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Research Article

Extensions of Certain Classical Summation Theorems for the Series ${}_2F_1$, ${}_3F_2$, and ${}_4F_3$ with Applications in Ramanujan's Summations

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Motivated by the extension of classical Gauss's summation theorem for the series ${}_2F_1$ given in the literature, the authors aim at presenting the extensions of various other classical summation theorems such as those of Kummer, Gauss's second, and Bailey for the series ${}_2F_1$, Watson, Dixon and Whipple for the series ${}_3F_2$, and a few other hypergeometric identities for the series ${}_3F_2$ and ${}_4F_3$. As applications, certain very interesting summations due to Ramanujan have been generalized. The results derived in this paper are simple, interesting, easily established, and may be useful.

1. Introduction

In 1812, Gauss [1] systematically discussed the series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)} z^2 + \cdots , \tag{1.1}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \end{cases} \tag{1.2}$$

It is noted that the series (1.1) and its natural generalization ${}_pF_q$ in (1.6) are of great importance to mathematicians and physicists. This series (1.1) has been known as the Gauss series or the ordinary hypergeometric series and may be regarded as a generalization of the elementary geometric series. In fact (1.1) reduces to the geometric series in two cases, when $a = c$ and $b = 1$ also when $b = c$ and $a = 1$. The series (1.1) is represented by the notation ${}_2F_1[a, b; c; z]$ or

$$ {}_2F_1\begin{bmatrix} a, \ b & \\ & ; \ z \\ c & \end{bmatrix}, \tag{1.3}$$

which is usually referred to as Gauss hypergeometric function. In (1.1), the three elements a, b, and c are described as the parameters of the series, and z is called the variable of the series. All four of these quantities may be real or complex with an exception that c is neither zero nor a negative integer. Also, in (1.1), it is easy to see that if any one of the numerator parameters a or b or both is a negative integer, then the series reduces to a polynomials, that is, the series terminates.

The series (1.1) is absolutely convergent within the unit circle when $|z| < 1$ provided that $c \neq 0, -1, -2, \ldots$. Also when $|z| = 1$, the series is absolutely convergent if $\Re(c - a - b) > 0$, conditionally convergent if $-1 < \Re(c - a - b) \leq 0$, $z \neq 1$ and divergent if $\Re(c - a - b) \leq -1$.

Further, if in (1.1), we replace z by z/b and let $b \to \infty$, then $((b)_n z^n / b^n) \to z^n$, and we arrive to the following Kummer's series

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a}{1 \cdot c} z + \frac{a(a+1)}{1 \cdot 2c(c+1)} z^2 + \cdots . \tag{1.4}$$

This series is absolutely convergent for all values of a, c, and z, real or complex, excluding $c = 0, -1, -2, \ldots$ and is represented by the notation ${}_1F_1(a; c; z)$ or

$$ {}_1F_1\begin{bmatrix} a, & \\ & ; \ z \\ c & \end{bmatrix}, \tag{1.5}$$

which is called a confluent hypergeometric function.

Gauss hypergeometric function ${}_2F_1$ and its confluent case ${}_1F_1$ form the core special functions and include, as their special cases, most of the commonly used functions. Thus ${}_2F_1$ includes, as its special cases, Legendre function, the incomplete beta function, the complete elliptic functions of first and second kinds, and most of the classical orthogonal polynomials. On the other hand, the confluent hypergeometric function includes, as its special cases, Bessel functions, parabolic cylindrical functions, and Coulomb wave function.

Also, the Whittaker functions are slightly modified forms of confluent hypergeometric functions. On account of their usefulness, the functions ${}_2F_1$ and ${}_1F_1$ have already been explored to considerable extent by a number of eminent mathematicians, for example, C. F. Gauss, E. E. Kummer, S. Pincherle, H. Mellin, E. W. Barnes, L. J. Slater, Y. L. Luke, A. Erdélyi, and H. Exton.

A natural generalization of ${}_2F_1$ is the generalized hypergeometric series ${}_pF_q$ defined by

$$
{}_pF_q\begin{bmatrix} a_1 & \cdots & a_p & \\ & & & ;\ z \\ b_1 & \cdots & b_q & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{1.6}
$$

The series (1.6) is convergent for all $|z| < \infty$ if $p \le q$ and for $|z| < 1$ if $p = q + 1$ while it is divergent for all z, $z \neq 0$ if $p > q + 1$. When $|z| = 1$ with $p = q + 1$, the series (1.6) converges absolutely if

$$
\Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) > 0, \tag{1.7}
$$

conditionally convergent if

$$
-1 < \Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) \le 0, \quad z \neq 1 \tag{1.8}
$$

and divergent if

$$
\Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) \le -1. \tag{1.9}
$$

It should be remarked here that whenever hypergeometric and generalized hypergeometric functions can be summed to be expressed in terms of Gamma functions, the results are very important from a theoretical and an applicable point of view. Only a few summation theorems are available in the literature and it is well known that the classical summation theorems such as of Gauss, Gauss's second, Kummer, and Bailey for the series ${}_2F_1$, and Watson, Dixon, and Whipple for the series ${}_3F_2$ play an important role in the theory of generalized hypergeometric series. It has been pointed out by Berndt [2], that very interesting summations due to Ramanujan can be obtained quite simply by employing the above mentioned classical summation theorems. Also, in a well-known paper by Bailey [3], a large number of very interesting results involving products of generalized hypergeometric series have been developed. In [4] a generalization of Kummer's second theorem was given from which the well-known Preece identity and a well-known quadratic transformation due to Kummer were derived.

2. Known Classical Summation Theorems

As already mentioned that the classical summation theorems such as those of Gauss, Kummer, Gauss's second, and Bailey for the series ${}_2F_1$ and Watson, Dixon, and Whipple for the series ${}_3F_2$ play an important role in the theory of hypergeometric series. These theorems are included in this section so that the paper may be self-contained.

In this section, we will mention classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$. These are the following.

Gauss theorem [5]:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & & ;\ 1 \\ & c \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.1} $$

provided $\Re(c-a-b) > 0$.

Kummer theorem [5]:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & & ;\ -1 \\ & 1+a-b \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma(1+(1/2)a)}{\Gamma(1+(1/2)a-b)\Gamma(1+a)}. \tag{2.2} $$

Gauss's second theorem [5]:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & & ;\ \dfrac{1}{2} \\ & \dfrac{1}{2}(a+b+1) \end{matrix}\right] = \frac{\Gamma(1/2)\Gamma((1/2)a+(1/2)b+(1/2))}{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}. \tag{2.3} $$

Bailey theorem [5]:

$$ {}_2F_1\left[\begin{matrix} a, & 1-a \\ & & ;\ \dfrac{1}{2} \\ & c \end{matrix}\right] = \frac{\Gamma((1/2)c)\Gamma((1/2)c+(1/2))}{\Gamma((1/2)c+(1/2)a)\Gamma((1/2)c-(1/2)a+(1/2))}. \tag{2.4} $$

Watson theorem [5]:

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ \dfrac{1}{2}(a+b+1), & 2c \end{matrix}\right] \\ &= \frac{\Gamma(1/2)\Gamma(c+(1/2))\Gamma((1/2)a+(1/2)b+(1/2))\Gamma(c-(1/2)a-(1/2)b+(1/2))}{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))\Gamma(c-(1/2)a+(1/2))\Gamma(c-(1/2)b+(1/2))} \end{aligned} \tag{2.5} $$

provided $\Re(2c-a-b) > -1$.

Dixon theorem [5]:

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ & 1+a-b, & 1+a-c \end{matrix}\right]
= \frac{\Gamma(1+(1/2)a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+(1/2)a-b-c)}{\Gamma(1+a)\Gamma(1+(1/2)a-b)\Gamma(1+(1/2)a-c)\Gamma(1+a-b-c)} \tag{2.6}
$$

provided $\Re(a-2b-2c) > -2$.

Whipple theorem [5]:

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ & e, & f \end{matrix}\right]
= \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma((1/2)a+(1/2)e)\Gamma((1/2)a+(1/2)f)\Gamma((1/2)b+(1/2)e)\Gamma((1/2)b+(1/2)f)} \tag{2.7}
$$

provided $\Re(c) > 0$ and $\Re(e+f-a-b-c) > 0$ with $a+b=1$ and $e+f=2c+1$.

Other hypergeometric identities [5]:

$$
{}_3F_2\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b \\ & & & ;\ 1 \\ \frac{1}{2}a, & 1+a-b \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma((1/2)a+(1/2))}{\Gamma(1+a)\Gamma((1/2)a-b+(1/2))}, \tag{2.8}
$$

$$
{}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & & & & ;\ 1 \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}\right]
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma((1/2)a+(1/2))\Gamma((1/2)a-b-c+(1/2))}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma((1/2)a-b+(1/2))\Gamma((1/2)a-c+(1/2))} \tag{2.9}
$$

provided $\Re(a-2b-2c) > -1$.

It is not out of place to mention here that Ramanujan independently discovered a great number of the primary classical summation theorems in the theory of hypergeometric series. In particular, he rediscovered well-known summation theorems of Gauss, Kummer, Dougall, Dixon, Saalschütz, and Thomae as well as special cases of the well-known Whipple's transformation. Unfortunately, Ramanujan left us little knowledge as to know how he made his beautiful discoveries about hypergeometric series.

3. Ramanujan's Summations

The classical summation theorems mentioned in Section 2 have wide applications in the theory of generalized hypergeometric series and other connected areas. It has been pointed out by Berndt [2] that a large number of very interesting summations due to Ramanujan can be obtained quite simply by employing the above mentioned theorems.

We now mention here certain very interesting summations by Ramanujan [2].

(i) For $\Re(x) > 1/2$,

$$1-\frac{(x-1)}{(x+1)}+\frac{(x-1)(x-2)}{(x+1)(x+2)}-\cdots=\frac{x}{2x-1}, \tag{3.1}$$

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2+\cdots=\frac{\sqrt{\pi}}{\sqrt{2}\Gamma^2(3/4)}. \tag{3.2}$$

(ii) For $\Re(x) > 0$,

$$1+\frac{(x-1)}{(x+1)}+\frac{(x-1)(x-2)}{(x+1)(x+2)}+\cdots=\frac{2^{2x-1}\Gamma^2(x+1)}{\Gamma(2x+1)}, \tag{3.3}$$

$$1+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{2^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2+\frac{1}{2^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2+\cdots=\frac{\sqrt{\pi}}{\Gamma^2(3/4)}. \tag{3.4}$$

(iii) For $\Re(x) > 0$,

$$1-\frac{1}{3}\frac{(x-1)}{(x+1)}+\frac{1}{5}\frac{(x-1)(x-2)}{(x+1)(x+2)}-\cdots=\frac{2^{4x}\Gamma^4(x+1)}{4x\Gamma^2(2x+1)}. \tag{3.5}$$

(iv) For $\Re(x) > 1/4$,

$$1+\frac{(x-1)^2}{(x+1)^2}+\left[\frac{(x-1)(x-2)}{(x+1)(x+2)}\right]^2+\cdots=\frac{2x}{4x-1}\frac{\Gamma^4(x+1)\Gamma(4x+1)}{\Gamma^4(2x+1)}. \tag{3.6}$$

(v) For $\Re(x) > 1$,

$$1 - 3\frac{(x-1)}{(x+1)} + 5\frac{(x-1)(x-2)}{(x+1)(x+2)} - \cdots = 0, \tag{3.7}$$

$$\begin{aligned} 1 + \frac{1}{5}\left(\frac{1}{2}\right)^2 + \frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \cdots &= \frac{\pi^2}{4\Gamma^4(3/4)}, \\ 1 + \frac{1}{5^2}\left(\frac{1}{2}\right) + \frac{1}{9^2}\left(\frac{1\cdot 3}{2\cdot 4}\right) + \cdots &= \frac{\pi^{5/2}}{8\sqrt{2}\Gamma^2(3/4)}, \\ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \cdots &= \frac{\pi}{\Gamma^4(3/4)}. \end{aligned} \tag{3.8}$$

(vi) For $\Re(x) < 2/3$,

$$1 + \left(\frac{x}{1!}\right)^3 + \left(\frac{x(x+1)}{2!}\right)^3 + \cdots = \frac{6\sin(\pi x/2)\sin(\pi x)\Gamma^3((1/2)x+1)}{\pi^2 x^2 \Gamma((3/2)x+1)(1+2\cos\pi x)}. \tag{3.9}$$

(vii) For $\Re(x) > 1/2$,

$$1 + 3\frac{(x-1)}{(x+1)} + 5\frac{(x-1)(x-2)}{(x+1)(x+2)} + \cdots = x. \tag{3.10}$$

(viii) For $\Re(x) > 1/2$,

$$1 + 3\left[\frac{(x-1)}{(x+1)}\right]^2 + 5\left[\frac{(x-1)(x-2)}{(x+1)(x+2)}\right]^2 + \cdots = \frac{x^2}{2x-1}. \tag{3.11}$$

We now come to the derivations of these summation in brief.

It is easy to see that the series (3.1) corresponds to

$${}_2F_1\left[\begin{matrix} 1, \quad 1-x \\ 1+x \end{matrix}\; ;\; 1\right] \tag{3.12}$$

which is a special case of Gauss's summation theorem (2.1) for $a = 1$, $b = 1 - x$ and $c = 1 + x$.

The series (3.2) corresponds to

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; -1\right] \tag{3.13} $$

which is a special case of Kummer's summation theorem (2.2) for $a = b = 1/2$. Similarly the series (3.3) corresponds to

$$ {}_2F_1\left[\begin{matrix} 1, \quad 1-x \\ 1+x \end{matrix} ; -1\right] \tag{3.14} $$

which is a special case of Kummer's summation theorem (2.2) for $a = 1$, $b = 1 - x$.

The series (3.4) corresponds to

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}\right] \tag{3.15} $$

which is a special case of Gauss's second summation theorem (2.3) for $a = b = 1/2$ or Bailey's summation theorem (2.4) for $a = 1/2$ and $c = 1$.

Also, it can easily be seen that the series (3.5) to (3.9) correspond to each of the following series:

$$ {}_3F_2\left[\begin{matrix} 1, \frac{1}{2}, 1-x \\ \frac{3}{2}, 1+x \end{matrix} ; 1\right], \quad {}_3F_2\left[\begin{matrix} 1, 1-x, 1-x \\ 1+x, 1+x \end{matrix} ; 1\right], \quad {}_3F_2\left[\begin{matrix} 1, \frac{3}{2}, 1-x \\ \frac{1}{2}, 1+x \end{matrix} ; 1\right], $$
$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{4} \\ 1, \frac{5}{4} \end{matrix} ; 1\right], \quad {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \\ \frac{5}{4}, \frac{5}{4} \end{matrix} ; 1\right], $$
$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; 1\right], \quad {}_3F_2\left[\begin{matrix} x, x, x \\ 1, 1 \end{matrix} ; 1\right], \tag{3.16} $$

which are special cases of classical Dixon's theorem (2.6) for (i) $a = 1$, $b = 1/2$, $c = 1 - x$, (ii) $a = 1$, $b = c = 1 - x$, (iii) $a = 1$, $b = 3/2$, $c = 1 - x$, (iv) $a = b = 1/2$, $c = 1/4$, (v) $a = 1/2$, $b = c = 1/4$, (vi) $a = b = c = 1/2$, and (vii) $a = b = c = x$, respectively.

The series (3.10) which corresponds to

$$ {}_3F_2\left[\begin{matrix} 1, & \frac{3}{2}, & 1-x \\ & & & ; \ -1 \\ \frac{1}{2}, & 1+x & \end{matrix}\right] \tag{3.17} $$

is a special case of (2.8) for $a = 1$, $b = 1 - x$, and the series (3.11) which corresponds to

$$ {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & 1-x, & 1-x \\ & & & & ; \ 1 \\ \frac{1}{2}, & 1+x, & 1+x & \end{matrix}\right] \tag{3.18} $$

is a special case of (2.9) for $a = 1$, $b = c = 1 - x$.

Thus by evaluating the hypergeometric series by respective summation theorems, we easily obtain the right hand side of the Ramanujan's summations.

Recently good progress has been done in the direction of generalizing the above-mentioned classical summation theorems (2.2)–(2.7) (see [6]). In fact, in a series of three papers by Lavoie et al. [7–9], a large number of very interesting contiguous results of the above mentioned classical summation theorems (2.2)–(2.7) are given. In these papers, the authors have obtained explicit expressions of

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & & ; \ -1 \\ 1+a-b+i & \end{matrix}\right], \tag{3.19} $$

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & & ; \ \frac{1}{2} \\ \frac{1}{2}(a+b+i+1) & \end{matrix}\right], \tag{3.20} $$

$$ {}_2F_1\left[\begin{matrix} a, & 1-a+i \\ & & ; \ \frac{1}{2} \\ c & \end{matrix}\right] \tag{3.21} $$

each for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, and

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ; \ 1 \\ \frac{1}{2}(a+b+i+1), & 2c+j & \end{matrix}\right] \tag{3.22} $$

for $i, j = 0, \pm 1, \pm 2$

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b+i, & 1+a-c+i+j \end{matrix} ; 1\right] \tag{3.23} $$

for $i = -3, -2, -1, 0, 1, 2$; $j = 0, 1, 2, 3$, and

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ e, & f \end{matrix} ; 1\right] \tag{3.24} $$

for $a + b = 1 + i + j$, $e + f = 2c + 1 + i$ for $i, j = 0, \pm 1, \pm 2, \pm 3$.

Notice that, if we denote (3.23) by $f_{i,j}$, the natural symmetry

$$ f_{i,j}(a, b, c) = f_{i+j,-j}(a, c, b) \tag{3.25} $$

makes it possible to extend the result to $j = -1, -2, -3$.

It is very interesting to mention here that, in order to complete the results (3.23) of 7×7 matrix, very recently Choi [10] obtained the remaining ten results.

For $i = 0$, the results (3.19), (3.20), and (3.21) reduce to (2.2), (2.3), and (2.4), respectively, and for $i = j = 0$, the results (3.22), (3.23), and (3.24) reduce to (2.5), (2.6), and (2.7), respectively.

On the other hand the following very interesting result for the series ${}_3F_2$ (written here in a slightly different form) is given in the literature (e.g., see [11])

$$ {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ c+1, & d \end{matrix} ; 1\right] = \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)} \left[(c-a-b) + \frac{ab}{d}\right] \tag{3.26} $$

provided $\Re(c - a - b) > 0$ and $\Re(d) > 0$.

For $d = c$, we get Gauss's summation theorem (2.1). Thus (3.26) may be regarded as the extension of Gauss's summation theorem (2.1).

Miller [12] very recently rederived the result (3.26) and obtained a reduction formula for the Kampé de Fériet function. For comment of Miller's paper [12], see a recent paper by Kim and Rathie [13].

The aim of this research paper is to establish the extensions of the above mentioned classical summation theorem (2.2) to (2.9). In the end, as an application, certain very interesting summations, which generalize summations due to Ramanujan have been obtained.

The results are derived with the help of contiguous results of the above mentioned classical summation theorems obtained in a series of three research papers by Lavoie et al. [7–9].

The results derived in this paper are simple, interesting, easily established, and may be useful.

4. Results Required

The following summation formulas which are special cases of the results (2.2) to (2.7) obtained earlier by Lavoie et al. [7–9] will be required in our present investigations.

(i) Contiguous Kummer's theorem [9]:

$$
{}_2F_1\left[\begin{matrix} a, \quad b \\ 2+a-b \end{matrix} ; -1\right] = \frac{\Gamma(1/2)\Gamma(2+a-b)}{2^a(1-b)} \times\left[\frac{1}{\Gamma((1/2)a+(1/2))\Gamma((1/2)a-b+1)} - \frac{1}{\Gamma((1/2)a)\Gamma((1/2)a-b+(3/2))}\right],
$$

$$
{}_2F_1\left[\begin{matrix} a, \quad b \\ 3+a-b \end{matrix} ; -1\right] = \frac{\Gamma(1/2)\Gamma(3+a-b)}{2^a(1-b)(2-b)} \times\left[\frac{(1+a-b)}{\Gamma((1/2)a+(1/2))\Gamma((1/2)a-b+2)} - \frac{2}{\Gamma((1/2)a)\Gamma((1/2)a-b+(3/2))}\right]. \tag{4.1}
$$

(ii) Contiguous Gauss's Second theorem [9]:

$$
{}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+3) \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma(1/2)\Gamma((1/2)a+(1/2)b+(3/2))\Gamma((1/2)a-(1/2)b-(1/2))}{\Gamma((1/2)a-(1/2)b+(3/2))} \times\left[\frac{(1/2)(a+b-1)}{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))} - \frac{2}{\Gamma((1/2)a)\Gamma((1/2)a)}\right]. \tag{4.2}
$$

(iii) Contiguous Bailey's theorem [9]:

$$
{}_2F_1\left[\begin{matrix} a,\ 3-a \\ \\ c \end{matrix}\ ;\ \frac{1}{2}\right] = \frac{\Gamma(1/2)\Gamma(c)\Gamma(1-a)}{2^{c-3}\Gamma(3-a)} \times \left[\frac{(c-2)}{\Gamma((1/2)c-(1/2)a+(1/2))\Gamma((1/2)c+(1/2)a-1)} - \frac{2}{\Gamma((1/2)c-(1/2)a)\Gamma((1/2)c+(1/2)a-(3/2))}\right]. \tag{4.3}
$$

(iv) Contiguous Watson's theorem [7]:

$$
{}_3F_2\left[\begin{matrix} a,\quad b,\quad c \\ \\ \frac{1}{2}(a+b+1),\ 2c+1 \end{matrix}\ ;\ 1\right] = \frac{2^{a+b-2}\Gamma(c+(1/2))\Gamma((1/2)a+(1/2)b+(1/2))\Gamma(c-(1/2)a-(1/2)b+(1/2))}{\Gamma(1/2)\Gamma(a)\Gamma(b)} \times \left[\frac{\Gamma((1/2)a)\Gamma((1/2)b)}{\Gamma(c-(1/2)a+(1/2))\Gamma(c-(1/2)b+(1/2))} - \frac{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}{\Gamma(c-(1/2)a+1)\Gamma(c-(1/2)b+1)}\right] \tag{4.4}
$$

provided that $\Re(2c-a-b) > -1$.

$$
{}_3F_2\left[\begin{matrix} a,\quad b,\ c \\ \\ \frac{1}{2}(a+b+3),\ 2c \end{matrix}\ ;\ 1\right] = \frac{2^{a+b+1}\Gamma(c+(1/2))\Gamma((1/2)a+(1/2)b+(3/2))\Gamma(c-(1/2)a-(1/2)b-(1/2))}{(a-b-1)(a-b+1)\Gamma(1/2)\Gamma(a)\Gamma(b)} \times \left[\frac{a(2c-a)+b(2c-b)-2c+1}{8}\frac{\Gamma((1/2)a)\Gamma((1/2)b)}{\Gamma(c-(1/2)a+(1/2))\Gamma(c-(1/2)b+(1/2))} - \frac{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}{\Gamma(c-(1/2)a)\Gamma(c-(1/2)b)}\right] \tag{4.5}
$$

provided that $\Re(2c-a-b)>1$.

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & ;\ 1 \\ \frac{1}{2}(a+b+3), & 2c-1 & \end{matrix}\right]
= \frac{2^{a+b-1}\Gamma(c-(1/2))\Gamma((1/2)a+(1/2)b+(3/2))\Gamma(c-(1/2)a-(1/2)b-(1/2))}{(a-b-1)(a-b+1)\Gamma(1/2)\Gamma(a)\Gamma(b)}
\times\left[\frac{(a+b-1)\Gamma((1/2)a)\Gamma((1/2)b)}{\Gamma(c-(1/2)a-(1/2))\Gamma(c-(1/2)b-(1/2))} - \frac{(4c-a-b-3)\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}{\Gamma(c-(1/2)a)\Gamma(c-(1/2)b)}\right] \tag{4.6}
$$

provided that $\Re(2c-a-b)>1$.

(v) Contiguous Dixon's theorem [8]:

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & ;\ 1 \\ 2+a-b, & 2+a-c & \end{matrix}\right] = \frac{2^{-2c+1}\Gamma(2+a-b)\Gamma(2+a-c)}{(b-1)(c-1)\Gamma(a-2c+2)\Gamma(a-b-c+2)}
\times\left[\frac{\Gamma((1/2)a-c+(3/2))\Gamma((1/2)a-b-c+2)}{\Gamma((1/2)a+(1/2))\Gamma((1/2)a-b+1)} - \frac{\Gamma((1/2)a-c+1)\Gamma((1/2)a-b-c+(5/2))}{\Gamma((1/2)a)\Gamma((1/2)a-b+(3/2))}\right] \tag{4.7}
$$

provided that $\Re(a-2b-2c)>-4$.

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & ;\ 1 \\ 2+a-b, & 1+a-c & \end{matrix}\right] = \frac{2^{-2b+1}\Gamma(1+a-c)\Gamma(2+a-b)}{(b-1)\Gamma(a-2b+2)\Gamma(a-b-c+2)}
\times\left[\frac{\Gamma((1/2)a-b+1)\Gamma((1/2)a-b-c+(3/2))}{\Gamma((1/2)a)\Gamma((1/2)a-c+(1/2))} - \frac{\Gamma((1/2)a-b+(3/2))\Gamma((1/2)a-b-c+2)}{\Gamma((1/2)a+(1/2))\Gamma((1/2)a-c+1)}\right] \tag{4.8}
$$

provided that $\Re(a-2b-2c) > -3$.

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ 2+a-c, & 3+a-b & \end{matrix}\; ; \; 1\right]
= \frac{2^{-2b+2}\Gamma(2+a-c)\Gamma(3+a-b)}{(b-1)(b-2)(c-1)\Gamma(a-2b+3)\Gamma(a-b-c+3)}
\times\left[\frac{(a-2c-b+3)\Gamma((1/2)a-b+2)\Gamma((1/2)a-b-c+(5/2))}{\Gamma((1/2)a)\Gamma((1/2)a-c+(3/2))}
-\frac{(a-b+1)\Gamma((1/2)a-b+(3/2))\Gamma((1/2)a-b-c+3)}{\Gamma((1/2)a+(1/2))\Gamma((1/2)a-c+1)}\right] \tag{4.9}
$$

provided that $\Re(a-2b-2c) > -3$.

(vi) Contiguous Whipple's theorem [9]:

$$
{}_3F_2\left[\begin{matrix} a, & 1-a, & c \\ e, & 2c+2-e & \end{matrix}\; ; \; 1\right]
= \frac{\Gamma(e)\Gamma(2c+2-e)\Gamma(e-c-1)}{2^{2a}\Gamma(e-a)\Gamma(e-c)\Gamma(2c-e-a+2)}
\times\left[\frac{\Gamma((1/2)e-(1/2)a+(1/2))\Gamma(c-(1/2)e-(1/2)a+1)}{\Gamma((1/2)e+(1/2)a-(1/2))\Gamma(c-(1/2)e+(1/2)a+1)}
-\frac{\Gamma((1/2)e-(1/2)a)\Gamma(c-(1/2)e-(1/2)a+(3/2))}{\Gamma((1/2)e+(1/2)a)\Gamma(c-(1/2)e+(1/2)a+(1/2))}\right] \tag{4.10}
$$

provided that $\Re(c) > 0$.

$$
{}_3F_2\left[\begin{matrix} a, & 3-a, & c \\ e, & 2c+2-e & \end{matrix}\; ; \; 1\right]
= \frac{\Gamma(e)\Gamma(2c-e+2)\Gamma(e-c-1)}{2^{2a-2}(c-1)(a-1)(a-2)\Gamma(e-a)\Gamma(e-c)\Gamma(2c-e-a+2)}
\times\left[\frac{(2c-e)\Gamma((1/2)e-(1/2)a+(1/2))\Gamma(c-(1/2)e-(1/2)a+1)}{\Gamma((1/2)e+(1/2)a-(3/2))\Gamma(c-(1/2)e+(1/2)a)}
-\frac{(e-2)\Gamma((1/2)e-(1/2)a)\Gamma(c-(1/2)e-(1/2)a+(3/2))}{\Gamma((1/2)e+(1/2)a-1)\Gamma(c-(1/2)e+(1/2)a-(1/2))}\right] \tag{4.11}
$$

provided that $\Re(c) > 0$.

5. Main Summation Formulas

In this section, the following extensions of the classical summation theorems will be established. In all these theorems we have $\Re(d) > 0$.

(i) Extension of Kummer's theorem:

$$
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ 2+a-b, & d & \end{matrix};\ -1\right]
= \frac{\Gamma(1/2)\Gamma(2+a-b)}{2^a(1-b)}\left[\frac{((1+a-b)/d)-1}{\Gamma((1/2)a)\Gamma((1/2)a-b+(3/2))}+\frac{(1-(a/d))}{\Gamma((1/2)a+(1/2))\Gamma((1/2)a-b+1)}\right]. \tag{5.1}
$$

(ii) Extension of Gauss's second theorem:

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{1}{2}(a+b+3), & d & \end{matrix};\ \frac{1}{2}\right]
&= \frac{\Gamma(1/2)\Gamma((1/2)a+(1/2)b+(3/2))\Gamma((1/2)a-(1/2)b-(1/2))}{\Gamma((1/2)a-(1/2)b+(3/2))} \\
&\quad\times\left\{\frac{[(1/2)(a+b+1)-(ab/d)]}{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}+\frac{[((a+b+1)/d)-2]}{\Gamma((1/2)a)\Gamma((1/2)b)}\right\}.
\end{aligned} \tag{5.2}
$$

(iii) Extension of Bailey's theorem:

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a, & 1-a, & d+1 \\ c+1, & d & \end{matrix};\ \frac{1}{2}\right]
&= \frac{\Gamma(1/2)\Gamma(c+1)}{2^c}\left\{\frac{(2/d)}{\Gamma((1/2)c+(1/2)a)\Gamma((1/2)c-(1/2)a+(1/2))}\right. \\
&\quad\left.+\frac{(1-(c/d))}{\Gamma((1/2)c-(1/2)a+1)\Gamma((1/2)c+(1/2)a+(1/2))}\right\}.
\end{aligned} \tag{5.3}
$$

(iv) Extension of Watson's theorem:
First Extension:

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ \frac{1}{2}(a+b+1), & 2c+1, & d & \end{matrix} ;\ 1\right]
= \frac{2^{a+b-2}\Gamma(c+(1/2))\Gamma((1/2)a+(1/2)b+(1/2))\Gamma(c-(1/2)a-(1/2)b+(1/2))}{\Gamma(1/2)\Gamma(a)\Gamma(b)}
\times\left\{\frac{\Gamma((1/2)a)\Gamma((1/2)b)}{\Gamma(c-(1/2)a+(1/2))\Gamma(c-(1/2)b+(1/2))}
+\frac{((2c-d)/d)\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}{\Gamma((1/2)c-(1/2)a+1)\Gamma(c-(1/2)b+1)}\right\} \tag{5.4}
$$

provided $\Re(2c-a-b) > -1$.

Second Extension:

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ \frac{1}{2}(a+b+3), & 2c, & d & \end{matrix} ;\ 1\right]
= \frac{2^{a+b-2}\Gamma(c+(1/2))\Gamma((1/2)a+(1/2)b+(3/2))\Gamma(c-(1/2)a-(1/2)b-(1/2))}{(a-b-1)(a-b+1)\Gamma(1/2)\Gamma(a)\Gamma(b)}
\times\left\{\alpha\frac{\Gamma((1/2)a)\Gamma((1/2)b)}{\Gamma(c-(1/2)a+(1/2))\Gamma(c-(1/2)b+(1/2))}+\beta\frac{\Gamma((1/2)a+(1/2))\Gamma((1/2)b+(1/2))}{\Gamma(c-(1/2)a)\Gamma(c-(1/2)b)}\right\} \tag{5.5}
$$

provided $\Re(2c-a-b) > 1$, α and β are given by

$$
\alpha = a(2c-a)+b(2c-b)-2c+1-\frac{ab}{d}(4c-a-b-1),
\qquad
\beta = 8\left[\frac{1}{2d}(a+b+1)-1\right]. \tag{5.6}
$$

(v) Extension of Dixon's theorem:

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ 2+a-b, & 1+a-c, & d & \end{matrix} ;\ 1\right]
= \frac{\alpha}{(b-1)}\frac{\Gamma(1+a-c)\Gamma(2+a-b)\Gamma((3/2)+(1/2)a-b-c)\Gamma(1/2)}{2^{a}\,\Gamma((1/2)a)\Gamma((1/2)a-c+(1/2))\Gamma(2+a-b-c)\Gamma((1/2)a-b+(3/2))}
+\frac{\beta}{(b-1)}\frac{2^{-a-1}\Gamma(1/2)\Gamma(1+a-c)\Gamma(1+a-b)\Gamma(1+(1/2)a-b-c)}{\Gamma((1/2)a+(1/2))\Gamma(1+(1/2)a-b)\Gamma(1+(1/2)a-c)\Gamma(1+a-b-c)} \tag{5.7}
$$

provided $\Re(a-2b-2c) > -2$, α and β are given by

$$\alpha = 1 - \frac{1}{d}(1+a-b),$$
$$\beta = \frac{1+a-b}{1+a-b-c}\left[\frac{a}{d}(1+a-b-2c) - 2\left(\frac{1}{2}a-b-c+1\right)\right]. \tag{5.8}$$

(vi) Extension of Whipple's theorem:

$$\begin{aligned}
{}_4F_3\left[\begin{matrix} a, & 1-a, & c, & d+1 \\ & & & & ;\ 1 \\ e+1, & 2c-e+1, & d & \end{matrix}\right]
&= \frac{2^{-2a}\Gamma(e+1)\Gamma(e-c)\Gamma(2c-e+1)}{\Gamma(e-a+1)\Gamma(e-c+1)\Gamma(2c-e-a+1)} \\
&\times\left\{\left(1-\frac{2c-e}{d}\right)\frac{\Gamma((1/2)e-(1/2)a+1)\Gamma(c-(1/2)e-(1/2)a+(1/2))}{\Gamma((1/2)e+(1/2)a)\Gamma(c-(1/2)e+(1/2)a+(1/2))}\right. \\
&\left.+\left(\frac{e}{d}-1\right)\frac{\Gamma((1/2)e-(1/2)a+(1/2))\Gamma(c-(1/2)e-(1/2)a+1)}{\Gamma((1/2)e+(1/2)a+(1/2))\Gamma(c-(1/2)e+(1/2)a)}\right\}
\end{aligned} \tag{5.9}$$

provided $\Re(c) > 0$.

(vii) Extension of (2.8):

$$\begin{aligned}
{}_3F_2\left[\begin{matrix} a, & b, & 1+d \\ & & & ;\ -1 \\ 1+a-b, & d & \end{matrix}\right]
&= \left(1-\frac{a}{2d}\right)\frac{\Gamma(1+a-b)\Gamma(1+(1/2)a)}{\Gamma(1+a)\Gamma(1+(1/2)a-b)} + \left(\frac{a}{2d}\right)\frac{\Gamma(1+a-b)\Gamma((1/2)a+(1/2))}{\Gamma(1+a)\Gamma((1/2)a-b+(1/2))}.
\end{aligned} \tag{5.10}$$

(viii) Extension of (2.9):

$$\begin{aligned}
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ & & & & ;\ 1 \\ 1+a-b, & 1+a-c, & d & \end{matrix}\right]
&= \left(1-\frac{a}{2d}\right)\frac{\Gamma(1+(1/2)a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+(1/2)a-b-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+(1/2)a-b)\Gamma(1+(1/2)a-c)} \\
&+ \left(\frac{a}{2d}\right)\frac{\Gamma((1/2)+(1/2)a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma((1/2)+(1/2)a-b-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma((1/2)+(1/2)a-b)\Gamma((1/2)+(1/2)a-c)}
\end{aligned} \tag{5.11}$$

provided $\Re(a-2b-2c) > -1$.

5.1. Derivations

In order to derive (5.1), it is just a simple exercise to prove the following relation:

$$
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ 2+a-b, & d & \end{matrix} ; -1\right] = {}_2F_1\left[\begin{matrix} a, & b \\ 2+a-b & \end{matrix} ; -1\right] - \frac{ab}{d(2+a-b)}\,{}_2F_1\left[\begin{matrix} a+1, & b+1 \\ 3+a-b & \end{matrix} ; -1\right]. \tag{5.12}
$$

Now, it is easy to see that the first and second ${}_2F_1$ on the right-hand side of (5.12) can be evaluated with the help of contiguous Kummer's theorems (4.1), and after a little simplification, we arrive at the desired result (5.1).

In the exactly same manner, the results (5.2) to (5.11) can be established with the help of the following relations:

$$
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{1}{2}(a+b+3), & d & \end{matrix} ; \frac{1}{2}\right] = {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+3) & \end{matrix} ; \frac{1}{2}\right] + \frac{ab}{d(a+b+3)}\,{}_2F_1\left[\begin{matrix} a+1, & b+1 \\ \frac{1}{2}(a+b+5) & \end{matrix} ; \frac{1}{2}\right],
$$

$$
{}_3F_2\left[\begin{matrix} a, & 1-a, & d+1 \\ c+1, & d & \end{matrix} ; \frac{1}{2}\right] = {}_2F_1\left[\begin{matrix} a, & 1-a \\ c+1 & \end{matrix} ; \frac{1}{2}\right] + \frac{a(1-a)}{2d(1+c)}\,{}_2F_1\left[\begin{matrix} a+1, & 2-a \\ c+2 & \end{matrix} ; \frac{1}{2}\right],
$$

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ \frac{1}{2}(a+b+1), & 2c+1, & d & \end{matrix} ; 1\right] = {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 & \end{matrix} ; 1\right] + \frac{2abc}{d(2c+1)(a+b+1)}\,{}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3), & 2c+2 & \end{matrix} ; 1\right],
$$

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ & & & & ;\ 1 \\ \frac{1}{2}(a+b+3), & 2c, & d \end{matrix}\right] \\
&= {}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ \frac{1}{2}(a+b+3), & 2c \end{matrix}\right] + \frac{ab}{d(a+b+3)}\, {}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ & & & ;\ 1 \\ \frac{1}{2}(a+b+5), & 2c+1 \end{matrix}\right], \\
&{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ & & & & ;\ 1 \\ 2+a-b, & 1+a-c, & d \end{matrix}\right] \\
&= {}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ 2+a-b, & 1+a-c \end{matrix}\right] + \frac{abc}{d(2+a-b)(1+a-c)}\, {}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ & & & ;\ 1 \\ 3+a-b, & 2+a-c \end{matrix}\right], \\
&{}_4F_3\left[\begin{matrix} a, & 1-a, & c, & d+1 \\ & & & & ;\ 1 \\ e+1, & 2c-e+1, & d \end{matrix}\right] \\
&= {}_3F_2\left[\begin{matrix} a, & 1-a, & c \\ & & & ;\ 1 \\ e+1, & 2c-e+1 \end{matrix}\right] + \frac{ac(1-a)}{d(e+1)(2c-e+1)}\, {}_3F_2\left[\begin{matrix} a+1, & 2-a, & c+1 \\ & & & ;\ 1 \\ e+2, & 2c-e+2 \end{matrix}\right], \\
&{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ & & & ;\ -1 \\ 1+a-b, & d \end{matrix}\right] \\
&= {}_2F_1\left[\begin{matrix} a, & b \\ & & ;\ -1 \\ 1+a-b \end{matrix}\right] - \frac{ab}{d(1+a-b)}\, {}_2F_1\left[\begin{matrix} a+1, & b+1 \\ & & ;\ -1 \\ 2+a-b \end{matrix}\right], \\
&{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ & & & & ;\ 1 \\ 1+a-b, & 1+a-c, & d \end{matrix}\right] \\
&= {}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ 1+a-b, & 1+a-c \end{matrix}\right] + \frac{abc}{d(1+a-b)(1+a-c)}\, {}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ & & & ;\ 1 \\ 2+a-b, & 2+a-c \end{matrix}\right]
\end{aligned}
\tag{5.13}
$$

and using the results (4.2); (2.4), (4.3); (2.5), (4.4); (4.5), (4.6); (4.8), (4.9); (4.10), (4.11); (2.2), (4.1), and (2.6), (4.7), respectively.

5.2. Special Cases

(1) In (5.1), if we take $d = 1 + a - b$, we get Kummer's theorem (2.2).

(2) In (5.2), if we take $d = (1/2)(a + b + 1)$, we get Gauss's second theorem (2.3).

(3) In (5.3), if we take $d = c$, we get Bailey's theorem (2.4).

(4) In (5.4), if we take $d = 2c$, we get Watson's theorem (2.5).

(5) In (5.5), if we take $d = (1/2)(a + b + 1)$, we again get Watson's theorem (2.5).

(6) In (5.7), if we take $d = 1 + a - b$, we get Dixon's theorem (2.6).

(7) In (5.9), if we take $d = e$, we get Whipple's theorem (2.7).

(8) In (5.10), if we take $d = (1/2)a$, we get (2.8).

(9) In (5.11), if we take $d = (1/2)a$, we get (2.9).

6. Generalizations of Summations Due to Ramanujan

In this section, the following summations, which generalize Ramanujan's summations (3.1) to (3.11), will be established.

In all the summations, we have $d > 0$.

(i) For $\Re(x) > 1/2$:

$$1 - \left(\frac{x-1}{x+2}\right)\left(\frac{d+1}{d}\right) + \left(\frac{(x-1)(x-2)}{(x+2)(x+3)}\right)\left(\frac{d+2}{d}\right) - \cdots = \frac{(x+1)}{2dx(2x-1)}\{d(2x-1) + (1-x)\}, \tag{6.1}$$

$$1 - \left(\frac{1}{2}\right)^2\left(\frac{d+1}{2d}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{d+2}{3d}\right) - \cdots = \sqrt{2\pi}\left[\left(\frac{1}{d} - 1\right)\frac{4}{\Gamma^2(1/4)} + \left(1 - \frac{1}{2d}\right)\frac{1}{\Gamma^2(3/4)}\right]. \tag{6.2}$$

(ii) For $\Re(x) > 0$:

$$\begin{aligned} 1 &- \left(\frac{x-1}{x+2}\right)\left(\frac{d+1}{d}\right) + \left(\frac{(x-1)(x-2)}{(x+2)(x+3)}\right)\left(\frac{d+2}{d}\right) - \cdots \\ &= \frac{\Gamma(1/2)\Gamma(2+x)}{2x}\left[\left(\frac{1+x}{d} - 1\right)\frac{1}{\Gamma(1/2)\Gamma(1+x)} + \left(1 - \frac{1}{d}\right)\frac{1}{\Gamma(x+(1/2))}\right], \end{aligned} \tag{6.3}$$

$$1 + \frac{1}{2}\left(\frac{1}{2}\right)^2\left(\frac{d+1}{2d}\right) + \frac{1}{2^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{d+2}{3d}\right) - \cdots = \sqrt{\pi}\left[\frac{1}{d\Gamma^2(3/4)} + \left(1 - \frac{1}{d}\right)\frac{8}{\Gamma^2(1/4)}\right]. \tag{6.4}$$

(iii) For $\Re(x) > 0$:

$$1 - \frac{1}{3}\left(\frac{x-1}{x+2}\right)\left(\frac{d+1}{d}\right) + \frac{1}{5}\left(\frac{(x-1)(x-2)}{(x+2)(x+3)}\right)\left(\frac{d+2}{d}\right) - \cdots$$
$$= \left(\frac{1+x}{d} - 1\right)\frac{(x+1)}{2x(2x+1)} + \frac{\pi}{4}\left(2 - \frac{1}{d}\right)\frac{\Gamma(x)\Gamma(x+2)}{(2x+1)\Gamma^2(x+(1/2))}. \tag{6.5}$$

(iv) For $\Re(x) > 1/4$:

$$1 + \frac{(x-1)^2}{(x+1)(x+2)}\left(\frac{d+1}{d}\right) + \frac{(x-1)^2(x-2)^2}{(x+1)(x+2)^2(x+3)}\left(\frac{d+2}{d}\right) + \cdots$$
$$= \left(\frac{1+x}{d} - 1\right)\frac{2^{2x-1}\Gamma(x+2)\Gamma(x+(3/2))}{\sqrt{\pi}\Gamma(2x+1)} - \frac{\sqrt{\pi}}{8}\frac{(x+1)\Gamma^2(x)\Gamma(2x+1)}{\Gamma(2x)\Gamma^2(x+(1/2))}\left[\frac{1}{d}(3x-1) - 4x + 1\right]. \tag{6.6}$$

(v) For $\Re(x) > 1$:

$$1 - 3\frac{(x-1)}{(x+2)}\left(\frac{d+1}{d}\right) + 5\frac{(x-1)(x-2)}{(x+2)(x+3)}\left(\frac{d+2}{d}\right) + \cdots$$
$$= \frac{1}{4x}\frac{\Gamma(x+2)\Gamma(x+3)}{\Gamma^2(x+(1/2))}\left(1 - \frac{1+x}{d}\right),$$

$$1 + \frac{1}{5}\left(\frac{1}{2}\right)^2\left(\frac{d+1}{2d}\right) + \frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{d+2}{3d}\right) + \cdots$$
$$= \frac{3}{4\pi}\left(\frac{1}{d} - 1\right) + \frac{\pi^2}{3\Gamma^4(3/4)}\left(1 - \frac{1}{4d}\right),$$

$$1 + \frac{1}{5}\left(\frac{d+1}{9d}\right)\left(\frac{1}{2}\right) + \frac{1}{9^2}\left(\frac{5(d+2)}{13d}\right)\left(\frac{1\cdot 3}{2\cdot 4}\right) + \cdots \tag{6.7}$$
$$= \frac{5\pi^{3/2}}{48\sqrt{2}\Gamma^2(3/4)}\left(\frac{5}{4d} - 1\right) - \frac{5}{48\sqrt{2}}\frac{\pi^{5/2}}{\Gamma^2(3/4)}\left(\frac{3}{8d} - \frac{3}{2}\right),$$

$$1 + \left(\frac{1}{2}\right)^3\left(\frac{d+1}{2d}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^3\left(\frac{d+2}{3d}\right) + \cdots$$
$$= \frac{\pi}{\Gamma^4(3/4)} - \frac{3}{2}\left(1 - \frac{1}{d}\right)\frac{\Gamma^2(3/4)}{\pi^3}.$$

(vi) For $\Re(x) < 2/3$:

$$1 + \frac{n^3}{2}\left(\frac{d+1}{d}\right)\frac{1}{1!} + \left(\frac{n(n+1)^3}{3.4}\right)\left(\frac{d+2}{d}\right)\frac{1}{2!} + \cdots$$
$$= \frac{(1-(1/d))}{(n-1)}\frac{\Gamma(1/2)\Gamma((3/2)-(2n/2))}{2^n\Gamma(n/2)\Gamma((1/2)-(n/2))\Gamma((3/2)-(n/2))\Gamma(2-n)}$$
$$+ \frac{2^{-n-1}}{(n-1)^2}\frac{[(n/d)(1-2n)-(2-3n)]\Gamma(1/2)\Gamma(1-(3n/2))}{\Gamma^2(1-(n/2))\Gamma((n/2)+(1/2))\Gamma(1-n)}. \quad (6.8)$$

(vii) For $\Re(x) > 1/2$:

$$1 - 3\frac{(x-1)}{(x+2)}\left(\frac{d+1}{d}\right) + 5\frac{(x-1)(x-2)}{(x+2)(x+3)}\left(\frac{d+2}{d}\right) - \cdots = \frac{x}{2d} + \left(1-\frac{1}{2d}\right)\frac{\Gamma(1+x)\sqrt{\pi}}{2\Gamma(x+(1/2))}. \quad (6.9)$$

(viii) For $\Re(x) > 1/2$:

$$1 + \frac{(x-1)^2}{(x+1)^2}\left(\frac{d+1}{d}\right) + \frac{(x-1)^2(x-2)^2}{(x+1)^2(x+2)^2}\left(\frac{d+2}{d}\right) + \cdots$$
$$= \left(1-\frac{1}{2d}\right)\frac{\sqrt{\pi}\Gamma^2(1+x)\Gamma(2x-(1/2))}{2\Gamma(2x)\Gamma^2(x+(1/2))} + \frac{1}{2d}\frac{\Gamma^2(1+x)\Gamma(2x-1)}{\Gamma^2(x)\Gamma(2x)}. \quad (6.10)$$

6.1. Derivations

The series (6.1) corresponds to

$$ {}_3F_2\left[\begin{matrix} 1, & 1-x, & 1+d \\ 2+x, & d & \end{matrix} ; 1\right] \quad (6.11)$$

which is a special case of extended Gauss's summation theorem (3.26) for $a = 1, b = 1-x$ and $c = 1+x$.

The series (6.2) corresponds to

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & d+1 \\ 2, & d & \end{matrix} ; -1\right] \quad (6.12)$$

which is a special case of extended Kummer's summation theorem (5.1) for $a = b = 1/2$. Similarly the series (6.3) corresponds to

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & 1-x, & d+1 \\ 2+x, & d & \end{matrix} ; -1\right] \quad (6.13)$$

which is a special case of extended Kummer's theorem (5.1) for $a = 1$ and $b = 1-x$.

The series (6.4) corresponds to

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & d+1 \\ & & \\ 2, & d & \end{matrix} ; \frac{1}{2}\right] \tag{6.14} $$

which is a special case of extended Gauss's second summation theorem (5.2) for $a = b = 1/2$ or extended Bailey's summation theorem (5.3) for $a = 1/2$, $c = 1$.

Also, it can be easily seen that the series (6.5) to (6.8) which correspond to

$$ \begin{gathered} {}_4F_3\left[\begin{matrix} 1, & \frac{1}{2}, & 1-x, & 1+d \\ & & & \\ \frac{3}{2}, & 2+x, & d & \end{matrix} ; 1\right], \\ {}_4F_3\left[\begin{matrix} 1, & 1-x, & 1+x, & 1+d \\ & & & \\ 1+x, & 2+x, & d & \end{matrix} ; 1\right], \\ {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & 1-x, & 1+d \\ & & & \\ \frac{1}{2}, & 1+x, & d & \end{matrix} ; 1\right], \\ {}_4F_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{4}, & 1+d \\ & & & \\ 2, & \frac{5}{4}, & d & \end{matrix} ; 1\right], \\ {}_4F_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & 1+d \\ & & & \\ \frac{5}{4}, & \frac{5}{4}, & d & \end{matrix} ; 1\right], \\ {}_4F_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & 1+d \\ & & & \\ 2, & 1, & d & \end{matrix} ; 1\right], \\ {}_4F_3\left[\begin{matrix} n, & n, & n, & 1+d \\ & & & \\ 1, & 2, & d & \end{matrix} ; 1\right] \end{gathered} \tag{6.15} $$

are special cases of extended Dixon's theorem (5.7).

The series (6.9) corresponds to

$$
{}_3F_2\left[\begin{matrix} 1, & 1-x, & d+1 \\ 1+x, & d & \end{matrix} ; -1\right] \tag{6.16}
$$

which is a special case of (5.10) for $a = 1$ and $b = 1 - x$. And the series (6.10) corresponds to

$$
{}_4F_3\left[\begin{matrix} 1, & 1-x, & 1-x, & 1+d \\ 1+x, & 1+x, & d & \end{matrix} ; 1\right] \tag{6.17}
$$

which is a special case of (5.11) for $a = 1, b = 1 - x = c$.

7. Concluding Remarks

(1) Various other applications of these results are under investigations and will be published later.

(2) Further generalizations of the extended summation theorem (5.1) to (5.9) in the forms

$$
\begin{gathered}
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ 2+a-b+i, & d & \end{matrix} ; -1\right], \\
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{1}{2}(a+b+3+i), & d & \end{matrix} ; \frac{1}{2}\right], \\
{}_3F_2\left[\begin{matrix} a, & 1-a+i, & d+1 \\ c+1, & d & \end{matrix} ; 1\right]
\end{gathered} \tag{7.1}
$$

each for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, and

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ & & & & ;\ 1 \\ \frac{1}{2}(a+b+i+1), & 2c+j, & d & \end{matrix}\right],
$$
$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ & & & & ;\ 1 \\ 2+a-b+i, & 1+a-c+i+j, & d & \end{matrix}\right], \tag{7.2}
$$

each for $i, j = 0, \pm 1, \pm 2, \pm 3$, and

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & 1+d \\ & & & & ;\ 1 \\ e, & f, & d & \end{matrix}\right], \tag{7.3}
$$

where $a + b = 1 + i + j$, $e + f = 2c + j$ for $i, j = 0, \pm 1, \pm 2, \pm 3$ are also under investigations and will be published later.

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