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Research Article

(L, M)-Fuzzy σ-Algebras

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The notion of (L, M)-fuzzy σ-algebras is introduced in the lattice value fuzzy set theory. It is a generalization of Klement's fuzzy σ-algebras. In our definition of (L, M)-fuzzy σ-algebras, each L-fuzzy subset can be regarded as an L-measurable set to some degree.

1. Introduction and Preliminaries

In 1980, Klement established an axiomatic theory of fuzzy σ-algebras in [1] in order to prepare a measure theory for fuzzy sets. In the definition of Klement's fuzzy σ-algebra (X, σ), σ was defined as a crisp family of fuzzy subsets of a set X satisfying certain set of axioms. In 1991, Biacino and Lettieri generalized Klement's fuzzy σ-algebras to L-fuzzy setting [2].

In this paper, when both L and M are complete lattices, we define an (L, M)-fuzzy σ-algebra on a nonempty set X by means of a mapping $\sigma : L^X \to M$ satisfying three axioms. Thus each L-fuzzy subset of X can be regarded as an L-measurable set to some degree.

When σ is an (L, M)-fuzzy σ-algebra on X, (X, σ) is called an (L, M)-fuzzy measurable space. An $(L, 2)$-fuzzy σ-algebra is also called an L-σ-algebra. A Klement σ-algebra can be viewed as a stratified $[0, 1]$-σ-algebra. A Biacino-Lettieri L-σ-algebra can be viewed as a stratified L-σ-algebra. A $(2, M)$-fuzzy σ-algebra is also called an M-fuzzifying σ-algebra. A crisp σ-algebra can be regarded as a $(2, 2)$-fuzzy σ-algebra.

Throughout this paper, both L and M denote complete lattices, and L has an order-reversing involution$'$. X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. We often do not distinguish a crisp subset A of X and its character function χ_A. The smallest element and the largest element in M are denoted by $\perp_M$ and $\top_M$, respectively.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [3]. $\{a \in M : a \prec b\}$ is called the greatest minimal family of b in the sense of [4], denoted by

$\beta(b)$. Moreover, for $b \in M$, we define $\alpha(b) = \{a \in M : a \prec^{op} b\}$. In a completely distributive lattice M, there exist $\alpha(b)$ and $\beta(b)$ for each $b \in M$, and $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ (see [4]).

In [4], Wang thought that $\beta(0) = \{0\}$ and $\alpha(1) = \{1\}$. In fact, it should be that $\beta(0) = \emptyset$ and $\alpha(1) = \emptyset$.

For a complete lattice L, $A \in L^X$ and $a \in L$, we use the following notation:

$$A_{[a]} = \{x \in X : A(x) \geqslant a\}. \tag{1.1}$$

If L is completely distributive, then we can define

$$A^{[a]} = \{x \in X : a \notin \alpha(A(x))\}. \tag{1.2}$$

Some properties of these cut sets can be found in [5–10].

Theorem 1.1 (see [4]). *Let M be a completely distributive lattice and $\{a_i : i \in \Omega\} \subseteq M$. Then*

(1) $\alpha(\bigwedge_{i\in\Omega} a_i) = \bigcup_{i\in\Omega} \alpha(a_i)$, *that is, α is an $\bigwedge - \bigcup$ map;*

(2) $\beta(\bigvee_{i\in\Omega} a_i) = \bigcup_{i\in\Omega} \beta(a_i)$, *that is, β is a union-preserving map.*

For $a \in L$ and $D \subseteq X$, we define two L-fuzzy sets $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \qquad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases} \tag{1.3}$$

Then for each L-fuzzy set A in L^X, it follows that

$$A = \bigvee_{a\in L} (a \wedge A_{[a]}). \tag{1.4}$$

Theorem 1.2 (see [5, 7, 10]). *If L is completely distributive, then for each L-fuzzy set A in L^X, we have*

(1) $A = \bigvee_{a\in L}(a \wedge A_{[a]}) = \bigwedge_{a\in L}(a \vee A^{[a]})$;

(2) *for all* $a \in L$, $A_{[a]} = \bigcap_{b\in\beta(a)} A_{[b]}$;

(3) *for all* $a \in L$, $A^{[a]} = \bigcap_{a\in\alpha(b)} A^{[b]}$.

For a family of L-fuzzy sets $\{A_i : i \in \Omega\}$ in L^X, it is easy to see that

$$\left(\bigwedge_{i\in\Omega} A_i\right)_{[a]} = \bigcap_{i\in\Omega} (A_i)_{[a]}. \tag{1.5}$$

If L is completely distributive, then it follows [7] that

$$\left(\bigwedge_{i\in\Omega} A_i\right)^{[a]} = \bigcap_{i\in\Omega} (A_i)^{[a]}. \tag{1.6}$$

Definition 1.3. Let X be a nonempty set. A subset σ of $[0,1]^X$ is called a Klement fuzzy σ-algebra if it satisfies the following three conditions:

(1) for any constant fuzzy set α, $\alpha \in \sigma$;

(2) for any $A \in [0,1]^X$, $1 - A \in \sigma$;

(3) for any $\{A_n : n \in \mathbb{N}\} \subseteq \sigma$, $\bigvee_{n\in\mathbb{N}} A_n \in \sigma$.

The fuzzy sets in σ are called fuzzy measurable sets, and the pair (X, σ) a fuzzy measurable space.

Definition 1.4. Let L be a complete lattice with an order-reversing involution $'$ and X a nonempty set. A subset σ of L^X is called an L-σ-algebra if it satisfies the following three conditions:

(1) for any $a \in L$, constant L-fuzzy set $a \wedge \chi_X \in \sigma$;

(2) for any $A \in L^X$, $A' \in \sigma$;

(3) for any $\{A_n : n \in \mathbb{N}\} \subseteq \sigma$, $\bigvee_{n\in\mathbb{N}} A_n \in \sigma$.

The L-fuzzy sets in σ are called L-measurable sets, and the pair (X, σ) an L-measurable space.

2. (L, M)-Fuzzy σ-Algebras

L. Biacino and A. Lettieri defined that an L-σ-algebra σ is a crisp subset of L^X. Now we consider an M-fuzzy subset σ of L^X.

Definition 2.1. Let X be a nonempty set. A mapping $\sigma : L^X \to M$ is called an (L, M)-fuzzy σ-algebra if it satisfies the following three conditions:

(LMS1) $\sigma(\chi_\emptyset) = \top_M$;

(LMS2) for any $A \in L^X$, $\sigma(A) = \sigma(A')$;

(LMS3) for any $\{A_n : n \in \mathbb{N}\} \subseteq L^X$, $\sigma(\bigvee_{n\in\mathbb{N}} A_n) \geq \bigwedge_{n\in\mathbb{N}} \sigma(A_n)$.

An (L, M)-fuzzy σ-algebra σ is said to be stratified if and only if it satisfies the following condition:

(LMS1)* $\forall a \in L$, $\sigma(a \wedge \chi_X) = \top_M$.

If σ is an (L, M)-fuzzy σ-algebra, then (X, σ) is called an (L, M)-fuzzy measurable space.

An $(L, \mathbf{2})$-fuzzy σ-algebra is also called an L-σ-algebra, and an $(L, \mathbf{2})$-fuzzy measurable space is also called an L-measurable space.

A $(\mathbf{2}, M)$-fuzzy σ-algebra is also called an M-fuzzifying σ-algebra, and a $(\mathbf{2}, M)$-fuzzy measurable space is also called an M-fuzzifying measurable space.

Obviously a crisp measurable space can be regarded as a $(\mathbf{2}, \mathbf{2})$-fuzzy measurable space.

If σ is an (L, M)-fuzzy σ-algebra, then $\sigma(A)$ can be regarded as the degree to which A is an L-measurable set.

Remark 2.2. If a subset σ of L^X is regarded as a mapping $\sigma : L^X \to \mathbf{2}$, then σ is an L-σ-algebra if and only if it satisfies the following conditions:

(LS1) $\chi_\emptyset \in \sigma$;

(LS2) $A \in \sigma \Rightarrow A' \in \sigma$;

(LS3) for any $\{A_n : n \in \mathbb{N}\} \subseteq \sigma$, $\bigvee_{n\in\mathbb{N}} A_n \in \sigma$.

Thus we easily see that a Klement σ-algebra is exactly a stratified $[0,1]$-σ-algebra, and a Biacino-Lettieri L-σ-algebra is exactly a stratified L-σ-algebra.

Moreover, when $L = \mathbf{2}$, a mapping $\sigma : \mathbf{2}^X \to M$ is an M-fuzzifying σ-algebra if and only if it satisfies the following conditions:

(MS1) $\sigma(\emptyset) = \top_M$;

(MS2) for any $A \in \mathbf{2}^X$, $\sigma(A) = \sigma(A')$;

(MS3) for any $\{A_n : n \in \mathbb{N}\} \subseteq 2^X$, $\sigma(\bigvee_{n\in\mathbb{N}} A_n) \geq \bigwedge_{n\in\mathbb{N}} \sigma(A_n)$.

Example 2.3. Let (X, σ) be a crisp measurable space. Define$\chi_\sigma : \mathbf{2}^X \to [0,1]$ by

$$\chi_\sigma(A) = \begin{cases} 1, & A \in \sigma; \\ 0, & A \notin \sigma. \end{cases} \tag{2.1}$$

Then it is easy to prove that (X, χ_σ) is a $[0,1]$-fuzzifying measurable space.

Example 2.4. Let X be a nonempty set and $\sigma : \mathbf{2}^X \to [0,1]$ a mapping defined by

$$\sigma(A) = \begin{cases} 1, & A \in \{\emptyset, X\}; \\ 0.5, & A \notin \{\emptyset, X\}. \end{cases} \tag{2.2}$$

Then it is easy to prove that (X, σ) is a $[0,1]$-fuzzifying measurable space. If $A \in \mathbf{2}^X$ with $A \notin \{\emptyset, X\}$, then 0.5 is the degree to which A is measurable.

Example 2.5. Let X be a nonempty set and $\sigma : [0,1]^X \to [0,1]$ a mapping defined by

$$\sigma(A) = \begin{cases} 1, & A \in \{\chi_\emptyset, \chi_X\}; \\ 0.5, & A \notin \{\chi_\emptyset, \chi_X\}. \end{cases} \tag{2.3}$$

Then it is easy to prove that (X, σ) is a $([0,1],[0,1])$-fuzzy measurable space. If $A \in [0,1]^X$ with $A \notin \{\chi_\emptyset, \chi_X\}$, then 0.5 is the degree to which A is $[0,1]$-measurable.

Proposition 2.6. *Let (X, σ) be an (L, M)-fuzzy measurable spaces. Then for any $\{A_n : n \in \mathbb{N}\} \subseteq L^X$, $\sigma(\bigwedge_{n\in\mathbb{N}} A_n) \geq \bigwedge_{n\in\mathbb{N}} \sigma(A_n)$.*

Proof. This can be proved from the following fact:

$$\sigma\left(\bigwedge_{n\in\mathbb{N}} A_n\right) = \sigma\left(\bigvee_{n\in\mathbb{N}} (A_n)'\right) \geq \bigwedge_{n\in\mathbb{N}} \sigma((A_n)') = \bigwedge_{n\in\mathbb{N}} \sigma(A_n). \tag{2.4}$$

The next two theorems give characterizations of an (L, M)-fuzzy σ-algebra. □

Theorem 2.7. *A mapping $\sigma : L^X \to M$ is an (L, M)-fuzzy σ-algebra if and only if for each $a \in M \setminus \{\perp_M\}$, $\sigma_{[a]}$ is an L-σ-algebra.*

Proof. The proof is obvious and is omitted. □

Corollary 2.8. *A mapping $\sigma : \mathbf{2}^X \to M$ is an M-fuzzifying σ-algebra if and only if for each $a \in M \setminus \{\perp_M\}$, $\sigma_{[a]}$ is a σ-algebra.*

Theorem 2.9. *If M is completely distributive, then a mapping $\sigma : L^X \to M$ is an (L, M)-fuzzy σ-algebra if and only if for each $a \in \alpha(\perp_M)$, $\sigma^{[a]}$ is an L-σ-algebra.*

Proof.

Necessity. Suppose that $\sigma : L^X \to M$ is an (L, M)-fuzzy σ-algebra and $a \in \alpha(\perp_M)$. Now we prove that $\sigma^{[a]}$ is an L-σ-algebra.

(LS1) By $\sigma(\chi_\emptyset) = \top_M$ and $\alpha(\top_M) = \emptyset$, we know that $a \notin \alpha(\sigma(\chi_\emptyset))$; this implies that $\chi_\emptyset \in \sigma^{[a]}$.

(LS2) If $A \in \sigma^{[a]}$, then $a \notin \alpha(\sigma(A)) = \alpha(\sigma(A'))$; this shows that $A' \in \sigma^{[a]}$.

(LS3) If $\{A_i : i \in \Omega\} \subseteq \sigma^{[a]}$, then for all $i \in \Omega$, $a \notin \alpha(\sigma(A_i))$. Hence $a \notin \bigcup_{i\in\Omega} \alpha(\sigma(A_i))$. By $\sigma(\bigvee_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \sigma(A_i)$, we know that

$$\alpha\left(\sigma\left(\bigvee_{i\in\Omega} A_i\right)\right) \subseteq \alpha\left(\bigwedge_{i\in\Omega} \sigma(A_i)\right) = \bigcup_{i\in\Omega} \alpha(\sigma(A_i)). \tag{2.5}$$

This shows that $a \notin \alpha(\sigma(\bigvee_{i\in\Omega} A_i))$. Therefore, $\bigvee_{i\in\Omega} A_i \in \sigma^{[a]}$. The proof is completed. □

Corollary 2.10. *If M is completely distributive, then a mapping $\sigma : \mathbf{2}^X \to M$ is an M-fuzzifying σ-algebra if and only if for each $a \in \alpha(\perp_M)$, $\sigma^{[a]}$ is a σ-algebra.*

Now we consider the conditions that a family of L-σ-algebras forms an (L, M)-fuzzy σ-algebra. By Theorem 1.2, we can obtain the following result.

Corollary 2.11. *If M is completely distributive, and σ is an (L, M)-fuzzy σ-algebra, then*

(1) $\sigma_{[b]} \subseteq \sigma_{[a]}$ *for any* $a, b \in M \setminus \{\perp_M\}$ *with* $a \in \beta(b)$;

(2) $\sigma^{[b]} \subseteq \sigma^{[a]}$ *for any* $a, b \in \alpha(\perp_M)$ *with* $b \in \alpha(a)$.

Theorem 2.12. *Let M be completely distributive, and let $\{\sigma^a : a \in \alpha(\perp_M)\}$ be a family of L-σ-algebras. If $\sigma^a = \bigcap\{\sigma^b : a \in \alpha(b)\}$ for all $a \in \alpha(\perp_M)$, then there exists an (L, M)-fuzzy σ-algebra σ such that $\sigma^{[a]} = \sigma^a$.*

Proof. Suppose that $\sigma^a = \bigcap\{\sigma^b : a \in \alpha(b)\}$ for all $a \in \alpha(\perp_M)$. Define $\sigma : L^X \to M$ by

$$\sigma(A) = \bigwedge_{a \in M} (a \vee \sigma^a(A)) = \bigwedge\{a \in M : A \notin \sigma^a\}. \tag{2.6}$$

By Theorem 1.2, we can obtain that $\sigma^{[a]} = \sigma^a$. □

Corollary 2.13. *Let M be completely distributive, and let $\{\sigma^a : a \in \alpha(\perp_M)\}$ be a family of σ-algebras. If $\sigma^a = \bigcap\{\sigma^b : a \in \alpha(b)\}$ for all $a \in \alpha(\perp_M)$, then there exists an M-fuzzifying σ-algebra σ such that $\sigma^{[a]} = \sigma^a$.*

Theorem 2.14. *Let M be completely distributive, and let $\{\sigma_a : a \in M \setminus \{\perp_M\}\}$ be a family of L-σ-algebra. If $\sigma_a = \bigcap\{\sigma_b : b \in \beta(a)\}$ for all $a \in M \setminus \{\perp_M\}$, then there exists an (L, M)-fuzzy σ-algebra σ such that $\sigma_{[a]} = \sigma_a$.*

Proof. Suppose that $\sigma_a = \bigcap\{\sigma_b : b \in \beta(a)\}$ for all $a \in M \setminus \{\perp_M\}$. Define $\sigma : L^X \to M$ by

$$\sigma(A) = \bigvee_{a \in M} (a \wedge \sigma_a(A)) = \bigvee\{a \in M : A \in \sigma_a\}. \tag{2.7}$$

By Theorem 1.2, we can obtain $\sigma_{[a]} = \sigma_a$. □

Corollary 2.15. *Let M be completely distributive, and let $\{\sigma_a : a \in M \setminus \{\perp_M\}\}$ be a family of σ-algebra. If $\sigma_a = \bigcap\{\sigma_b : b \in \beta(a)\}$ for all $a \in M \setminus \{\perp_M\}$, then there exists an M-fuzzifying σ-algebra σ such that $\sigma_{[a]} = \sigma_a$.*

Theorem 2.16. *Let $\{\sigma_i : i \in \Omega\}$ be a family of (L, M)-fuzzy σ-algebra on X. Then $\underset{i \in \Omega}{\wedge} \sigma_i$ is an (L, M)-fuzzy σ-algebra on X, where $\bigwedge_{i \in \Omega} \sigma_i : L^X \to M$ is defined by $(\bigwedge_{i \in \Omega} \sigma_i)(A) = \bigwedge_{i \in \Omega} \sigma_i(A)$.*

Proof. This is straightforward. □

3. (L, M)-Fuzzy Measurable Functions

In this section, we will generalize the notion of measurable functions to fuzzy setting.

Theorem 3.1. *Let (Y, τ) be an (L, M)-fuzzy measurable space and $f : X \to Y$ a mapping. Define a mapping $f_L^{\leftarrow}(\tau) : L^X \to M$ by for all $A \in L^X$,*

$$f_L^{\leftarrow}(\tau)(A) = \bigvee \{\tau(B) : f_L^{\leftarrow}(B) = A\}, \quad \text{where } \forall x \in X,\ f_L^{\leftarrow}(B)(x) = B(f(x)). \tag{3.1}$$

Then $(X, f_L^{\leftarrow}(\tau))$ is an (L, M)-fuzzy measurable space.

Proof. **(LMS1)** holds from the following equality:

$$f_L^{\leftarrow}(\tau)(\chi_{\emptyset}) = \bigvee \{\tau(B) : f_L^{\leftarrow}(B) = \chi_{\emptyset}\} = \tau(\chi_{\emptyset}) = \top_M. \tag{3.2}$$

(LMS2) can be shown from the following fact: for all $A \in L^X$,

$$\begin{aligned} f_L^{\leftarrow}(\tau)(A) &= \bigvee \{\tau(B) : f_L^{\leftarrow}(B) = A\} \\ &= \bigvee \{\tau(B') : f_L^{\leftarrow}(B') = f_L^{\leftarrow}(B)' = A'\} \\ &= f_L^{\leftarrow}(\tau)(A'). \end{aligned} \tag{3.3}$$

(LMS3) for any $\{A_n : n \in \mathbb{N}\} \subseteq L^X$, by

$$\begin{aligned} f_L^{\leftarrow}(\tau)\left(\bigvee_{n\in\mathbb{N}} A_n\right) &= \bigvee \left\{\tau(B) : f_L^{\leftarrow}(B) = \bigvee_{n\in\mathbb{N}} A_n\right\} \\ &\geq \bigvee \left\{\tau\left(\bigvee_{n\in\mathbb{N}} B_n\right) : f_L^{\leftarrow}(B_n) = A_n\right\} \\ &\geq \bigwedge_{n\in\mathbb{N}} f_L^{\leftarrow}(\tau)(A_n) \end{aligned} \tag{3.4}$$

we can prove (LMS3). □

Definition 3.2. Let (X, σ) and (Y, τ) be (L, M)-fuzzy measurable spaces. A mapping $f : X \to Y$ is called (L, M)-fuzzy measurable if $\sigma(f_L^{\leftarrow}(B)) \geq \tau(B)$ for all $B \in L^Y$.

An $(L, \mathbf{2})$-fuzzy measurable mapping is called an L-measurable mapping, and a $(\mathbf{2}, M)$-fuzzy measurable mapping is called an M-fuzzifying measurable mapping.

Obviously a Klement fuzzy measurable mapping can be viewed as an $[0,1]$-measurable mapping.

The following theorem gives a characterization of (L, M)-fuzzy measurable mappings.

Theorem 3.3. *Let (X,σ) and (Y,τ) be two (L,M)-fuzzy measurable spaces. A mapping $f : X \to Y$ is (L,M)-fuzzy measurable if and only if $f_L^{\leftarrow}(\tau)(A) \leq \sigma(A)$ for all $A \in L^X$.*

Proof.

Necessity. If $f : X \to Y$ is (L,M)-fuzzy measurable, then $\sigma(f_L^{\leftarrow}(B)) \geq \tau(B)$ for all $B \in L^Y$. Hence for all $B \in L^Y$, we have

$$f_L^{\leftarrow}(\tau)(A) = \bigvee\{\tau(B) : f_L^{\leftarrow}(B) = A\} \\ \leq \bigvee\{\sigma(f_L^{\leftarrow}(B)) : f_L^{\leftarrow}(B) = A\} \\ = \sigma(A). \tag{3.5}$$

Sufficiency. If $f_L^{\leftarrow}(\tau)(A) \leq \sigma(A)$ for all $A \in L^X$, then $\tau(B) \leq f_L^{\leftarrow}(\tau)(f_L^{\leftarrow}(B)) \leq \sigma(f_L^{\leftarrow}(B))$ for all $B \in L^Y$; this shows that $f : X \to Y$ is (L,M)-fuzzy measurable. □

The next three theorems are trivial.

Theorem 3.4. *If $f : (X,\sigma) \to (Y,\tau)$ and $f : (Y,\tau) \to (Z,\rho)$ are (L,M)-fuzzy measurable, then $g \circ f : (X,\sigma) \to (Z,\rho)$ is (L,M)-fuzzy measurable.*

Theorem 3.5. *Let (X,σ) and (Y,τ) be (L,M)-fuzzy measurable spaces. Then a mapping $f : (X,\sigma) \to (Y,\tau)$ is (L,M)-fuzzy measurable if and only if $f : (X,\sigma_{[a]}) \to (Y,\tau_{[a]})$ is L-measurable for any $a \in M \setminus \{\perp_M\}$.*

Theorem 3.6. *Let M be completely distributive, and let (X,σ) and (Y,τ) be (L,M)-fuzzy measurable spaces. Then a mapping $f : (X,\sigma) \to (Y,\tau)$ is (L,M)-fuzzy measurable if and only if $f : (X,\sigma^{[a]}) \to (Y,\tau^{[a]})$ is L-measurable for any $a \in \alpha(\perp_M)$.*

Corollary 3.7. *Let (X,σ) and (Y,τ) be M-fuzzifying measurable spaces. Then a mapping $f : (X,\sigma) \to (Y,\tau)$ is M-fuzzifying measurable if and only if $f : (X,\sigma_{[a]}) \to (Y,\tau_{[a]})$ is measurable for any $a \in M \setminus \{\perp_M\}$.*

Corollary 3.8. *Let M be completely distributive, and let (X,σ) and (Y,τ) be M-fuzzifying measurable spaces. Then a mapping $f : (X,\sigma) \to (Y,\tau)$ is M-fuzzifying measurable if and only if $f : (X,\sigma^{[a]}) \to (Y,\tau^{[a]})$ is measurable for any $a \in \alpha(\perp_M)$.*

4. (I,I)-Fuzzy σ-Algebras Generated by I-Fuzzifying σ-Algebras

In this section, $\mathcal{B}$ will be used to denote the σ-algebra of Borel subsets of $I = [0,1]$.

Theorem 4.1. *Let (X,σ) be an I-fuzzifying measurable space. Define a mapping $\zeta(\sigma) : I^X \to I$ by*

$$\zeta(\sigma)(A) = \bigwedge_{B\in\mathcal{B}} \sigma\Big(A^{-1}(B)\Big). \tag{4.1}$$

Then $\zeta(\sigma)$ is a stratified (I,I)-fuzzy σ-algebra, which is said to be the (I,I)-fuzzy σ-algebra generated by σ.

Proof. **(LMS1)** For any $B \in \mathcal{B}$ and for any $a \in I$, if $a \in B$, then $(a \wedge \chi_X)^{-1}(B) = X$; if $a \notin B$, then $(a \wedge \chi_X)^{-1}(B) = \emptyset$. However, we have that $\sigma((a \wedge \chi_X)^{-1}(B)) = 1$. This shows that $\zeta(\sigma)(a \wedge \chi_X) = 1$.

(LMS2) for all $A \in I^X$ and for all $B \in \mathcal{B}$, we have

$$\begin{aligned}
\zeta(\sigma)(A') &= \bigwedge_{B\in\mathcal{B}} \sigma\left((1-A)^{-1}(B)\right) \\
&= \bigwedge_{B\in\mathcal{B}} \sigma(\{x \in X : 1 - A(x) \in B\}) \\
&= \bigwedge_{B\in\mathcal{B}} \sigma(\{x \in X : \exists b \in B, \text{ s.t. } A(x) = 1 - b\}) \\
&= \bigwedge_{B\in\mathcal{B}} \sigma\left(A^{-1}(B)\right) \\
&= \zeta(\sigma)(A).
\end{aligned} \tag{4.2}$$

(LMS3) for any $\{A_n : n \in \mathbb{N}\} \subseteq L^X$ and for all $B \in \mathcal{B}$, by

$$\begin{aligned}
\zeta(\sigma)\left(\bigvee_{n\in\mathbb{N}} A_n\right) &= \bigwedge_{B\in\mathcal{B}} \sigma\left(\left(\bigvee_{n\in\mathbb{N}} A_n\right)^{-1}(B)\right) \\
&= \bigwedge_{B\in\mathcal{B}} \sigma\left(\bigcup_{n\in\mathbb{N}} A_n^{-1}(B)\right) \\
&\geq \bigwedge_{B\in\mathcal{B}} \bigwedge_{n\in\mathbb{N}} \sigma\left(A_n^{-1}(B)\right) \\
&= \bigwedge_{n\in\mathbb{N}} \bigwedge_{B\in\mathcal{B}} \sigma\left(A_n^{-1}(B)\right) = \bigwedge_{n\in\mathbb{N}} \zeta(\sigma)(A_n),
\end{aligned} \tag{4.3}$$

we obtain $\zeta(\sigma)(\bigvee_{n\in\mathbb{N}} A_n) \geq \bigwedge_{n\in\mathbb{N}} \zeta(\sigma)(A_n)$. □

Corollary 4.2. *Let (X,σ) be a measurable space. Define a subset $\zeta(\sigma) \subseteq I^X$(can be viewed as a mapping $\zeta(\sigma) : I^X \to \mathbf{2}$) by*

$$\zeta(\sigma) = \left\{A \in I^X : \forall B \in \mathcal{B}, A^{-1}(B) \in \sigma\right\}. \tag{4.4}$$

Then $\zeta(\sigma)$ is a stratified I-σ-algebra.

From Corollary 4.2, we see that the functor ζ in Theorem 4.1 is a generalization of Klement functor ζ.

Theorem 4.3. *Let (X,σ) and (Y,τ) be two I-fuzzifying measurable spaces, and $f : X \to Y$ is a map. Then $f : (X,\sigma) \to (Y,\tau)$ is I-fuzzifying measurable if and only if $f : (X,\zeta(\sigma)) \to (Y,\zeta(\tau))$ is (I,I)-fuzzy measurable.*

Proof.

Necessity. Suppose that $f : (X,\sigma) \to (Y,\tau)$ is I-fuzzifying measurable. Then $\sigma(f^{-1}(A)) \geq \tau(A)$ for any $A \in \mathbf{2}^X$. In order to prove that $f : (X,\zeta(\sigma)) \to (Y,\zeta(\tau))$ is (I,I)-fuzzy measurable, we need to prove that $\zeta(\sigma)(f_L^{\leftarrow}(A)) \geq \zeta(\tau)(A)$ for any $A \in I^X$.

In fact, for any $A \in I^X$, by

$$\begin{aligned} \zeta(\sigma)\left(f_L^{\leftarrow}(A)\right) &= \bigwedge_{B\in\mathcal{B}} \sigma\left(\left(f_L^{\leftarrow}(A)\right)^{-1}(B)\right) = \bigwedge_{B\in\mathcal{B}} \sigma\left((A\circ f)^{-1}(B)\right) \\ &= \bigwedge_{B\in\mathcal{B}} \sigma(B\circ A\circ f) = \bigwedge_{B\in\mathcal{B}} \sigma\left(f^{-1}\left(A^{-1}(B)\right)\right) \\ &\geq \bigwedge_{B\in\mathcal{B}} \tau\left(A^{-1}(B)\right) = \zeta(\tau)(A), \end{aligned} \tag{4.5}$$

we can prove the necessity.

Sufficiency. Suppose that $f : (X,\zeta(\sigma)) \to (Y,\zeta(\tau))$ is (I,I)-fuzzy measurable. Then $\zeta(\sigma)(f_I^{\leftarrow}(A)) \geq \zeta(\tau)(A)$ for any $A \in I^X$. In particular, it follows that $\zeta(\sigma)(f_I^{\leftarrow}(A)) \geq \zeta(\tau)(A)$ for any $A \in \mathbf{2}^X$. In order to prove that $f : (X,\sigma) \to (Y,\tau)$ is I-fuzzifying measurable, we need to prove that $\sigma(f^{-1}(A)) \geq \tau(A)$ for any $A \in \mathbf{2}^X$. In fact, for any $A \in \mathbf{2}^X$ and for any $B \in \mathcal{B}$, if $0,1 \in B$, then $A^{-1}(B) = X$; if $0,1 \notin B$, then $A^{-1}(B) = \emptyset$; if only one of 0 and 1 is in B, then $A^{-1}(B) = A$ or $A^{-1}(B) = A'$. However, we have

$$\begin{aligned} \sigma\left(f_I^{\leftarrow}(A)\right) &= \sigma\left(f_I^{\leftarrow}(A)\right) \\ &= \sigma\left(f_I^{\leftarrow}(A)\right) \wedge \sigma\left(f_I^{\leftarrow}(A)'\right) \\ &= \bigwedge_{B\in\mathcal{B}} \sigma\left(\left(f_L^{\leftarrow}(A)\right)^{-1}(B)\right) \\ &= \zeta(\sigma)\left(f_L^{\leftarrow}(A)\right) \\ &\geq \zeta(\tau)(A) \\ &= \zeta(\tau)(A) \wedge \zeta(\tau)\left(A'\right) \\ &= \bigwedge_{B\in\mathcal{B}} \tau\left(A^{-1}(B)\right) = \tau(A). \end{aligned} \tag{4.6}$$

This shows that $f : (X,\sigma) \to (Y,\tau)$ is I-fuzzifying measurable. □

Corollary 4.4. *Let (X,σ) and (Y,τ) be two measurable spaces, and $f : X \to Y$ is a mapping. Then $f : (X,\sigma) \to (Y,\tau)$ is measurable if and only if $f : (X,\zeta(\sigma)) \to (Y,\zeta(\tau))$ is I-measurable.*

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