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Research Article

A Class of Two-Person Zero-Sum Matrix Games with Rough Payoffs

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We concentrate on discussing a class of two-person zero-sum games with rough payoffs. Based on the expected value operator and the trust measure of rough variables, the expected equilibrium strategy and r-trust maximin equilibrium strategy are defined. Five cases whether the game exists r-trust maximin equilibrium strategy are discussed, and the technique of genetic algorithm is applied to find the equilibrium strategies. Finally, a numerical example is provided to illustrate the practicality and effectiveness of the proposed technique.

1. Introduction

Game theory is widely applied in many fields, such as, economic and management problems, social policy, and international and national politics since it is proposed by von Neumann and Morgenstern [1]. Peski [2] presented the simple necessary and sufficient conditions for the comparison of information structures in zero-sum games and solved an problem, which is how to find the value of information in zero-sum games. Owena and McCormick [3] analyzed a "manhunting" game involving a mobile hider and consider a deductive search game involving a fugitive, then developed a model based on a base-line model. The traditional game theory assumes that all data of a game are known exactly by players. However, there are some games in which players are not able to evaluate exactly some data in our realistic situations. In these games, the imprecision is due to inaccuracy of information and vague comprehension of situations by players. For these uncertain games, many scholars have made contribution and got some techniques to find the equilibrium strategies of these games. Some scholars, such as, Berg and Engel [4], Ein-Dor and Kanter [5], Takahashi [6], discussed a two-person zero-sum matrix game with random payoffs. Xu [7] made use of linear programming method to discuss two-person zero-sum game with grey number payoff matrix. Harsanyi [8] made a great contribution in treating the imprecision of probabilistic nature in games by developing the theory of Bayesian games. Dhingra et al. [9] combined the cooperative game theory with fuzzy set theory to yield a new optimization method to herein as cooperative fuzzy games and proposed a computational technique to solve the multiple objective optimization problems. Then Espin et al. [10] proposed an innovative fuzzy logic approach to analyze n-person cooperative games and theoretically and experimentally examined the results by analyzing three-case studies.

Although many cooperative and noncooperative games with uncertain payoffs are researched much by many scholars, there is still a kind of games with uncertain payoffs to be discussed little, that is, games with rough payoffs. Since rough set theory is proposed and studied by Pawlak [11, 12], it is drastic to be applied into many fields, such as, data mining and neural network. Nurmi et al. [13] introduced three uncertainty events in social choice such as the impreciseness of a probabilistic, fuzzy, and rough type, further explored difficult issues of how diverse types of impreciseness can be combined, and in particular the combination of roughness with randomness and fuzziness in voting games. Liu [14] proposed a new concept of rough variable which is a measurable function from rough space to R. Based on the concept of rough variable, a game with rough payoffs is studied in this paper.

In game theory, it is an important task to define the concepts of equilibrium strategies and investigate their properties. However, in these games with uncertain payoffs, there are no concepts of equilibrium strategies to be accepted widely. Campos [15] has proposed several methods to solve fuzzy matrix games based on linear programming but has not defined explicit concepts of equilibrium strategies. As the extension of the idea of Campos [15], Nishizaki and Sakawa [16] discussed multiobjective matrix game with fuzzy payoffs. Maeda [17] has defined Nash equilibrium strategies based on possibility and necessity measures and investigated its properties.

In this paper, based on the concept of rough variable proposed by Liu [14], we discuss a simplest game, namely, the game in which the number of players is two and rough payoffs which one player receives are equal to rough payoffs which the other player loses. We defined two kinds of concepts of maximin equilibrium strategies and investigate their properties. The rest of this paper is organized as follows. In Section 2, we recalls some definitions and properties about two-person zero-sum game and the rough variable. Then two concepts of equilibrium strategies of two-person zero-sum game with rough payoffs are introduced and then their properties are deduced in Section 3. In Section 4, we proposed the technique of GA to solve some complicated game problems with rough payoffs which can be converted into crisp programming problem. Then a numerical example is discussed to show the effectiveness of the prosed theory and algorithm in Section 5. Finally, the conclusion has been made in Section 6.

2. Basic Concepts of Two-Person Zero-Sum Game and Rough Variable

In this section, let us recall the basic definitions of the two-person zero-sum game in [18]. The concept and properties of rough variable proposed by Liu [14] is also reviewed.

2.1. Two-Person Zero-Sum Game

In the game theory, the decision makers realize sufficiently the affection of their actions to others. The two-person zero-sum game is the simplest case of game theory in which how much one player receives is equal to how much the other loses. When we assume that both players give pure, mixed strategies (see Parthasarathy and Raghavan [19]), such a game has been well resolved. But in our realistic world, there are also some noncooperative cases though more cooperation may exist in games. In reality, the non-cooperation between players may be vague. This paper mainly deals with the kind of games with rough payoffs.

In the two-person zero-sum game, what one player receives is equal to how much the other loses which could be illustrated by the following $m \times n$ matrix:

$$P = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \tag{2.1}$$

where P denotes the payoff matrix of player I, x_{ij} is the payoff of player I when player I proposes the strategy i, and player II proposes the strategy j. Then, the payoff matrix of player II is $-P$.

Definition 2.1. A vector x in $\Re^m$ is said to be a mixed strategy of player I if it satisfies the following condition:

$$x^T e_m = 1, \tag{2.2}$$

where the components of $x = (x_1, x_2, \ldots, x_m)^T$ are greater than or equal to 0; e_m is an $m \times 1$ vector, whose every component is equal to 1. The mixed strategy of the player II is defined similarly. Particularly, a strategy $s_k = (0, 0, \ldots, 1, \ldots, 0)$ is called a pure strategy of player I. Thereinto, the kth component of s_k is only equal to 1, the other components are equal to 0.

Definition 2.2. If the mixed strategies x and y are proposed by players I and II, respectively, then the expected payoff of player I is defined by

$$x^T P y = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j. \tag{2.3}$$

According to Definition 2.2, we have the definition of optimal strategies of players.

Definition 2.3. In one two-person zero-sum game, player I's mixed strategy x^* and player II's mixed strategy y^* are said to be optimal strategies if $x^T P y^* \leq x^{*T} P y^*$ and $x^{*T} P y^* \leq x^{*T} P y$ for any mixed strategies x and y.

2.2. Rough Variable

Since Pawlak [11] initialized the rough set theory, it has been well developed and applied in a wide variety of uncertainty surrounding real problems.

Definition 2.4 (Slowinski et al. [20]). Let U be a universe and X a set representing a concept. Then its lower approximation is defined by

$$\underline{X} = \left\{ x \in U \mid R^{-1}(x) \subset X \right\}, \tag{2.4}$$

and the upper approximation is defined by

$$\overline{X} = \bigcup_{x \in X} R(x), \tag{2.5}$$

where R is the similarity relationship on U. Obviously, we have $\underline{X} \subseteq X \subseteq \overline{X}$.

Definition 2.5 (Pawlak [11]). The collection of all sets having the same lower and upper approximations is called a rough set, denoted by $(\underline{X}, \overline{X})$. Its boundary is defined as follows:

$$Bn_R(X) = \overline{X} - \underline{X}. \tag{2.6}$$

Liu [14] also gave a new concept about rough variable. This paper mainly refers to this book. The following results will be used extensively.

Definition 2.6. Let Λ be a nonempty set, $\mathcal{A}$ a σ-algebra of subsets of Λ, Δ an element in $\mathcal{A}$, and π a nonnegative, real-valued, additive set function. Then $(\Lambda, \Delta, \mathcal{A}, \pi)$ is called a rough space.

Definition 2.7. A rough variable ξ on the rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$ is a function from Λ to the real line $\mathfrak{R}$ such that for every Borel set O of $\mathfrak{R}$, we have

$$\{\lambda \in \Lambda \mid \xi(\lambda) \in O\} \in \mathcal{A}. \tag{2.7}$$

The lower and the upper approximations of the rough variable ξ are then defined as follows:

$$\underline{\xi} = \{\xi(\lambda) \mid \lambda \in \Delta\}, \qquad \overline{\xi} = \{\xi(\lambda) \mid \lambda \in \Lambda\}. \tag{2.8}$$

Liu [14] also defined the trust measure of event A by $\mathrm{Tr}\{A\} = (1/2)(\mathrm{T\overline{r}}\{A\} + \mathrm{T\underline{r}}\{A\})$, where $\mathrm{T\overline{r}}\{A\}$ denotes the upper trust measure of event A, defined by $\mathrm{T\overline{r}}\{A\} = \pi\{A\}/\pi\{\Lambda\}$, and $\mathrm{T\underline{r}}\{A\}$ denotes the lower trust measure of event A, defined by $\mathrm{T\underline{r}}\{A\}) = \pi\{A \cap \Delta\}/\pi\{\Delta\}$.

When we do not have information enough to determine the measure π for a real-life problem, we can assumes that all elements in Λ are equally likely to occur. For this case, the measure π may be viewed as the Lebesgue measure. In this paper, we only consider the rough variable $\xi = ([a,b],[c,d])$ such $\xi(x) = x$ for all $x \in \Lambda$, where $c \le a < b \le d$.

Definition 2.8. Let ξ be a rough variable on the rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$. The expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Tr}\{\xi \geq r\} dr - \int_{-\infty}^0 \mathrm{Tr}\{\xi \leq r\} dr. \tag{2.9}$$

Remark 2.9. Let $\xi = ([a,b],[c,d])$ be a rough variable with $c \leq a \leq b \leq d$. Then we have

$$E[\xi] = \frac{1}{4}(a+b+c+d). \tag{2.10}$$

Remark 2.10. Assume that ξ and η are both variables with finite expected values. Then for any real numbers a and b, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{2.11}$$

3. Two Kinds of Equilibrium Strategies of Two-Person Zero-Sum Game with Rough Payoffs

Let consider the following example before defining the two-person zero-sum game with rough payoffs. When playing a Chinese poker, there are two teams which are constructed by two persons. Without loss of generality, we assume that Team A is the dealer, then its rule is as follows.

(1) If the score Team B gets is less than 40, Team A goes on being a dealer and rises of one grade, denoted as +1.

(2) If the score Team B gets is between 40 and 80, Team B becomes the dealer, denoted as 0.

(3) If the score Team B gets is more than 80, Team B becomes the dealer and rises of one grade, denoted as −1.

From the description, we know that the rule has determined a kind of classification which is regard as an equivalent relation by Pawlak [12] on the universe $[0,100]$. This means that obtaining 45 or 75 expresses the same meaning, and they are equivalent or indiscernible. Thus, the rough variable $\xi = ([40,80],[0,100])$ is applied to describe the above process and its trust measure expresses the probability that Team A obtains +1, or 0, or −1 in every game. In the following part, we will only consider the rough variable which is combined by the payoff.

Let the rough variable ξ_{ij} represent the payoff that the player I receives or player II loses, then a rough payoff matrix is presented as follows to denote a two-person zero-sum game:

$$P = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{bmatrix}. \tag{3.1}$$

When player I and player II, respectively, choose the mixed strategies x and y, the rough payoffs of player I are

$$x^T P y = \sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_j. \tag{3.2}$$

3.1. Basic Definition of Two Kinds of Equilibrium Strategies

Because of the vagueness of rough payoffs, it is difficult for players to choose the optimal strategy. Naturally, we consider how to maximize players' or minimize the opponent's rough expected payoffs. Based on this idea, we propose the following maximin equilibrium strategy.

Definition 3.1. Let rough variable ξ_{ij} $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ represent the payoffs that the player receives or player II loses when player I gives the strategy i and player II gives the strategy j. Then (x^*, y^*) is called a rough expected maximin equilibrium strategy if

$$E\left[x^T P y^*\right] \leq E\left[x^{*T} P y^*\right] \leq E\left[x^{*T} P y\right], \tag{3.3}$$

where P is defined by (3.1).

Remark 3.2. Since the rough variables ξ are independent, then for any mixed strategies x and y, according to Remark 2.10, we have that

$$E\left[x^T P y\right] = E\left[\sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_j\right] = \sum_{j=1}^{n} \sum_{i=1}^{m} E\left[\xi_{ij}\right] x_i y_j. \tag{3.4}$$

According to the definition of trust measure of rough variable, we can get another way to convert the rough variable into a crisp number. Then we propose another definition of Nash equilibrium to this game.

Definition 3.3. Let rough variable ξ_{ij} $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ represent the payoffs that the player receives or player II loses when player I gives the pure strategy i and player II gives the pure strategy j. r is the predetermined level of the payoffs, $r \in R$. Then (x^*, y^*) is called the r-trust maximin equilibrium strategy if

$$\mathrm{Tr}\left\{x^T P y^* \geq r\right\} \leq \mathrm{Tr}\left\{x^{*T} P y^* \geq r\right\} \leq \mathrm{Tr}\left\{x^{*T} P y \geq r\right\}, \tag{3.5}$$

where P is defined by (3.1).

3.2. The Existence of Two Kinds of Equilibrium Strategies

In the following part, we will introduce the equilibrium strategy under the expected operator and the trust measure, respectively.

3.2.1. The Existence of Expected Maximin Equilibrium Strategies

When the players' payoffs are crisp numbers, we know that the game surely has a mixed Nash equilibrium point. Then we will discuss if there is an expected maximin equilibrium strategy when the payoffs ξ_{ij} are characterized as rough variables.

Lemma 3.4. *Let ξ_{ij} $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be rough variables with finite expected values. Then strategy (x^*, y^*) is an expected maximin equilibrium strategy to the game if and only if for every pure strategy s_k $(k = 1,2,\dots,m)$ of player I and s_t $(t = 1,2,\dots,n)$ of player II, one has*

$$E\left[s_k^T P y^*\right] \le E\left[x^{*T} P y^*\right] \le E\left[x^{*T} P s_t\right], \tag{3.6}$$

where P is defined by (3.1).

Proof. The necessity is apparent. Now we only consider the sufficiency. According to (3.6), for every $k = 1,2,\dots,m$,

$$E\left[s_k^T P y^*\right] \le E\left[x^{*T} P y^*\right]. \tag{3.7}$$

Suppose that $x = (x_1, x_2, \dots, x_m)$ is any one mixed strategy of player I. In (3.7), for every k, we multiply x_k to every inequality. Then

$$\begin{aligned}
&E\left[s_k^T P y^*\right] x_k \le E\left[x^{*T} P y^*\right] x_k \\
&\Longrightarrow \sum_{k=1}^{m} E\left[s_k^T P y^*\right] x_k \le \sum_{k=1}^{m} E\left[x^{*T} P y^*\right] x_k \\
&\Longrightarrow E\left[x^T P y^*\right] \le E\left[x^{*T} P y^*\right] \sum_{k=1}^{m} x_k \\
&\Longrightarrow E\left[x^T P y^*\right] \le E\left[x^{*T} P y^*\right].
\end{aligned} \tag{3.8}$$

Similarly, we can prove

$$E\left[x^{*T} P y^*\right] \le E\left[x^{*T} P y\right]. \tag{3.9}$$

Thus, the strategy (x^*, y^*) is an expected maximin equilibrium strategy to the game. This completes the proof. □

Theorem 3.5. *In a two-person zero-sum game, rough variables ξ_{ij} $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ represent the payoffs player I receives or player II loses, and the payoff matrix P is defined by* (3.1). *Then there at least exists an expected maximin equilibrium strategy to the game.*

Proof. Suppose that $x = (x_1, x_2, \ldots, x_m)$ is any one mixed strategy of player I. For every pure strategy s_k $(k = 1, 2, \ldots, m)$ and any strategy y, we define

$$\rho_k(x) = \max\left\{0, E\left[s_k^T P y\right] - E\left[x^T P y\right]\right\}. \tag{3.10}$$

For every x_k $(k = 1, 2, \ldots, m)$, we define

$$f_k = \frac{x_k + \rho_k(x)}{1 + \sum_{k=1}^{m} \rho_k(x)}. \tag{3.11}$$

Obviously, $f_k \geq 0$ $(k = 1, 2, \ldots, m)$, $\sum_{k=1}^{m} f_k = 1$. Thus, $f = (f_1, f_2, \ldots, f_m)$ is a mixed strategy of player I and is a continous function about x. According to Brouwer's fixed-point theorem, there exists a point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)$ satisfying

$$x_k^* = \frac{x_k^* + \rho_k(x^*)}{1 + \sum_{k=1}^{m} \rho_k(x^*)}, \quad k = 1, 2, \ldots, m. \tag{3.12}$$

Now, we will prove $E[s_k^T P y] \leq E[x^{*T} P y]$. For any mixed strategy $x = (x_1, x_2, \ldots, x_m)$, let $E[s_h^T P y] = \min_{k \in J} E[s_k^T P y]$, $(J = \{1, 2, \ldots, m\})$. Then

$$E\left[s_h^T P y\right] = E\left[s_h^T P y\right] \sum_{k=1}^{m} x_k \leq \sum_{k=1}^{m} x_k E\left[s_k^T P y\right] = E\left[x^T P y\right]. \tag{3.13}$$

Namely, there exists a pure strategy s_h satisfying $E[s_h^T P y] \leq E[x^T P y]$. Thus, for $x^* = (x_1^*, x_2^*, \ldots, x_m^*)$, there exists a pure strategy s_h such that $E[s_h^T P y] \leq E[x^{*T} P y]$. According to (3.10), we have $\rho_h(x^*) = 0$.

For the strategy x_h^* $(x_h^* > 0)$ of player I, according to (3.12), we have

$$\begin{aligned} x_h^* &= \frac{x_h^* + \rho_h(x^*)}{1 + \sum_{k=1}^{m} \rho_k(x^*)} = \frac{x_h^*}{1 + \sum_{k=1}^{m} \rho_k(x^*)} \\ &\Longrightarrow \sum_{k=1}^{m} \rho_k(x^*) = 0. \end{aligned} \tag{3.14}$$

By the definition of $\rho_k(x^*)$, we know that $\rho_k(x^*)$ is nonegative. Therefore, for every $k = 1, 2, \ldots, m$, $\rho_k(x^*) = 0$. Then

$$\begin{aligned} &E\left[s_k^T P y\right] - E\left[x^{*T} P y\right] \leq 0, \\ &\Longrightarrow E\left[s_k^T P y\right] \leq E\left[x^{*T} P y\right], \quad k = 1, 2, \ldots, m. \end{aligned} \tag{3.15}$$

Similarly, we can prove $E[x^{*T}Py^*] \leq E[x^{*T}Ps_t]$ for every pure strategy s_t $(t = 1,2,\ldots,n)$ of player II. By Lemma 3.4, we know that (x^*, y^*) is an expected maximin equilibrium strategy to the game. This completes the proof. □

3.3. The Existence of r-Trust Maximin Equilibrium Strategies

Through the proof of Theorem 3.5, we know that there at least exists an expected maximin equilibrium strategy to any two-person zero-sum game with rough payoffs. Now we will discuss the existence of r-trust maximin equilibrium strategy to this kind of game.

Lemma 3.6. *Let rough variable ξ_{ij} $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,n)$ represent the payoffs that the player receives or player II loses when player I gives the strategy i and player II gives the strategy j. Suppose that r is a given number and rough variable $\xi_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ $(c_{ij} \leq b_{ij} \leq a_{ij} \leq d_{ij})$, then one has*

$$\mathrm{Tr}\left\{x^T P y \geq r\right\} = \begin{cases} 0 & \text{if } x^T D y \leq r, \\ \dfrac{x^T D y - r}{2(x^T D y - x^T C y)} & \text{if } x^T B y \leq r \leq x^T D y, \\ \dfrac{1}{2}\left(\dfrac{x^T D y - r}{x^T D y - x^T C y} + \dfrac{x^T B y - r}{x^T B y - x^T A y}\right) & \text{if } x^T A y \leq r \leq x^T B y, \\ \dfrac{1}{2}\left(\dfrac{x^T D y - r}{x^T D y - x^T C y} + 1\right) & \text{if } x^T C y \leq r \leq x^T A y, \\ 1 & \text{if } r \leq x^T C y, \end{cases} \tag{3.16}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$
$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix}. \tag{3.17}$$

Proof. According to [14], we have

$$
\begin{aligned}
x^T P y &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \xi_{ij} y_j \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\left[x_i y_j a_{ij}, x_i y_j b_{ij}\right], \left[x_i y_j c_{ij}, x_i y_j d_{ij}\right] \right) \\
&= \left(\left[\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j a_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j b_{ij} \right], \left[\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j c_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j d_{ij} \right] \right).
\end{aligned}
\tag{3.18}
$$

Suppose that

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}, \\
C &= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \qquad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix},
\end{aligned}
\tag{3.19}
$$

then

$$
x^T P y = \left(\left[x^T A y, x^T B y\right], \left[x^T C y, x^T D y\right] \right).
\tag{3.20}
$$

thus we have

$$
\operatorname{Tr}\left\{ x^T P y \geq r \right\} = \begin{cases} 0 & \text{if } x^T D y \leq r, \\ \dfrac{x^T D y - r}{2\left(x^T D y - x^T C y\right)} & \text{if } x^T B y \leq r \leq x^T D y, \\ \dfrac{1}{2}\left(\dfrac{x^T D y - r}{x^T D y - x^T C y} + \dfrac{x^T B y - r}{x^T B y - x^T A y} \right) & \text{if } x^T A y \leq r \leq x^T B y, \\ \dfrac{1}{2}\left(\dfrac{x^T D y - r}{x^T D y - x^T C y} + 1 \right) & \text{if } x^T C y \leq r \leq x^T A y, \\ 1 & \text{if } r \leq x^T C y. \end{cases}
\tag{3.21}
$$

This completes the proof. □

Because of $c_{ij} \le b_{ij} \le a_{ij} \le d_{ij}$, we can easily have $x^T C y \le x^T A y \le x^T B y \le x^T D y$. Then let us discuss the existence of r-trust maximin equilibrium strategy and consider two simple cases firstly.

Theorem 3.7. *If $r < \min\{c_{ij}\}$ for all $i = 1, 2 \ldots, m$, $j = 1, 2, \ldots, n$, then all strategies (x, y) are r-trust maximin equilibrium strategies.*

Proof. Suppose $l = \min\{c_{ij}\}$, then

$$x^T C y = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i y_j \ge \sum_{i=1}^{m} \sum_{j=1}^{n} l x_i y_j = l. \tag{3.22}$$

Because $r < \min\{c_{ij}\} = l$, then all strategies (x, y) satisfy $x^T C y \ge r$, according to Lemma 3.6, we have

$$\operatorname{Tr}\left\{x^T P y \ge r\right\} = 1 \quad \forall (x, y). \tag{3.23}$$

We choose any strategy (x^*, y^*), for all strategy (x, y),

$$\operatorname{Tr}\left\{x^T P y^* \ge r\right\} = \operatorname{Tr}\left\{x^{*T} P y^* \ge r\right\} = \operatorname{Tr}\left\{x^{*T} P y \ge r\right\} = 1. \tag{3.24}$$

Thus, all strategies (x, y) are r-trust maximin equilibrium strategies. This completes the proof. □

Theorem 3.8. *If $r > \max\{d_{ij}\}$ for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, then all strategies (x, y) are r-trust maximin equilibrium strategies.*

Proof. The proof is similar with that of Theorem 3.7. □

After discussing two particular cases, let us consider the usual case if there exists r-trust maximin equilibrium strategy (x, y).

Theorem 3.9. *In a two-person zero-sum game, rough variables ξ_{ij} $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ represent the payoffs player I receives or player II loses, and the payoff matrix P is defined by* (3.1). *For a predetermined number r, if for all (x, y), they cannot satisfy anyone of the following conditions: (1) $x^T D y \le r$, (2) $x^T B y \le r \le x^T D y$, (3) $x^T A y \le r \le x^T B y$, (4) $x^T C y \le r \le x^T A y$, (5) $r \le x^T C y$, then there does not exist one r-trust maximin equilibrium strategy.*

Proof. Let us only discuss one of five cases, the others are considered similarly. Suppose $S = \{(x, y) \mid x^T B y \ge r \ge x^T A y,\ \sum_{i=1}^{m} x_i = 1,\ \sum_{j=1}^{n} y_j = 1,\ 0 \le x_i, y_j \le 1\}$ and $Q = \{(x, y) \mid x^T A y \ge r \ge x^T C y,\ \sum_{i=1}^{m} x_i = 1,\ \sum_{j=1}^{n} y_j = 1,\ 0 \le x_i, y_j \le 1\}$. If not all $(x, y) \in S$, then without loss of generality we can suppose other $(x, y) \in Q$. If there exists a r-trust maximin equilibrium strategy (x^*, y^*) in S, according to Lemma 3.6, we have

$$\operatorname{Tr}\left\{x^{*T} P y^* \ge r\right\} = \frac{1}{2}\left(\frac{x^{*T} D y^* - r}{x^{*T} D y^* - x^{*T} C y^*} + \frac{x^{*T} B y^* - r}{x^{*T} B y^* - x^{*T} A y^*}\right). \tag{3.25}$$

Since $Q \neq \Phi$, then for the strategy y^*, there exists strategy $(x, y^*) \in Q$ such that $x^T C y^* < r < x^T A y^*$, then according to Lemma 3.6, we have

$$\mathrm{Tr}\left\{x^T P y^* \geq r\right\} = \frac{1}{2}\left(\frac{x^T D y^* - r}{x^T D y^* - x^T C y^*} + 1\right). \tag{3.26}$$

It is apparent that $\mathrm{Tr}\{x^{*T} P y^* \geq r\} \leq \mathrm{Tr}\{x^T C y^* > r\}$. Namely, $\mathrm{Tr}\{x^{*T} P y^* \geq r\} \ngtr \mathrm{Tr}\{x^T C y^* > r\}$. This is in conflict with the definition of r-trust maximin equilibrium strategy.

Similarly, if $(x^*, y^*) \in Q$, according to Lemma 3.6, we have

$$\mathrm{Tr}\left\{x^{*T} P y^* \geq r\right\} = \frac{1}{2}\left(\frac{x^{*T} D y^* - r}{x^{*T} D y^* - x^{*T} C y^*} + 1\right). \tag{3.27}$$

Since $S \neq \Phi$, then for the strategy x^*, there exists strategy (x^*, y) such that $x^{*T} B y \geq r \geq x^{*T} A y$, then

$$\mathrm{Tr}\left\{x^{*T} P y \geq r\right\} = \frac{1}{2}\left(\frac{x^{*T} D y - r}{x^{*T} D y - x^{*T} C y} + \frac{x^{*T} B y - r}{x^{*T} B y - x^{*T} A y}\right). \tag{3.28}$$

It is apparent that $\mathrm{Tr}\{x^{*T} P y^* \geq r\} \geq \mathrm{Tr}\{x^T C y > r\}$. Namely, $\mathrm{Tr}\{x^{*T} P y^* \geq r\} \nless \mathrm{Tr}\{x^{*T} C y > r\}$. This is in conflict with the definition of r-trust maximin equilibrium strategy too. Then there does not exist a r-trust maximin equilibrium strategy in this case.

The other cases can be proved in the same way. This completes the proof. □

According to Theorem 3.9, we know that this game exists r-trust maximin equilibrium strategy (x^*, y^*) only if all strategies (x, y) are in some section, for example, $x^T B y \leq r \leq x^T D y$. Next let us discuss the following case that all strategies (x, y) are in some section.

Theorem 3.10. *For all strategies* (x, y) *satisfying* $x^T B y \leq r \leq x^T D y$*, the game exists a* r*-trust maximin equilibrium strategy if and only if linear programming problems* (3.29) *and* (3.30) *have optimal solutions, where problems* (3.29) *and* (3.30) *are separately characterized as follows:*

$$\begin{aligned} \max \quad & \frac{1}{2}\left(q^T D y - r p\right) \\ \text{s.t.} \quad & \begin{cases} q^T B y - r p \leq 0, \\ q^T D y - r p \geq 0, \\ q_1 + q_2 + \cdots + q_m = p, \\ q_i \geq 0, \quad i = 1, 2, \ldots, m, \end{cases} \end{aligned} \tag{3.29}$$

where $p = 1/(x^T Dy - x^T Cy)$, $q^T = px^T$, y *is any fixed vector,*

$$\min \quad \frac{1}{2}\left(x^T Dt - rs\right)$$
$$\text{s.t.} \quad \begin{cases} x^T Bt - rs \le 0, \\ x^T Dt - rs \ge 0, \\ t_1 + t_2 + \cdots + t_n = s, \\ t_i \ge 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{3.30}$$

where $s = 1/(x^T Dy - x^T Cy)$, $t = py$, x *is any fixed vector.*

Proof. For all strategies (x, y) satisfying $x^T By \le r \le x^T Dy$, the trust measure function of payoffs matrix P is characterized by the following equation:

$$\operatorname{Tr}\left\{x^T Py \ge r\right\} = \frac{x^T Dy - r}{2(x^T Dy - x^T Cy)}. \tag{3.31}$$

Suppose $M = \{(x, y) \mid x^T By \le r \le x^T Dy, \sum_{i=1}^{m} x_i = 1, \sum_{j=1}^{n} y_j = 1, 0 \le x_i, y_j \le 1\}$. According to the definition of r-trust maximin equilibrium strategy, whether the game equilibrium has an equilibrium strategy in M is equal to the following two problems.

For any fixed y,

$$\max \quad \frac{x^T Dy - r}{2(x^T Dy - x^T Cy)}$$
$$\text{s.t.} \quad \begin{cases} x^T By \le r, \\ x^T Dy \ge r, \\ x_1 + x_2 + \cdots + x_m = 1, \\ 0 \le x_i \le 1, \quad i = 1, 2, \ldots, m. \end{cases} \tag{3.32}$$

For any fixed x,

$$\min \quad \frac{x^T Dy - r}{2(x^T Dy - x^T Cy)}$$
$$\text{s.t.} \quad \begin{cases} x^T By \le r, \\ x^T Dy \ge r, \\ y_1 + y_2 + \cdots + y_n = 1, \\ 0 \le y_i \le 1, \quad i = 1, 2, \ldots, n. \end{cases} \tag{3.33}$$

We know that only if the two problems have optimal solution, the game exists an equilibrium strategy. Because problems (3.32) and (3.33) are similar, then let us only discuss problem (3.32). For a fixed y, let $p = 1/2(x^T Dy - x^T Cy)$, $q^T = px^T$. Then problem (3.32) is converted into a linear programming problem

$$\begin{aligned} \max \quad & \frac{1}{2}\left(q^T Dy - rp\right) \\ \text{s.t.} \quad & \begin{cases} q^T By - rp \leq 0, \\ q^T Dy - rp \geq 0, \\ q_1 + q_2 + \cdots + q_m = p, \\ q_i \geq 0, \quad i = 1, 2, \ldots, m. \end{cases} \end{aligned} \tag{3.34}$$

Similarly, problem (3.33) can be converted into the following problem, for any fixed x,

$$\begin{aligned} \min \quad & \frac{1}{2}\left(x^T Dt - rs\right) \\ \text{s.t.} \quad & \begin{cases} x^T Bt - rs \leq 0, \\ x^T Dt - rs \geq 0, \\ t_1 + t_2 + \cdots + t_n = s, \\ t_i \geq 0, \quad i = 1, 2, \ldots, n, \end{cases} \end{aligned} \tag{3.35}$$

where $s = 1/(x^T Dy - x^T Cy)$, $t = py$. □

For problems (3.29) and (3.30), we can make use of MATLAB to get optimal solution of the programming problem by turning them a bi-level programming problem. Here, we do not give the detail description. Then we go on to discuss another case that $x^T Cy \leq r \leq x^T Ay$. Similarly, we can get the following conclusion.

Theorem 3.11. *For all strategies (x, y) satisfying $x^T Cy \leq r \leq x^T Ay$, the game exists a r-trust maximin equilibrium strategy if and only if linear programming problems* (3.36) *and* (3.37) *have optimal solutions, where problems* (3.36) *and* (3.37) *are characterized as follows:*

$$\begin{aligned} \max \quad & \frac{1}{2}\left(q^T Dy - rp + 1\right) \\ \text{s.t.} \quad & \begin{cases} q^T Ay - rp \geq 0, \\ q^T Cy - rp \leq 0, \\ q_1 + q_2 + \cdots + q_m = p, \\ q_i \geq 0, \quad i = 1, 2, \ldots, m, \end{cases} \end{aligned} \tag{3.36}$$

where $p = 1/(x^T Dy - x^T Cy)$, $q^T = px^T$, y *is any fixed vector,*

$$\min \quad \frac{1}{2}\left(x^T Dt - rs + 1\right)$$
$$\text{s.t.} \quad \begin{cases} x^T At - rs \geq 0, \\ x^T Ct - rs \leq 0, \\ t_1 + t_2 + \cdots + t_n = s, \\ t_i \geq 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{3.37}$$

where $s = 1/(x^T Dy - x^T Cy)$, $t = sy$, x *is any fixed vector.*

Proof. It can be proved similarly as Theorem 3.10. □

We have discussed many simple cases; there is still a more complicated case that $x^T Ay \leq r \leq x^T By$. For this case, according to Lemma 3.6, we have that

$$\operatorname{Tr}\left\{x^T Py \geq r\right\} = \frac{1}{2}\left(\frac{x^T Dy - r}{x^T Dy - x^T Cy} + \frac{x^T By - r}{x^T By - x^T Ay}\right). \tag{3.38}$$

Right now, the problem to find if the game has the r-trust maximin equilibrium strategy is converted into the following two problems:

$$\max \quad \frac{1}{2}\left(\frac{x^T Dy - r}{x^T Dy - x^T Cy} + \frac{x^T By - r}{x^T By - x^T Ay}\right)$$
$$\text{s.t.} \quad \begin{cases} x^T Ay \leq r, \\ x^T By \geq r, \\ x_1 + x_2 + \cdots + x_m = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, m, \end{cases} \tag{3.39}$$

where y is any fixed vector, and

$$\min \quad \frac{1}{2}\left(\frac{x^T Dy - r}{x^T Dy - x^T Cy} + \frac{x^T By - r}{x^T By - x^T Ay}\right)$$
$$\text{s.t.} \quad \begin{cases} x^T Ay \leq r, \\ x^T By \geq r, \\ y_1 + y_2 + \cdots + y_n = 1, \\ y_i \geq 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{3.40}$$

where x is any fixed vector. The two problems are nonlinear programming problems. We can make use of many traditional methods to solve them, for example, methods of feasible directions (see Polak [21]), Frank-Wolfe methods (see Meyer and Podrazik [22]). However, the solutions we got through these methods are usually local optimal solutions, not the global optimal solutions. In the following section, we will introduce an algorithm to solve the nonlinear programming such as problems (3.39) and (3.40). Then we can get the r-trust minmax equilibrium strategy of the game.

4. Genetic Algorithm

For many complex problems such as problem (3.39) and (3.40), it is difficult to obtain its optimal solution by the traditional technique. Therefore, GA is an efficient tool to obtain the efficient solution by its global searching ability. Take problem (3.39) as an example and we will list the detailed procedure to illustrate how the genetic algorithm introduced by Gen and Cheng [23] works. Let $H(x,y) = (1/2)((x^TDy - r)/(x^TDy - x^TCy) + (x^TBy - r)/(x^TBy - x^TAy))$ express the objective function and $X = \{(x,y) \mid x^TAy \leq r, x^TBy \geq r, \sum_{i=1}^{m} x_i = 1, \sum_{i=1}^{m} y_i = 1, y_i, x_i \geq 0, i = 1,2,\ldots,m\}$ express the feasible set.

(1) Initializing process: the initial population is formed by $N_{\text{pop-size}}$ chromosomes associated with basic feasible solutions of problem (3.39). Hence any general procedure to get them can be applied. Therefore, we can take the solution $(x,y) = (x_1,\ldots,x_m;y_1,\ldots,y_n)^T \in X$ as a chromosome. Randomly generate the feasible solution (x,y) in X. Repeat the above process $N_{\text{pop-size}}$ times, then we have N initial feasible chromosomes $(x^1,y^1),(x^2,y^2),\ldots,(x^{N_{\text{pop-size}}},y^{N_{\text{popsize}}})$.

(2) Evaluation function: in this case, we only attempt to obtain the best solution, which is absolutely superior to all other alternatives by comparing the objective function. Then we can construct the evaluation function by the following procedure: (i) compute the objective value $H(x^i,y^i)$, (ii) the evaluation function is constructed as follows:

$$\text{eval}\left(x^i,y^i\right) = \frac{H(x^i,y^i)}{\sum_{i=1}^{N_{\text{pop-size}}} H(x^i,y^i)} \tag{4.1}$$

which expresses the evaluation value of the ith chromosome in current generation.

(3) Selection process: The selection process is based on spinning the roulette wheel $N_{\text{pop-size}}$ times. Each time a single chromosome for a new population is selected in the following way. Calculate the cumulative probability q_i for each chromosome (x^i,y^i):

$$q_0 = 0, \qquad q_i = \sum_{j=1}^{i} \text{eval}\left(x^j,y^j\right), \quad i = 1,2,\ldots,\text{pop-size}. \tag{4.2}$$

Generate a random number r in $[0,q_{\text{pop-size}}]$ and select the chromosomes (x^i,y^i) such that $q_{i-1} < r \leq q_i$ $(1 \leq 1 \leq N_{\text{pop-size}})$. Repeat the above process $N_{\text{pop-size}}$ times and obtain $N_{\text{pop-size}}$ copies of chromosomes.

(4) Crossover operation: the goal of crossover is to exchange information between two parent chromosomes in order to produce two new offspring for the next population. The uniform crossover of Genetic operator proposed by Li et al. [24] in this paper. The detail

is as follows. Generate a random number $c \in (0,1)$ and if $c < P_c$, then the chromosome (x^i, y^i) is selected as a parament, where the parameter P_c which is the probability of crossover operation. Repeat this process $N_{\text{pop-size}}$ times and we get $P_c \cdot N_{\text{pop-size}}$ parent chromosomes to undergo the crossover operation. The crossover operator on (x^1, y^1) and (x^2, y^2) will produce two children as follows:

$$\begin{pmatrix} X^1 \\ Y^1 \end{pmatrix} = c\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} + (1-c)\begin{pmatrix} x^2 \\ y^2 \end{pmatrix},$$
$$\begin{pmatrix} X^2 \\ Y^2 \end{pmatrix} = c\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} + (1-c)\begin{pmatrix} x^1 \\ y^1 \end{pmatrix}. \tag{4.3}$$

The children of chromosomes (X^1, Y^1) and (X^2, Y^2) can be generated as above. They are feasible if they are both in X and then we replace the parents with them. Or else we keep the feasible one if it exists. Redo the crossover operator until we obtaine two feasible children or a given number of cycles is finished.

(5) Mutation operation: a mutation operator is a random process where one genotype is replaced by another to generate a new chromosome. Each genotype has the probability of mutation, P_m, to change from 0 to 1. Let (x^i, y^i) be selected as parent. Choose a mutation direction $\mathbf{d} \in \mathbf{R}^{m+n}$ randomly. M is an appropriate large positive number. We replace the parent (x^i, y^i) with the child

$$\begin{pmatrix} X^i \\ Y^i \end{pmatrix} = \begin{pmatrix} x^i \\ y^i \end{pmatrix} + M \cdot \mathbf{d}. \tag{4.4}$$

If (X^i, Y^i) are infeasible, we set M as a random number between 0 and M until it is feasible and then replace (x^i, y^i) with it.

Above all, it can be simply summarized in Procedure 1.

5. Numerical Example

Game theory is widely applied in many fields, such as, economic and management problems, social policy, and international and national politics; sometimes players should consider the state of uncertainty. A kind of games are usually characterized by rough payoffs. In this section, we give an example of two-person zero-sum game with rough payoffs to illustrate the effectiveness of the algorithm introduced above. There is a game between player I and player II. When player I gives strategy i and player II gives strategy j, player II will give some money to player I which is at least between c_{ij} and a_{ij}, or at most between b_{ij} and d_{ij}. The payoff matrix of player I is as follows

$$P = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix}, \tag{5.1}$$

Input: GA parameters: $N_{\text{pop-size}}$, P_c, P_m and the cycle number $\text{Gen}_{\max}$
Output: optimal sampling level, NPV
begin
  $g \leftarrow 0$;
  initialization( ) by checking the feasiblity;
  evaluation(0);
  while ($g \leq \text{Gen}_{\max}$) **do**
    selection( );
    crossover( );
    mutation( );
    evaluation(g);
    $g \leftarrow g + 1$;
  end
  Output: the optimal solution
End

PROCEDURE 1: Procedure of genetic algorithm.

where rough variables ξ_{ij} $(i = 1, 2, 3,\ j = 1, 2, 3)$ are characterized as

$$\begin{gathered}
\xi_{11} = ([15, 25], [10, 28]), \qquad \xi_{12} = ([13.5, 22], [8, 25]), \qquad \xi_{13} = ([15, 20], [11.2, 21]), \\
\xi_{21} = ([17, 30], [9, 35]), \qquad \xi_{22} = ([16.2, 26], [12, 28]), \qquad \xi_{23} = ([13, 27], [10, 30]), \\
\xi_{31} = ([18, 20], [11, 24]), \qquad \xi_{32} = ([18, 24], [12, 29]), \qquad \xi_{33} = ([13, 20], [12, 25]).
\end{gathered} \tag{5.2}$$

Firstly, let us consider the expected maximin equilibrium strategy of this game. According to Remarks 2.9 and 3.2, we have that

$$E\left[x^T P y\right] = E\left[\sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_j\right] = \sum_{j=1}^{n} \sum_{i=1}^{m} E[\xi_{ij}] x_i y_j = x^T P' y, \tag{5.3}$$

where

$$P' = \begin{bmatrix} E[\xi_{11}] & E[\xi_{12}] & E[\xi_{13}] \\ E[\xi_{21}] & E[\xi_{22}] & E[\xi_{23}] \\ E[\xi_{31}] & E[\xi_{32}] & E[\xi_{33}] \end{bmatrix} = \begin{bmatrix} 19.5 & 17.125 & 16.8 \\ 22.75 & 20.55 & 20 \\ 18.25 & 20.75 & 17.5 \end{bmatrix}. \tag{5.4}$$

Then, we can get the equilibrium strategy that when player I gives the mixed strategy $x = (0, 0, 1)$ and player II gives the mixed strategy $y = (0, 1, 0)$, player I gets the most payoff 20 which is the least payoffs player II loses.

Next, let us consider if this game has the r-trust maximin equilibrium strategy. According to Lemma 3.6, we have

$$A = \begin{bmatrix} 15 & 13.5 & 15 \\ 17 & 16.2 & 13 \\ 18 & 18 & 13 \end{bmatrix}, \qquad B = \begin{bmatrix} 25 & 22 & 20 \\ 30 & 26 & 27 \\ 20 & 24 & 20 \end{bmatrix},$$
$$C = \begin{bmatrix} 10 & 8 & 11.2 \\ 9 & 12 & 10 \\ 11 & 12 & 12 \end{bmatrix}, \qquad D = \begin{bmatrix} 28 & 25 & 21 \\ 35 & 28 & 30 \\ 24 & 29 & 25 \end{bmatrix}. \tag{5.5}$$

Then we give five predetermined numbers r and discuss if the game exists a r-trust maximin equilibrium strategy.

Case 1 ($r = 5$). Apparently, $\min c_{ij} = 8$ and $0 \le x, y \le 1$. Thus, for all (x, y), they satisfy $x^T C y \ge 5$. Based on Theorem 3.7, we know that all (x, y) are 5-trust maximin equilibrium strategy of this game.

Case 2 ($r = 40$). Similarly, $\max d_{ij} = 35$ and $0 \le x, y \le 1$. Thus, for all (x, y), they satisfy $x^T C y \le 40$. Based on Theorem 3.8, we know that all (x, y) are 40-trust maximin equilibrium strategy of this game.

Case 3 ($r = 25$). Because $\max b_{ij} = 30$ and $\min d_{ij} = 21$, for $0 \le x, y \le 1$, not all (x, y) satisfy $x^T B y \le 25 \le x^T D y$. According to Theorem 3.9, we know that this game does not exist a 25-trust maximin equilibrium strategy.

Case 4 ($r = 12.5$). Apparently, $\max c_{ij} = 12$ and $\min a_{ij} = 13$. Thus, for $0 \le x, y \le 1$, all (x, y) satisfy $x^T C y \le 12.5 \le x^T A y$. Based on Theorem 3.11, we can get the following two problems

$$\begin{aligned} &\max \quad \frac{1}{2}\left(\sum_{i=1}^{3} \sum_{j=1}^{3} q_i y_j d_{ij} - 12.5p + 1 \right) \\ &\text{s.t.} \quad \begin{cases} \displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} q_i y_j a_{ij} - 12.5p \ge 0, \\ \displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} q_i y_j c_{ij} - 12.5p \le 0, \\ q_1 + q_2 + q_3 = p, \\ q_1, q_2, q_3 \ge 0, \end{cases} \end{aligned} \tag{5.6}$$

where y is any fixed strategy, $p = 1/(x^T Dy - x^T Cy)$, $q^T = px^T$,

$$\min \quad \frac{1}{2}\left(\sum_{i=1}^{3}\sum_{j=1}^{3} x_i t_j d_{ij} - 12.5s + 1\right)$$
$$\text{s.t.} \quad \begin{cases} \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_i t_j a_{ij} - 12.5s \geq 0, \\ \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_i t_j c_{ij} - 12.5s \leq 0, \\ t_1 + t_2 + t_3 = s, \\ t_1, t_2, t_3 \geq 0, \end{cases} \tag{5.7}$$

where x is any fixed strategy, $s = 1/(x^T Dy - x^T Cy)$, $t = sy$.

Then we can make use of simplex method to get the optimal solutions of problems (5.6) and (5.7). The optimal solutions is $(x, y)^T = (0.378, 0.124, 0.498, 0.397, 0.469, 0.194)^T$.

Case 5 ($r = 19$). Apparently, $\max a_{ij} = 18$ and $\min b_{ij} = 20$. Thus, for $0 \leq x, y \leq 1$, all (x, y) satisfy $x^T Ay \leq 19 \leq x^T By$. Then we can get two nonlinear programming problems.

For any fixed y

$$\max \quad \frac{1}{2}\left(\frac{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j d_{ij} - 19}{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j (d_{ij} - c_{ij})} + \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j b_{ij} - 19}{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j (b_{ij} - a_{ij})}\right)$$
$$\text{s.t.} \quad \begin{cases} \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j a_{ij} \leq 19, \\ \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j b_{ij} \geq 19, \\ x_1 + x_2 + x_3 = 1, \\ x_i, y_j \geq 0. \end{cases} \tag{5.8}$$

For any fixed x,

$$\min \quad \frac{1}{2}\left(\frac{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j d_{ij} - 19}{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j (d_{ij} - c_{ij})} + \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j b_{ij} - 19}{\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j (b_{ij} - a_{ij})}\right)$$
$$\text{s.t.} \quad \begin{cases} \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j a_{ij} \leq 19, \\ \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j b_{ij} \geq 19, \\ y_1 + y_2 + y_3 = 1, \\ x_i, y_j \geq 0. \end{cases} \tag{5.9}$$

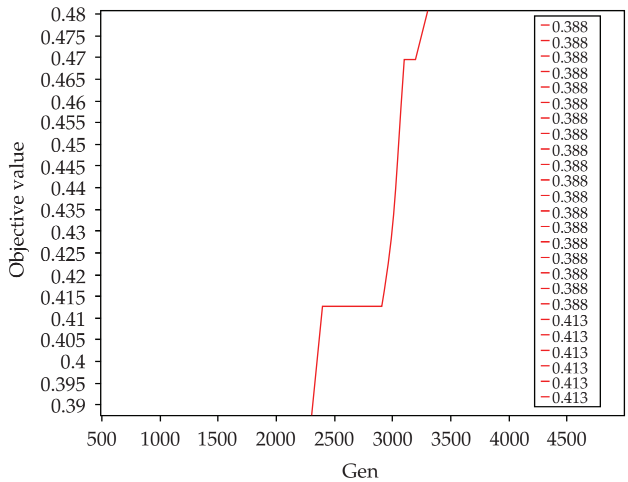

Figure 1: The results by GA.

For problems (5.8) and (5.9), we can solve them by using the rough simulation-based genetic algorithm. Let the number of chromosomes in every generation be 20, the probability of crossover 0.85 and of mutation 0.1. Then after a run of a genetic algorithm computer program (5000 generations), we get the optimal solution $(x, y)^T = (0.178, 0.251, 0.571, 0.349, 0.479, 0.172)^T$ and the objective value as shown in Figure 1.

6. Conclusion

In this paper, we have considered a class of two-person zero-sum matrix games with rough payoffs. Firstly, we have given the definition of the game with rough payoffs and then proposed two kinds of equilibrium strategy. Secondly, we have discussed wether the two-person zero-sum matrix games with rough payoffs exist the equilibrium strategy. Thirdly, we proposed the genetic algorithm to solve the most complicated case. It is an available and efficient way to search the equilibrium of this kind of games with rough payoffs. Lastly, the numerical example illustrated well our research methods. We have only considered one kind of games with uncertain payoffs. Of course, there are many other games with uncertain payoffs which need to be researched.

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