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Research Article

On Subclass of Analytic Univalent Functions Associated with Negative Coefficients

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M. H. Al-Abbadi and M. Darus (2009) recently introduced a new generalized derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$, which generalized many well-known operators studied earlier by many different authors. In this present paper, we shall investigate a new subclass of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ which is defined by new generalized derivative operator. Some results on coefficient inequalities, growth and distortion theorems, closure theorems, and extreme points of analytic functions belonging to the subclass are obtained.

1. Introduction and Definitions

Let $\mathcal{A}(x)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=x+1}^{\infty} a_k z^k, \quad a_k \text{ is complex number}, \tag{1.1}$$

and $x \in \mathbb{N} = \{1, 2, 3, \ldots\}$, which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$; note that $\mathcal{A}(1) = \mathcal{A}$ and $\mathcal{A}(x) \subseteq \mathcal{A}(1)$. Suppose that $\mathcal{S}(x)$ denote the subclass of $\mathcal{A}(x)$ consisting of functions that are univalent in U. Further, let $\mathcal{S}_\alpha^*(x)$ and $\mathcal{C}_\alpha(x)$ be the classes of $\mathcal{S}(x)$ consisting of functions, respectively, starlike of order α and convex of order α in U, for $0 \le \alpha < 1$. Let $\mathcal{T}(x)$ denote the subclass of $\mathcal{S}(x)$ consisting of functions of the form

$$f(z) = z - \sum_{k=x+1}^{\infty} |a_k| z^k, \quad x \in \mathbb{N} = \{1,2,3,\ldots\}, \tag{1.2}$$

defined on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{T}(x)$ is called a function with negative coefficient and the class $\mathcal{T}(1)$ was introduced and studied by Silverman [1]. In [1] Silverman investigated the subclasses of $\mathcal{T}(1)$ denoted by $S^*_{\mathcal{T}}(\alpha)$ and $C_{\mathcal{T}}(\alpha)$ for $0 \leq \alpha < 1$. That are, respectively, starlike of order α and convex of order α. Now $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C} \setminus \{0\}, \\ x(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1,2,\ldots\},\ x \in \mathbb{C}. \end{cases} \tag{1.3}$$

The authors in [2] have recently introduced a new generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ as follows.

Definition 1.1. For $f \in \mathcal{A} = \mathcal{A}(1)$, the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2} : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m}} c(n,k) a_k z^k, \quad (z \in U), \tag{1.4}$$

where $n, m \in \mathbb{N}_0 = \{0,1,2,\ldots\}, \lambda_2 \geq \lambda_1 \geq 0$, and $c(n,k) = \binom{n+k-1}{n} = (n+1)_{k-1}/(1)_{k-1}$.

(1) Special cases of this operator include the Ruscheweyh derivative operator in the cases $\mu^{n,1}_{\lambda_1,0} \equiv \mu^{n,m}_{0,0} \equiv \mu^{n,0}_{0,\lambda_2} \equiv R^n$ [3], the Salagean derivative operator $\mu^{0,m+1}_{1,0} \equiv S^n$ [4], the generalized Ruscheweyh derivative operator $\mu^{n,2}_{\lambda_1,0} \equiv R^n_{\lambda}$ [5], the generalized Salagean derivative operator introduced by Al-Oboudi $\mu^{0,m+1}_{\lambda_1,0} \equiv S^n_{\beta}$ [6], and the generalized Al-Shaqsi and Darus derivative operator $\mu^{\lambda,n+1}_{\beta,0} \equiv D^n_{\lambda,\beta}$ where ($n = \lambda, m = n+1, \lambda_1 = \beta$, and $\lambda_2 = 0$) can be found in [7]. It is easily seen that $\mu^{0,1}_{\lambda_1,0} f(z) = \mu^{0,m}_{0,0} f(z) = \mu^{0,0}_{0,\lambda_2} f(z) = \mu^{1,1}_{\lambda_1,1} f(z) = f(z)$, $\mu^{1,1}_{\lambda_1,0} f(z) = \mu^{1,m}_{0,0} f(z) = \mu^{1,0}_{0,\lambda_2} f(z) = \mu^{0,2}_{1,0} f(z) = z f'(z)$, and also $\mu^{n-1,0}_{\lambda_1,0} f(z) = \mu^{n-1,m}_{0,0} f(z)$ where $n = 1,2,3,\ldots$.

By making use of the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$, the authors introduce a new subclass as follows.

Definition 1.2. For $0 \leq \alpha < 1$, $(n, m \in \mathbb{N}_0 = \{0,1,2,\ldots\})$ and $\lambda_2 \geq \lambda_1 \geq 0$, let $\mathcal{H}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ be the subclass of $\mathcal{S}(x)$ consisting of functions f satisfying

$$\operatorname{Re}\left(\frac{z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)'}{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)} \right) > \alpha, \quad (z \in U), \tag{1.5}$$

where

$$\mu_{\lambda_1,\lambda_2}^{n,m} f(z) = z + \sum_{k=x+1}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k) a_k z^k, \tag{1.6}$$

$\{x = 1,2,3,\ldots,(z \in U)\}$, and $\mu_{\lambda_1,\lambda_2}^{n,m} f(z) \neq 0$.

Further, we define the class $\mathcal{T}\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$ by

$$\mathcal{T}\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha) = \mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha) \cap \mathcal{T}(x), \quad \{x = 1,2,3,\ldots\}, \tag{1.7}$$

for $0 \leq \alpha < 1$, $(n,m \in \mathbb{N}_0 = \{0,1,2,\ldots\})$, and $\lambda_2 \geq \lambda_1 \geq 0$.

Also note that various subclasses of $\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$ and $\mathcal{T}\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$ have been studied by many authors by suitable choices of $n, m, \lambda_1, \lambda_2$, and x. For example,

$$\mathcal{T}\mathcal{H}_{0,\lambda_2}^{0,0}(1,\alpha) \equiv \mathcal{T}\mathcal{H}_{\lambda_1,0}^{0,1}(1,\alpha) \equiv \mathcal{T}\mathcal{H}_{0,0}^{0,m}(1,\alpha) \equiv \mathcal{T}\mathcal{H}_{\lambda_1,1}^{1,1}(1,\alpha) \equiv S_{\mathcal{T}}^{*}(\alpha), \tag{1.8}$$

starlike of order α with negative coefficients. And

$$\mathcal{T}\mathcal{H}_{0,\lambda_2}^{1,0}(1,\alpha) \equiv \mathcal{T}\mathcal{H}_{\lambda_1,0}^{1,1}(1,\alpha) \equiv \mathcal{T}\mathcal{H}_{0,0}^{1,m}(1,\alpha) \equiv \mathcal{T}\mathcal{H}_{1,0}^{0,2}(1,\alpha) \equiv C_{\mathcal{T}}(\alpha), \tag{1.9}$$

class of convex function of order α with negative coefficients. Also

$$\begin{aligned}
&\mathcal{T}\mathcal{H}_{0,\lambda_2}^{0,0}(x,\alpha) \equiv \mathcal{T}\mathcal{H}_{\lambda_1,0}^{0,1}(x,\alpha) \equiv \mathcal{T}\mathcal{H}_{0,0}^{0,m}(x,\alpha) \equiv \mathcal{T}\mathcal{H}_{\lambda_1,1}^{1,1}(x,\alpha) \equiv S_{\mathcal{T}}^{*}(x,\alpha), \\
&\mathcal{T}\mathcal{H}_{0,\lambda_2}^{1,0}(x,\alpha) \equiv \mathcal{T}\mathcal{H}_{\lambda_1,0}^{1,1}(x,\alpha) \equiv \mathcal{T}\mathcal{H}_{0,0}^{1,m}(x,\alpha) \equiv \mathcal{T}\mathcal{H}_{1,0}^{0,2}(x,\alpha) \equiv C_{\mathcal{T}}(x,\alpha), \\
&\mathcal{T}\mathcal{H}_{\lambda_1,0}^{0,2}(x,\alpha) \equiv P(x,\lambda_1,\alpha) \quad (0 \leq \lambda_1 < 1), \\
&\mathcal{T}\mathcal{H}_{\lambda_1,0}^{1,2}(x,\alpha) \equiv C(x,\lambda_1,\alpha) \quad (0 \leq \lambda_1 < 1).
\end{aligned} \tag{1.10}$$

The classes $S_{\mathcal{T}}^{*}(x,\alpha)$ and $C_{\mathcal{T}}(x,\alpha)$ were studied by Chatterjea [8] (see also Srivastava et al. [9]), whereas the classes $P(x,\lambda_1,\alpha)$ and $C(x,\lambda_1,\alpha)$ were, respectively, studied by Altintaş [10] and Kamali and Akbulut [11]. When $\lambda_1 = \lambda_2 = 0$ or $m = 1, \lambda_2 = 0$, or $m = 0, \lambda_1 = 0$ in the class $\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$, we have the class $R_n(\alpha)$ introduced and studied by Ahuja [12]. Finally we note that when $m = 2, \lambda_2 = 0$ in the class $\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$ we have the class $\mathcal{K}_{\lambda}^{n}(x,\alpha)$ introduced and studied by Al-Shaqsi and Darus [13].

2. Coefficient Inequalities

In this section, we provide a necessary and sufficient condition for a function f analytic in U to be in $\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$ and in $\mathcal{T}\mathcal{H}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$.

Theorem 2.1. *For* $0 \leq \alpha < 1$ *and* $\lambda_2 \geq \lambda_1 \geq 0$, *let* $f \in \mathcal{S}(x)$ *be defined by* (1.1). *If*

$$\sum_{k=x+1}^{\infty} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k)|a_k| \leq 1-\alpha, \quad x = 1,2,\ldots, \tag{2.1}$$

then $f \in \mathcal{H}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$, *where* $n \in \mathbb{N} = \{1,2,\ldots\}$ *and* $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$.

Proof. Assume that (2.1) holds true. Then we shall prove condition (1.5). It is sufficient to show that

$$\left| \frac{z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)'}{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)} - 1 \right| \leq 1-\alpha, \quad (z \in U). \tag{2.2}$$

So, we have that

$$\begin{aligned}
\left| \frac{z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)'}{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)} - 1 \right| &= \left| \frac{z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' - \mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)} \right| \\
&= \left| \frac{\sum_{k=x+1}^{\infty}\left((k-1)(1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)a_k z^k}{z + \sum_{k=x+1}^{\infty}\left((1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)a_k z^k} \right| \\
&\qquad |z| < 1, \\
&\leq \frac{\sum_{k=x+1}^{\infty}\left((k-1)(1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)|a_k|}{1 - \sum_{k=x+1}^{\infty}\left((1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)|a_k|},
\end{aligned} \tag{2.3}$$

and expression (2.3) is bounded by $(1-\alpha)$.

Hence (2.2) holds if

$$\begin{aligned}
&\sum_{k=x+1}^{\infty} \frac{(k-1)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k)|a_k| \\
&\quad \leq (1-\alpha)\left[1 - \sum_{k=x+1}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k)|a_k|\right],
\end{aligned} \tag{2.4}$$

which is equivalent to

$$\sum_{k=x+1}^{\infty} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k)|a_k| \leq (1-\alpha), \tag{2.5}$$

by (2.1). Thus $f \in \mathcal{H}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$. Note that the denominator in (2.3) is positive provided that (2.1) holds. □

Theorem 2.2. *Let f be defined by* (1.2) *and* $1-\sum_{k=x+1}^{\infty}(k(1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m)c(n,k)|a_k| \geq 0$ $(x=1,2,\ldots)$. *Then $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ if and only if* (2.1) *is satisfied.*

Proof. We only prove the right-hand side, since the other side can be justified using similar arguments in proof of Theorem 2.1. Since $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ by condition (1.5), we have that

$$\begin{aligned} &\operatorname{Re}\left(\frac{z\left(\mu^{n,m}_{\lambda_1,\lambda_2}f(z)\right)'}{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}\right) \\ &\quad=\operatorname{Re}\left\{\frac{z-\sum_{k=x+1}^{\infty}\left(k(1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)|a_k|z^k}{z-\sum_{k=x+1}^{\infty}\left((1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)|a_k|z^k}\right\}>\alpha. \end{aligned} \tag{2.6}$$

Choose values of z on real axis so that $z(\mu^{n,m}_{\lambda_1,\lambda_2}f(z))'/\mu^{n,m}_{\lambda_1,\lambda_2}f(z)$ is real. Letting $z \to 1^-$ through real values, we have that

$$\frac{1-\sum_{k=x+1}^{\infty}\left(k(1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)|a_k|z^k}{1-\sum_{k=x+1}^{\infty}\left((1+\lambda_1(k-1))^{m-1}/(1+\lambda_2(k-1))^m\right)c(n,k)|a_k|z^k}>\alpha. \tag{2.7}$$

Thus we obtain

$$\sum_{k=x+1}^{\infty}\frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}c(n,k)|a_k| \leq 1-\alpha, \tag{2.8}$$

which is (2.1). Hence the proof is complete. □

The result is sharp with the extremal function f given by

$$f(z)=z-\frac{(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1}c(n,x+1)}z^{x+1}, \quad (x=1,2,\ldots). \tag{2.9}$$

Theorem 2.3. *Let the function f given by* (1.2) *be in the class $\mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$. Then*

$$|a_k| \leq \frac{(1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1}c(n,k)}, \tag{2.10}$$

where $0 \leq \alpha < 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $k \geq x+1$, and $x=1,2,3,\ldots$. Equality holds for the function given by (2.9).

Proof. Since $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$, then condition (2.1) gives

$$|a_k| \le \frac{(1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1}c(n,k)}, \tag{2.11}$$

for each $k = x+1$ where $x = 1,2,3,\ldots$.

Clearly the function given by (2.9) satisfies (2.10), and therefore, f given by (2.9) is in $\mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ for this function; the result is clearly sharp. □

3. Growth and Distortion Theorems

In this section, growth and distortion theorems will be considered and covering property for function in the class will also be given.

Theorem 3.1. *Let the function f given by* (1.2) *be in the class $\mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$. Then for $0 < |z| = r < 1$,*

$$\begin{aligned} &r - \frac{(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1}c(n,x+1)}r^{x+1} \\ &\le |f(z)| \le r + \frac{(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1}c(n,x+1)}r^{x+1}, \quad x = 1,2,\ldots, \end{aligned} \tag{3.1}$$

where $0 \le \alpha < 1$, $m \in \mathbb{N}_0 = \{0,1,2\ldots\}$, and $n \in \mathbb{N} = \{1,2,\ldots\}$.

Proof. We only prove the right-hand side inequality in (3.1), since the other inequality can be justified using similar arguments. Since $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ by Theorem 2.2, we have that

$$\sum_{k=x+1}^{\infty} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}c(n,k)|a_k| \le (1-\alpha). \tag{3.2}$$

Now

$$\begin{aligned} &\frac{(x+1-\alpha)(1+\lambda_1 x)^{m-1}}{(1+\lambda_2 x)^m}c(n,x+1)\left(\sum_{k=x+1}^{\infty}|a_k|\right) \\ &= \sum_{k=x+1}^{\infty}\frac{(x+1-\alpha)(1+\lambda_1 x)^{m-1}}{(1+\lambda_2 x)^m}c(n,x+1)|a_k|, \\ &\le \sum_{k=x+1}^{\infty}\frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}c(n,k)|a_k|, \\ &\le 1-\alpha, \quad x = 1,2,3,\ldots. \end{aligned} \tag{3.3}$$

And therefore,

$$\sum_{k=x+1}^{\infty} |a_k| \leq \frac{(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1} c(n,x+1)}, \quad x = 1,2,\ldots. \tag{3.4}$$

Since

$$f(z) = z - \sum_{k=x+1}^{\infty} |a_k| z^k, \quad x = 1,2,\ldots, \tag{3.5}$$

then we have that

$$\left|f(z)\right| = \left|z - \sum_{k=x+1}^{\infty} |a_k| z^k\right|. \tag{3.6}$$

After that,

$$\begin{aligned} \left|f(z)\right| &\leq |z| + |z|^{x+1} \sum_{k=x+1}^{\infty} |a_k||z|^{k-(x+1)} \\ &\leq r + r^{x+1} \sum_{k=x+1}^{\infty} |a_k|. \end{aligned} \tag{3.7}$$

By aid of inequality (3.4), it yields the right-hand side inequality of (3.1). Thus, this completes the proof. □

Theorem 3.2. *Let the function f given by* (1.2) *be in the class $\mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$. Then for $0 < |z| = r < 1$,*

$$\begin{aligned} 1 &- \frac{(x+1)(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1} c(n,x+1)} r^x \\ &\leq \left|f'(z)\right| \leq 1 + \frac{(x+1)(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1} c(n,x+1)} r^x, \quad x = 1,2,\ldots, \end{aligned} \tag{3.8}$$

where $0 \leq \alpha < 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, *and* $n \in \mathbb{N} = \{1,2,\ldots\}$.

Proof. Since $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$, by Theorem 2.2, we have that

$$\sum_{k=x+1}^{\infty} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k)|a_k| \leq 1-\alpha, \quad x = 1,2,3,\ldots. \tag{3.9}$$

Now

$$\begin{aligned}
&\frac{(x+1-\alpha)(1+\lambda_1 x)^{m-1}}{(1+\lambda_2 x)^m}c(n,x+1)\left(\sum_{k=x+1}^{\infty}k|a_k|\right)\\
&\quad=\sum_{k=x+1}^{\infty}\frac{(x+1-\alpha)(1+\lambda_1 x)^{m-1}}{(1+\lambda_2 x)^m}c(n,x+1)k|a_k|\\
&\quad\le (x+1)\sum_{k=x+1}^{\infty}\frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}c(n,k)|a_k|\\
&\quad\le (x+1)(1-\alpha),\quad (k\ge x+1, x=1,2,3,\ldots).
\end{aligned}\tag{3.10}$$

Hence

$$\sum_{k=x+1}^{\infty}k|a_k|\le\frac{(x+1)(1-\alpha)(1+\lambda_2 x)^m}{(x+1-\alpha)(1+\lambda_1 x)^{m-1}c(n,x+1)},\quad x=1,2,\ldots.\tag{3.11}$$

Since

$$f'(z)=1-\sum_{k=x+1}^{\infty}k|a_k|z^{k-1},\quad x=1,2,\ldots,\tag{3.12}$$

then we have that

$$1-|z|^x\sum_{k=x+1}^{\infty}k|a_k||z|^{k-(x+1)}\le\left|f'(z)\right|\le 1+|z|^x\sum_{k=x+1}^{\infty}k|a_k||z|^{k-(x+1)},\tag{3.13}$$

and therefore,

$$1-r^x\sum_{k=x+1}^{\infty}k|a_k|\le\left|f'(z)\right|\le 1+r^x\sum_{k=x+1}^{\infty}k|a_k|,\quad x=1,2,\ldots.\tag{3.14}$$

By using the inequality (3.11) in (3.14), we get Theorem 3.2. This completes the proof. □

4. Extreme Points

The extreme points of the class $\mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ are given by the following theorem.

Theorem 4.1. *Let* $f_x(z)=z$ *and*

$$f_k(z)=z-\frac{(1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1}c(n,k)}z^k,\tag{4.1}$$

where $0\le\alpha<1$, $\lambda_2\ge\lambda_1\ge 0$, $(n,m\in\mathbb{N}_0)$, $k=x+1,x+2,\ldots,$ *and* $x=1,2,3,\ldots.$

Then $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=x}^{\infty} \delta_k f_k(z), \tag{4.2}$$

where $\delta_k \geq 0$ *and* $\sum_{k=x}^{\infty} \delta_k = 1$.

Proof. Suppose that f can be expressed as in (4.2). Our goal is to show that $f \in \mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$. By (4.2), we have that

$$\begin{aligned}
f(z) &= \sum_{k=x}^{\infty} \delta_k f_k(z) \\
&= \delta_x f_x(z) + \sum_{k=x+1}^{\infty} \delta_k f_k(z) \\
&= \delta_x f_x(z) + \sum_{k=x+1}^{\infty} \delta_k \left[z - \frac{(1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)} z^k \right] \\
&= \sum_{k=x}^{\infty} \delta_k z - \sum_{k=x+1}^{\infty} \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)} z^k \\
&= z - \sum_{k=x+1}^{\infty} \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)} z^k.
\end{aligned} \tag{4.3}$$

Now

$$f(z) = z - \sum_{k=x+1}^{\infty} |a_k| z^k = z - \sum_{k=x+1}^{\infty} \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)} z^k, \tag{4.4}$$

so that

$$|a_k| = \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)}. \tag{4.5}$$

Now, we have that

$$\sum_{k=x+1}^{\infty} \delta_k = 1 - \delta_x \leq 1, \quad x = 1, 2, 3, \ldots. \tag{4.6}$$

Setting

$$\sum_{k=x+1}^{\infty} \delta_k = \sum_{k=x+1}^{\infty} \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)}{(1-\alpha)(1+\lambda_2(k-1))^m} \leq 1, \tag{4.7}$$

we arrive to

$$\sum_{k=x+1}^{\infty} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1-\alpha)(1+\lambda_2(k-1))^m} c(n,k)|a_k| \le 1. \tag{4.8}$$

And therefore,

$$\sum_{k=x+1}^{\infty} \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k)|a_k| \le 1-\alpha, \quad x=1,2,3,\dots. \tag{4.9}$$

It follows from Theorem 2.2 that $f \in \mathcal{TH}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$.

Conversely, let us suppose that $f \in \mathcal{TH}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$; our goal is, to get (4.2). From (4.2) and using similar last arguments, it is easily seen that

$$f(z) = z - \sum_{k=x+1}^{\infty} |a_k| z^k = z - \sum_{k=x+1}^{\infty} \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)} z^k, \tag{4.10}$$

which suffices to show that

$$|a_k| = \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)}. \tag{4.11}$$

Now, we have that $f \in \mathcal{TH}_{\lambda_1,\lambda_2}^{n,m}(x,\alpha)$, then by previous Theorem 2.3,

$$|a_k| \le \frac{(1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)}. \tag{4.12}$$

That is

$$\frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)|a_k|}{(1-\alpha)(1+\lambda_2(k-1))^m} \le 1. \tag{4.13}$$

Since $\sum_{k=x}^{\infty} \delta_k = 1$, we see $\delta_k \le 1$, for each $k = x, x+1, x+2, \dots$, and $x = 1,2,3,\dots$.

We can set that

$$\delta_k = \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)|a_k|}{(1-\alpha)(1+\lambda_2(k-1))^m}. \tag{4.14}$$

Thus, the desired result is that

$$|a_k| = \frac{\delta_k (1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1} c(n,k)}. \tag{4.15}$$

This completes the proof of the theorem. □

Corollary 4.2. *The extreme points of* $\mathcal{TH}^{n,m}_{\lambda_1,\lambda_2}(x,\alpha)$ *are the functions*

$$f_x(z) = z,$$
$$f_k(z) = z - \frac{(1-\alpha)(1+\lambda_2(k-1))^m}{(k-\alpha)(1+\lambda_1(k-1))^{m-1}c(n,k)}z^k, \tag{4.16}$$

where $0 \le \alpha < 1$, $(n, m \in \mathbb{N}_0 = \{0,1,2,\ldots\})$, $\lambda_2 \ge \lambda_1 \ge 0$, *and* $k = x+1, x+2, \ldots, (x = 1,2,3,\ldots)$.

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References

[1] H. Silverman, "Univalent functions with negative coefficients," *Proceedings of the American Mathematical Society*, vol. 51, pp. 109–116, 1975.
[2] M. H. Al-Abbadi and M. Darus, "Differential subordination for new generalized derivative operator," *Acta Universitatis Apulensis*, no. 20, pp. 265–280, 2009.
[3] St. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.
[4] G. Š. Sălăgean, "Subclasses of univalent functions," in *Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, vol. 1013 of *Lecture Notes in Mathematics*, pp. 362–372, Springer, Berlin, Germany, 1983.
[5] K. Al-Shaqsi and M. Darus, "On univalent functions with respect to k-symmetric points defined by a generalized Ruscheweyh derivatives operator," *Journal of Analysis and Applications*, vol. 7, no. 1, pp. 53–61, 2009.
[6] F. M. Al-Oboudi, "On univalent functions defined by a generalized Sălăgean operator," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 27, pp. 1429–1436, 2004.
[7] M. Darus and K. Al-Shaqsi, "Differential sandwich theorems with generalised derivative operator," *International Journal of Computational and Mathematical Sciences*, vol. 2, no. 2, pp. 75–78, 2008.
[8] S. K. Chatterjea, "On starlike functions," *Journal of Pure Mathematics*, vol. 1, pp. 23–26, 1981.
[9] H. M. Srivastava, S. Owa, and S. K. Chatterjea, "A note on certain classes of starlike functions," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 77, pp. 115–124, 1987.
[10] O. Altıntaş, "On a subclass of certain starlike functions with negative coefficients," *Mathematica Japonica*, vol. 36, no. 3, pp. 489–495, 1991.
[11] M. Kamali and S. Akbulut, "On a subclass of certain convex functions with negative coefficients," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 341–350, 2003.
[12] O. P. Ahuja, "Integral operators of certain univalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 8, no. 4, pp. 653–662, 1985.
[13] K. Al Shaqsi and M. Darus, "On certain subclass of analytic univalent functions with negative coefficients," *Applied Mathematical Sciences*, vol. 1, no. 21–24, pp. 1121–1128, 2007.