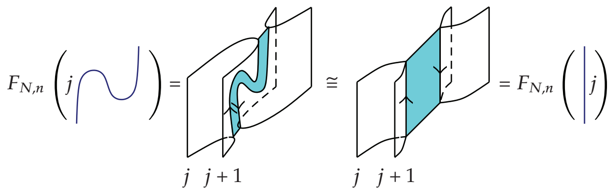
(4.7)

The case with the colors switched is given by reflection in a vertical plane. Notice that $\mathcal{F}_{N,n}$ respects the gradings of the morphisms.

Proposition 4.1. *$\mathcal{F}_{N,n}$ is a monoidal functor.*

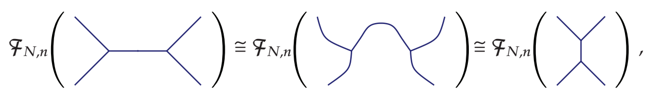
Proof. The assignment given by $\mathcal{F}_{N,n}$ clearly respects the monoidal structures of $\mathcal{SC}_{1,n}$ and $\mathbf{Foam}_N$. So we only need to show that $\mathcal{F}_{N,n}$ is a functor, that is, it respects the relations (2.5) to (2.22) of Section 2.

"*Isotopy Relations*". Relations (2.5) to (2.9) are straightforward to check and correspond to isotopies of their images under $\mathcal{F}_{N,n}$. For the sake of completeness we show the first equality in (2.5). We have



(4.8)

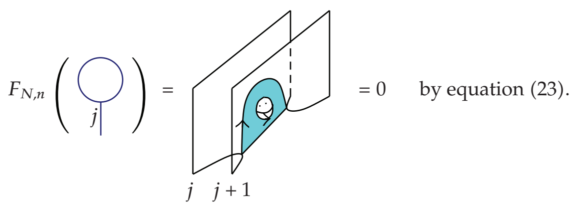
One Color Relations. For relation (2.10) we have



(4.9)

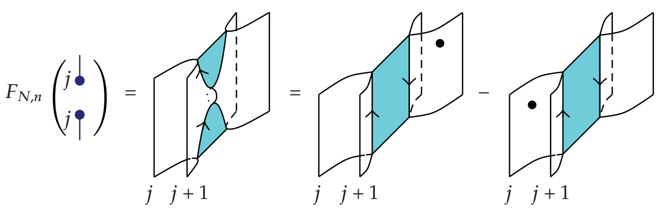
where the first equivalence follows from relations (2.5) and (2.7) and the second from isotopy of the foams involved.

For relation (2.11) we have



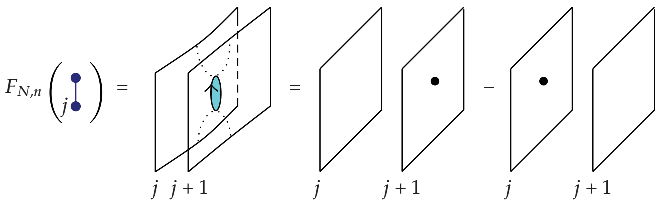
(4.10)

Relation (2.12) requires some more work. We have



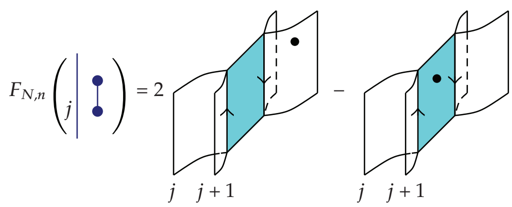
(4.11)

where the second equality follows from the (DR_1) relation. We also have

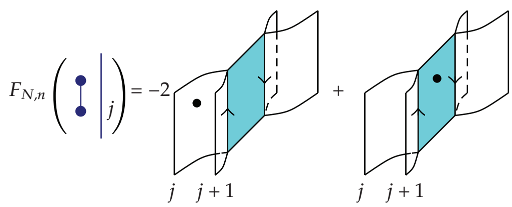


(4.12)

Using (3.12), we obtain

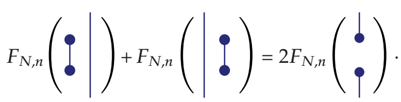


(4.13)



(4.14)

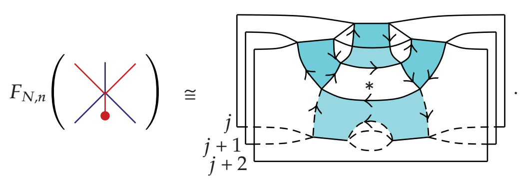
and, therefore, we have that



(4.15)

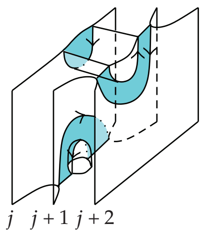
Two Distant Colors. Relations (2.13) to (2.15) correspond to isotopies of the foams involved and are straightforward to check.

Adjacent Colors. We prove the case where "blue" corresponds to j and "red" corresponds to $j+1$. The relations with colors reversed are proved the same way. To prove relation (2.16) we first notice that using the (MP) move we get



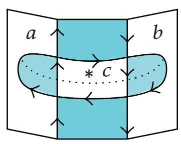
(4.16)

Apply (SqR_1) to the simple-double square tube perpendicular to the triple facet to obtain two terms. The first term contains a double-triple digon tube which is the left-hand side of the (DR_{3_2}) relation rotated by 180^0 around a vertical axis. Next apply the (DR_{3_2}) relation and use (MP) to remove the four singular vertices, which results in simple-triple bubbles (with dots) in the double facets. Using (3.19) to remove these bubbles gives



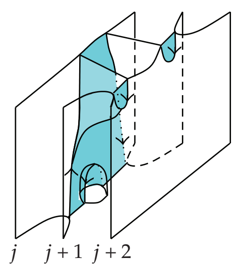
(4.17)

which is $\mathcal{F}_{N,n}$ . The second term contains

$$\sum_{a+b+c+d=N-3}$$ 

(4.18)

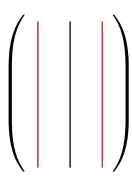
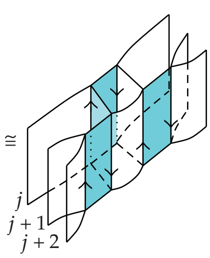
behind a simple facet with d dots (notice that all dots are on simple facets). Using the (MP) relation to get a simple-triple bubble in the double facet, followed by (RD_2) and (S_1) we obtain



(4.19)

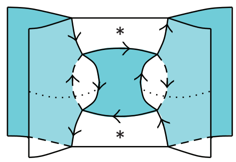
which equals $\mathcal{F}_{N,n}$ .

We now prove relation (2.17). We have an isotopy equivalence

$F_{N,n}$  $\cong$ 

(4.20)

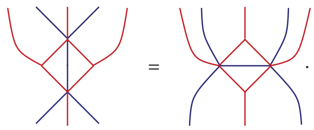
Notice that $\mathcal{F}_{N,n}$ is the l.h.s. of the (SqR_2) relation. The first term on the r.h.s. of (SqR_2) is isotopic to $-\mathcal{F}_{N,n}$. For the second term on the r.h.s. of (SqR_2) we notice that $\mathcal{F}_{N,n}$ contains



(4.21)

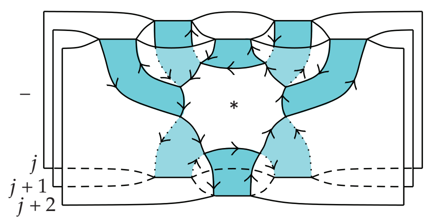
Applying (DR_{3_1}) followed by (MP) to remove the singular vertices creating simple-simple bubbles on the two double facets and using (3.20) to remove these bubbles, we conclude that $\mathcal{F}_{N,n}$ is the second term on the r.h.s. of (SqR_2).

We now prove relation (2.18) in the form



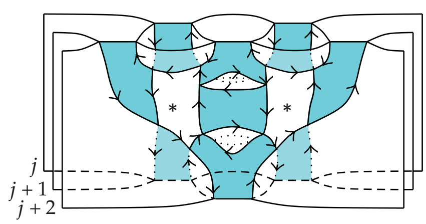
(4.22)

The image of the l.h.s. also contains a bit like the one in (4.21). Simplifying it like we did in the proof of (2.17), we obtain that $\mathcal{F}_{N,n}$ reduces to



(4.23)

For the r.h.s. we have



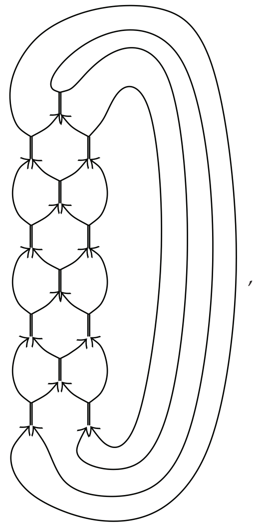
(4.24)

Using (DR_{3_1}) on the vertical digon, followed by (MP) and the Bubble relation (3.20), we obtain (4.23).

Relation (2.19) follows from straightforward computation and is left to the reader.

Relations Involving Three Colors. Relations (2.20) and (2.21) follow from isotopies of the foams involved. To show that $\mathcal{F}_{N,n}$ respects relation (2.22), we use a different type of argument. First of all, we note that the images under $\mathcal{F}_{N,n}$ of both sides of relation (2.22) are multiples of each other, because the graded vector space of morphisms in $\mathbf{Foam}_N$

between the bottom and top webs has dimension one in degree zero. Verifying this only requires computing the coefficient of $q^{-(4N-4)}$ (this includes the necessary shift!) in the MOY polynomial associated to the web



, (4.25)

which is a standard calculation left to the reader. To see that the multiplicity coefficient is equal to one, we close both sides of relation (2.22) simply by putting a dot on each open end. Using relations (2.14) and (2.16) to reduce these closed diagrams, we see that both sides give the same nonzero sum of disjoint unions of coloured *StartDot-EndDot* diagrams. Note that we have already proved that $\mathcal{F}_{N,n}$ respects relations (2.14) and (2.16). By applying foam relation (4.12) to the images of all nonzero terms in the sum, we obtain a nonzero sum of dotted sheets. This implies that both sides of (2.22) have the same image under $\mathcal{F}_{N,n}$. □

We have now proved that $\mathcal{F}_{N,n}$ is a monoidal functor for all $N \geq 4$. Our main result about the whole family of these functors, that is, for all $N \geq 4$ together, is the proposition below. It implies that all the defining relations in Soergel's category can be obtained from the corresponding relations between $sl(N)$ foams, when all $N \geq 4$ are considered, and that there are no other independent relations in Soergel's category corresponding to relations between foams.

Proposition 4.2. *Let $\underline{i}$, $\underline{j}$ be two arbitrary objects in $\mathcal{SC}_{1,n}$ and let $f \in Hom(\underline{i}, \underline{j})$ be arbitrary. If $\mathcal{F}_{N,n}(f) = 0$ for all $N \geq 4$, then $f = 0$.*

Proof. Let us first suppose that $\underline{\mathbf{i}} = \underline{\mathbf{j}} = \emptyset$. Suppose also that f has degree $2d$ and that $N \geq \max\{4, d+1\}$. Recall that, as shown in [5, Corollary 3], we know that $\mathrm{Hom}(\emptyset, \emptyset)$ is the free commutative polynomial ring generated by the StartDot-EndDots of all possible colors. So f is a polynomial in StartDot-EndDots, and therefore a sum of monomials. Let m be one of these monomials, no matter which one, and let m_j denote the power of the StartDot-EndDot with color j in m. Close $\mathcal{F}_{N,n}(f)$ by gluing disjoint discs to the boundaries of all open simple facets (i.e., the vertical ones with corners in the pictures). For each color j, put $N-1-m_j$ dots on the left simple open facet corresponding to j and also put $N-1$ dots on the rightmost simple open facet. Note that, after applying (RD_1), we get a linear combination of dotted simple spheres.

Only one term survives and is equal to ±1, because only in that term each sphere has exactly $N-1$ dots. This shows that $\mathcal{F}_{N,n}(f) \neq 0$, because it admits a nonzero closure.

Now let us suppose that $\underline{\mathbf{i}} = \emptyset$ and $\underline{\mathbf{j}}$ is arbitrary. By [4, Corollaries 4.11 and 4.12], we know that $\mathrm{Hom}(\emptyset, \underline{\mathbf{j}})$ is the free $\mathrm{Hom}(\emptyset, \emptyset)$-module of rank one generated by the disjoint union of StartDots coloured by $\underline{\mathbf{j}}$. Closing off the StartDots by putting dots on all open ends gives an element of $\mathrm{Hom}(\emptyset, \emptyset)$, whose image under $\mathcal{F}_{N,n}$ is nonzero for N big enough by the above. This shows that the generator of $\mathrm{Hom}(\emptyset, \underline{\mathbf{j}})$ has nonzero image under $\mathcal{F}_{N,n}$ for N big enough, because $\mathcal{F}_{N,n}$ is a functor.

Finally, the general case, for $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ arbitrary, can be reduced to the previous case by [4, Corollary 4.12]. □

Acknowledgments

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