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Research Article

The Solution by Iteration of a Composed K-Positive Definite Operator Equation in a Banach Space

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The equation $Lu = f$, where $L = A + B$, with A being a K-positive definite operator and B being a linear operator, is solved in a Banach space. Our scheme provides a generalization to the so-called method of moments studied in a Hilbert space by Petryshyn (1962), as well as Lax and Milgram (1954). Furthermore, an application of the inverse function theorem provides simultaneously a general solution to this equation in some neighborhood of a point x_o, where L is Fréchet differentiable and an iterative scheme which converges strongly to the unique solution of this equation.

1. Introduction

Let H_o be a dense subspace of a Hilbert space, H. An operator T with domain $D(T) \supseteq H_o$ is said to be continuously H_o-invertible if the range of T, $R(T)$ with T considered as an operator restricted to H_o is dense in H and T has a bounded inverse on $R(T)$. Let H be a complex and separable Hilbert space, and let A be a linear unbounded operator defined on a dense domain $D(A)$ in H with the property that there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$ and a constant $\alpha > 0$ such that

$$\langle Au, Ku \rangle \geq \alpha \|Ku\|^2, \quad u \in D(A). \tag{1.1}$$

Then A is called K-positive definite (see, e.g., [1]). If $K = I$ (the identity operator on H), then (1.1) reduces to $\langle Au, u \rangle \geq \alpha \|u\|^2$, and in this case A is called positive definite. Positive definite operators have been studied by various authors (see, e.g., [1–4]). It is clear that the class of K-pd operators contains, among others, the class of positive definite operators and also contains the class of invertible operators (when $K = A$) as its subclass.

The class of K-positive definite operators was first studied by Petryshyn, who proved, interalia, the following theorem (see [1]).

Theorem 1.1. *If A is a K-pd operator and $D(A) = D(K)$, then there exists a constant $\alpha > 0$ such that, for all $u \in D(K)$,*

$$\|Au\| \leq \alpha \|Ku\|. \tag{1.2}$$

Furthermore, the operator A is closed, $R(A) = H$, and the equation $Au = f$, $f \in H$, has a unique solution.

Chidume and Aneke extended the notion of a K-pd operator to certain Banach spaces (see [5]). Later, in 2001, we also extended the class of K-pd operators to include the Fréchet differentiable operators. A new notion—the asymptotically K-pd operators—was also introduced and studied in certain Banach spaces. We proved, among others, the following theorem.

Theorem 1.2 (see [6]). *Suppose that X is a real uniformly smooth Banach space. Suppose that A is an asymptotically K-positive definite operator defined in a neighborhood $U(x_o)$ of a real uniformly smooth Banach space, X. Define the sequence $\{x_n\}$ by $x_o \in U(x_o)$, $x_{n+1} = x_n + r_n$, $n \geq 0$, $r_n = K^{-1}y - K^{-1}Ax_n$, $y \in R(A)$. Then $\{x_n\}$ converges strongly to the unique solution of $Ax = y \in U(x_o)$.*

In this paper, we consider the composed equation

$$(A + B)u = f, \tag{1.3}$$

where A is K-pd and B is some linear operator in a Banach space E. Our interest is on the existence and uniqueness of solution to the above equation in a Banach space. We also consider an iterative scheme that converges to the unique solution of this equation in an arbitrary Banach space. Our method generalizes the so called method of moments, studied in Hilbert spaces by Petryshyn [1] and a host of other authors.

2. Preliminaries

Let E be a real normed linear space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single valued and if E is uniformly smooth (equivalently if E^* is uniformly convex) then J is uniformly continuous on bounded subsets of E. We will denote the single-valued duality mapping by j.

Lemma 2.1. *Let E be a real Banach space, and let J be the normalized duality map on E. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{2.2}$$

3. Main Result

Let E be an arbitrary Banach space and A a K-positive definite operator defined in a dense domain $D(A) \subseteq E$. Let B be a linear unbounded operator such that $D(B) \supseteq D(A)$. We prove that the equation

$$Lu = f, \tag{3.1}$$

where $L = A + B$, has a unique solution and construct an iterative scheme that converges to the unique solution of this equation. Let

$$Lu = (A+B)u = f. \tag{3.2}$$

Multiplying both sides of (3.2) by A^{-1}, we have

$$u + Tu = g, \tag{3.3}$$

where $T = A^{-1}B$, $g = A^{-1}f$. Since A is continuously invertible, the operator $T = A^{-1}B$ is completely continuous. Hence T is locally lipschitzian and accretive. It follows that (3.3) has a unique solution (see [7]).

If $A = B$, then $L = A + B = 2A$. In this case $\langle Lu, Ku\rangle = 2\langle Au, Ku\rangle \geq 2\alpha\|Ku\|^2 = \beta\|Ku\|^2$. Thus L is K-positive definite and so the equation $Lu = f$ has a unique solution (see [5]). Examples of such A are all positive operators when $K = I$ and are all invertible operators when $K = A$. If $A \neq B$, then let $E = l_2$, for instance, and define $A : l_2 \to l_2$ by $Ax = (ax_1, ax_2, ax_3, \ldots)$ for $x = (x_1, x_2, x_3, \ldots) \in l_2$ and $a > 0$. Let $K = I$, the identity operator, then $\langle Ax, x\rangle = a\sum_{i=1}^{\infty} x_i^2 = a\|x\|^2 > (1/2)a\|x\|^2$. Thus A is K-positive definite. Let B be any linear operator; in particular, let $B : l_2 \to l_2$ be defined by $Bx = (0, x_1, x_2, x_3, \ldots)$. Then by (3.2) and (3.3), the equation $Lu = f$, where $L = A + B$, has a unique solution.

Next we derive the solution to (3.2) from the inverse function theorem and construct an iterative scheme which converges to the unique solution of this equation.

Theorem 3.1 (the inverse function theorem). *Suppose that E, Y are Banach spaces and $L : E \to Y$ is such that L has uniformly continuous Fréchet derivatives in a neighborhood of some point u_o of E. Then if $L'(u_o)$ is a linear homeomorphism of E onto Y, then L is a local homeomorphism of a neighborhood $U(u_o)$ of u_o to a neighborhood $L(u_o)$.*

Proof. For a sketch of proof of this theorem, see [6].

By mimicking the proof of Theorem 3.1 of [6], we get that, if $\|g - Lu_o\|$ is sufficiently small, $Lu = g$ has a unique solution $u = u_o + \rho^*$, where ρ^* is the limit of the sequence $\rho_o = 0$, $\rho_{n+1} = Q\rho_n$, where Q is a contraction mapping of a sphere $S(0, \epsilon)$ in E into itself, for some ϵ

sufficiently small. It follows that the sequence $u_n = u_o + \rho_n$ converges to $u_o + \rho^*$, the unique solution of $Lu = g$ in $U(u_o)$. Now

$$\begin{aligned}
u_n &= u_o + \rho_n = u_o + Q\rho_{n-1} \\
&= u_o + \left[L'(u_o)\right]^{-1}\left[g - L(u_o) - R(u_o, \rho_{n-1}\right] \quad \text{from Taylor's theorem} \\
&= u_o + \left[L'(u_o)\right]^{-1}\left[g + L'(u_o)\rho_{n-1} - L(u_o + \rho_{n-1})\right] \\
&= u_o + \rho_{n-1} + \left[L'(u_o)\right]^{-1}\left[g - L(u_{n-1})\right] \\
&= u_{n-1} + \left[L'(u_o)\right]^{-1}\left[g - Lu_{n-1}\right].
\end{aligned} \tag{3.4}$$

Hence

$$u_{n+1} = u_n + \left[L'(u_o)\right]^{-1}\left[g - Lu_n\right]. \tag{3.5}$$

□

Special Cases

(1) If $B = I$, then (3.5) becomes

$$u_{n+1} = u_n + \left[A'(u_o)\right]^{-1}\left[g - Au_n + u_n\right]. \tag{3.6}$$

(2) If $B = 0$, then we have Corollary 3.2 of [6].

For the case $B = 0$, we prove the following theorem for an asymptotically K-positive definite operator. Recall (see [6], page 606) the definition of an asymptotically K-pd operator. For simplicity and ease of reference, we repeat the definition.

Definition 3.2. Let E be a Banach space, and let A be a linear unbounded operator defined on a dense domain $D(A) \subset E$. The operator A is called asymptotically K-positive definite if there exist a continuously $D(A)$-invertible closed linear operator K with $D(K) \supseteq D(A) \supseteq R(A)$ and a constant $c > 0$ such that, for $j(Ku) \in J(Ku)$,

$$\left\langle K^{n-1}Au, j(K^n u)\right\rangle \geq ck_n\|K^n u\|^2, \quad u \in D(A), \tag{3.7}$$

where $\{k_n\}$ is a real sequence such that $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$.

We now prove the following theorem for an asymptotically K-positive definite operator equation in an arbitrary Banach space, E.

Theorem 3.3. *Let E be a real Banach space. Suppose that A is an asymptotically K-positive definite operator defined in a neighborhood $U(x_o)$ of a real Banach space, E. Define the sequence x_n by $x_o \in D(A)$, $x_{n+1} = x_n + r_n$, $n \geq 0$, $r_n = K^{-1}f - K^{-1}Ar_n$, $f \in R(A)$. Then x_n converges strongly to the unique solution of $Ax = f$.*

Proof. By the linearity of K we have

$$Kr_{n+1} = Kr_n - Ar_n. \tag{3.8}$$

Using Lemma 2.1 and Definition 3.2, we obtain

$$\begin{aligned} \|K^n r_{n+1}\|^2 &= \left\|K^n r_n - K^{n-1}Ar_n\right\|^2 \\ &\leq \|K^n r_n\|^2 - 2\left\langle K^{n-1}Ar_n, j\left(K^n r_n - K^{n-1}Ar_n\right)\right\rangle \\ &\leq \|K^n r_n\|^2 - 2ck_n\|K^n r_{n+1}\|^2. \end{aligned} \tag{3.9}$$

It follows that

$$(1+2ck_n)\|K^n r_{n+1}\|^2 \leq \|K^n r_n\|^2 \tag{3.10}$$

or

$$\|K^n r_{n+1}\|^2 \leq (1+2ck_n)^{-1}\|K^n r_n\|^2. \tag{3.11}$$

The last inequality shows that the sequence Kr_n is monotonically decreasing and hence converges to a real number $\delta \geq 0$. Hence $\lim_{n\to\infty}\|K^n r_n\| = 0$. Since K is continuously invertible, then $r_n \to 0$, and since A has a bounded inverse, we have that $x_n \to A^{-1}f$, the unique solution of $Ax = f$, $f \in E$. □

Our next result is a generalization of Theorem 3.6 of Chidume and Aneke [6] to an arbitrary real Banach space.

Lemma 3.4 (Alber-Guerre [8]). *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers, and let $\{\alpha_k\}$ be a sequence of positive numbers satisfying the condition $\sum_1^\infty\{\alpha_k\} = \infty$ and $\gamma_n/\alpha_n \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\phi(\lambda_n) + \gamma_n, \quad n = 1,2,\ldots \tag{3.12}$$

be given where $\phi(\lambda)$ is a continuous and nondecreasing function from $\Re^+ \to \Re^+$ such that it is positive on $\Re^+ - \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty}\phi(t) = \infty$. Then $\lambda_n \to 0$, as $n \to \infty$.

Theorem 3.5. *Suppose that E is a real Banach space and A is an asymptotically K-positive definite operator defined in a neighbourhood $U(x_0)$ of a real Banach space, E. Suppose that A is Frećhet differentiable. Define the sequence $\{x_n\}$ by $x_0 \in U(x_0)$, $x_{n+1} = x_n + r_n$, $n \geq 0$, $r_n = K^{-1}y - K^{-1}Ax_n$, $y \in R(A)$, and $x_{n+1} - x_n \to 0$, as $n \to \infty$. Then $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = y \in U(x_0)$.*

Proof. By the linearity of K we have $Kr_{n+1} = Kr_n - Ar_n$. Using Lemma 2.1 and the definition of an asymptotically K-positive definite operator, we obtain

$$\begin{aligned}
\|K^n r_{n+1}\|^2 &\leq \left\|K^n r_n - K^{n-1}Ar_n\right\|^2 \\
&\leq \|K^n r_n\|^2 - 2\left\langle K^{n-1}Ar_n, j(K^n r_{n+1})\right\rangle \\
&\leq \|K^n r_n\|^2 - 2\left\langle K^{n-1}Ar_n, j(K^n r_n)\right\rangle - 2\left\langle K^{n-1}Ar_n, j(K^n r_{n+1} - j(K^n r_n)\right\rangle \\
&\leq \|K^n r_n\|^2 - 2ck_n\|K^n r_n\|^2 - 2\left\langle K^{n-1}Ar_n, j(K^n r_{n+1}) - j(K^n r_n)\right\rangle \\
&\leq \|K^n r_n\|^2 - 2ck_n\|K^n r_n\|^2 + 2\left\|K^{n-1}Ar_n\right\|\|j(K^n r_{n+1}) - j(K^n r_n)\|.
\end{aligned} \tag{3.13}$$

Now,

$$K^n r_{n+1} - K^n r_n = K^n(r_{n+1} - r_n) = K^n K^{-1} A(x_{n+1} - x_n). \tag{3.14}$$

Since $x_{n+1} - x_n \to 0$ and j is uniformly continuous, it follows that $\|j(K^n r_{n+1}) - j(K^n r_n)\| \to 0$ as $n \to \infty$. Since A is Fréchet differentiable, then $\|K^{n-1}Ar_n\|$ is necessarily bounded in $U(x_0)$, whence

$$\|K^n r_{n+1}\|^2 \leq \|K^n r_n\|^2 - 2ck_n\|K^n r_n\|^2 + o(r). \tag{3.15}$$

We invoke Alber-Guerre lemma, Lemma 3.4, with $\phi(t) = t$ and $\lambda_n = \|K^n r_n\|^2$. Thus $\|K^n r_n\| \to 0$ as $n \to \infty$. Since K has a bounded inverse; then $r_n \to 0$ as $n \to \infty$, that is, $Ax_n \to y$. Hence $x_n \to A^{-1}y$, the unique solution of $Ax = y$ in $U(x_0)$. □

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