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Research Article

On Strong Monomorphisms and Strong Epimorphisms

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J. Dydak and F. R. Ruiz del Portal defined strong monomorphism and strong epimorphism in procategories. They obtained a useful characterization of them and some results. In this paper, we aim to study these notions further and obtain some properties of them.

1. Introduction

Dydak and Ruiz del Portal in [1] studied isomorphisms in procategories and obtained the following characterization of isomorphisms in procategories.

Proposition 1.1. *Let $f : X \to Y$ be a morphism in pro-$\mathcal{C}$ where $\mathcal{C}$ is an arbitrary category. f is an isomorphism if and only if for any commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

with P, Q objects in $\mathcal{C}$, there is $h : Y \to P$ such that $g \circ h = b$ and $h \circ f = a$.

This characterization led them to introduce the notions of strong monomorphism and strong epimorphism in procategories. They studied them and obtained some results and a useful characterization of them.

In this paper, we study some properties of strong monomorphisms and strong epimorphisms in procategories.

2. Preliminaries

First we recall some basic facts about procategories. The main reference is [1] and for more details see [2].

Let $\mathcal{C}$ be an arbitrary category. Loosely speaking, the pro-category pro-$\mathcal{C}$ of $\mathcal{C}$ is the universal category with inverse limits containing $\mathcal{C}$ as a full subcategory. An object of pro-$\mathcal{C}$ is an inverse system in $\mathcal{C}$, denoted by $X = (X_\alpha, p_\alpha^\beta, A)$, consisting of a directed set A, called the *index set* (from now onward it will be denoted by $I(X)$), of $\mathcal{C}$ objects X_α for each $\alpha \in I(X)$, called the *terms* of X, and of $\mathcal{C}$ morphisms $p_\alpha^\beta : X_\beta \to X_\alpha$ for each related pair $\alpha < \beta$, called the *bonding morphisms* of X (from now onward it will be denoted by $p_\alpha^\beta(X)$).

If P is an object of $\mathcal{C}$ and X is an object of pro-$\mathcal{C}$, then a morphism $f : X \to P$ in pro-$\mathcal{C}$ is the direct limit of $\mathrm{Mor}(X_\alpha, P)$, $\alpha \in I(X)$, and so f can be represented by $g : X_\alpha \to P$. Note that the morphism from X to X_α represented by the identity $X_\alpha \to X_\alpha$ is called the *projection morphism* and denoted by $p(X)_\alpha$.

If X and Y are two objects in pro-$\mathcal{C}$ with identical index sets, then a morphism $f : X \to Y$ is called a *level morphism* if, for each $\alpha < \beta$, the following diagram

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_\beta} & Y_\beta \\ {\scriptstyle p(X)_\alpha^\beta}\downarrow & & \downarrow{\scriptstyle p(Y)_\alpha^\beta} \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \end{array}$$

commutes.

Recall that an object X of pro-$\mathcal{C}$ is *uniformly movable* if every $\alpha \in I(X)$ admits a $\beta > \alpha$ (a uniform movability index of α) such that there is a morphism $r : X_\beta \to X$ satisfying $p(X)_\alpha \circ r = p(X)_\alpha^\beta$ where $p(X)_\alpha : X \to X_\alpha$ is the projection morphism.

The following lemma is important and will be used later. Therefore, we include its proof for completeness.

Lemma 2.1. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-$\mathcal{C}$. For any commutative diagram with P, Q objects in $\mathcal{C}$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

one may find $\alpha \in I(X)$ and representatives $a_\alpha : X_\alpha \to P$ of a and $b_\alpha : Y_\alpha \to Q$ of b such that

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \\ {\scriptstyle a_\alpha}\downarrow & & \downarrow{\scriptstyle b_\alpha} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative.

Proof. Choose representatives $u : X_\beta \to P$ of a and $v : Y_\beta \to Q$ of b. Since $g \circ u \circ p(X)_\beta = g \circ a = b \circ f = v \circ p(Y)_\beta \circ f = v \circ f_\beta \circ p(X)_\beta$, there is $\alpha > \beta$ such that $g \circ u \circ p(X)_\beta^\alpha = v \circ f_\beta \circ p(X)_\beta^\alpha$. Put $a_\alpha = u \circ p(X)_\beta^\alpha$ and $b_\alpha = v \circ p(Y)_\beta^\alpha$. □

Recall that a morphism $f : X \to Y$ of a category $\mathcal{C}$ is called a *monomorphism* if $f \circ g = f \circ h$ implies $g = h$ for any two morphisms $g, h : Z \to X$. A morphism $f : X \to Y$ of a category $\mathcal{C}$ is called an *epimorphism* if $g \circ f = h \circ f$ implies $g = h$ for any two morphisms $g, h : Y \to Z$.

Next, we recall definitions of strong monomorphism and strong epimorphism and state some of their basic results obtained. The main reference is [1].

Definition 2.2. A morphism $f : X \to Y$ in pro-$\mathcal{C}$ is called a *strong monomorphism* (*strong epimorphism*, resp.) if for every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

with P, Q objects in $\mathcal{C}$, there is a morphism $h : Y \to P$ such that $h \circ f = a$ ($g \circ h = b$, resp.).

Note that if X and Y are objects of $\mathcal{C}$, then $f : X \to Y$ is a strong monomorphism (strong epimorphism, resp.) if and only if f has a left inverse (a right inverse, resp.).

The following result presents the relation between monomorphisms and strong monomorphisms and between epimorphisms and strong epimorphisms.

Lemma 2.3. *If f is a strong monomorphism (strong epimorphism, resp.) of pro-$\mathcal{C}$, then f is a monomorphism (epimorphism, resp.) of pro-$\mathcal{C}$.*

The following lemma is very useful.

Lemma 2.4. *If $g \circ f$ is a strong monomorphism (strong epimorphism, resp.), then f is a strong monomorphism (g is a strong epimorphism, resp.).*

The following theorems are characterizations of isomorphisms in pro-$\mathcal{C}$ in terms of strong monomorphisms and strong epimorphisms.

Theorem 2.5. *Let $f : X \to Y$ be a morphism in pro-$\mathcal{C}$. The following statements are equivalent.*

(i) *f is an isomorphism.*

(ii) *f is a strong monomorphism and an epimorphism.*

Theorem 2.6. *Let $f : X \to Y$ be a morphism in pro-$\mathcal{C}$ where $\mathcal{C}$ is a category with direct sums. The following statements are equivalent.*

(i) *f is an isomorphism.*

(ii) *f is a strong epimorphism and a monomorphism.*

The following useful characterization of strong monomorphisms and strong epimorphisms was obtained.

Proposition 2.7. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-$\mathcal{C}$. The following statements are equivalent.*

(i) *f is a strong monomorphism (strong epimorphism, resp.).*

(ii) *For each $\alpha \in I(X)$, there is a morphism $u_\alpha : Y \to X_\alpha$ such that $u_\alpha \circ f = p(X)_\alpha$ ($f_\alpha \circ u_\alpha = p(Y)_\alpha$, resp.).*

(iii) *For each $\alpha \in I(X)$, there is $\beta \in I(X)$, $\beta > \alpha$ and a morphism $g_{\alpha,\beta} : Y_\beta \to X_\alpha$ such that $g_{\alpha,\beta} \circ f_\beta = p(X)^\beta_\alpha$ ($f_\alpha \circ g_{\alpha,\beta} = p(Y)^\beta_\alpha$, resp.).*

The immediate consequence of this characterization is that both notions are preserved by functors pro-F if $F : \mathcal{C} \to \mathfrak{D}$.

3. Properties of Strong Monomorphisms and Strong Epimorphisms

Theorem 3.1. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-$\mathcal{C}$. If each f_α is a strong monomorphism of $\mathcal{C}$ for each α, then f is a strong monomorphism of pro-$\mathcal{C}$.*

Proof. Suppose that f_α is a strong monomorphism of $\mathcal{C}$ for each α. Suppose that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

is a commutative diagram in pro-$\mathcal{C}$ with P, Q objects in $\mathcal{C}$. By Lemma 2.1, we may find $\alpha \in I(X)$ and representatives $a_\alpha : X_\alpha \to P$ of a and $b_\alpha : Y_\alpha \to Q$ of b such that

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \\ {\scriptstyle a_\alpha}\downarrow & & \downarrow{\scriptstyle b_\alpha} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative, where for $\alpha > \beta$, $a_\alpha = u \circ p(X)^\alpha_\beta$, $u : X_\beta \to P$, and $a = u \circ p(X)_\beta$. Thus, there is $h : Y_\alpha \to P$ such that $h \circ f_\alpha = a_\alpha$ since f_α is a strong monomorphism of $\mathcal{C}$. There is $c = h \circ p(Y)_\alpha : Y \to P$. But $c \circ f = h \circ p(Y)_\alpha \circ f = h \circ f_\alpha \circ p(X)_\alpha = a_\alpha \circ p(X)_\alpha = u \circ p(X)^\alpha_\beta \circ p(X)_\alpha = u \circ p(X)_\beta = a$. Hence, f is a strong monomorphism of pro-$\mathcal{C}$. □

Similarly, we have the following result.

Theorem 3.2. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-$\mathcal{C}$. If each f_α is a strong epimorphism of $\mathcal{C}$ for each α, then f is a strong epimorphism of pro-$\mathcal{C}$.*

Proof. Suppose that f_α is a strong epimorphism of $\mathcal{C}$ for each α. Suppose that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

is a commutative diagram in pro-$\mathcal{C}$ with P, Q objects in $\mathcal{C}$. By Lemma 2.1, we may find $\alpha \in I(X)$ and representatives $a_\alpha : X_\alpha \to P$ of a and $b_\alpha : Y_\alpha \to Q$ of b such that

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \\ {\scriptstyle a_\alpha}\downarrow & & \downarrow{\scriptstyle b_\alpha} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative, where for $\alpha > \beta$, $b_\alpha = v \circ p(Y)^\alpha_\beta$, $v : Y_\beta \to Q$, and $b = v \circ p(Y)_\beta$. Thus, there is $h : Y_\alpha \to P$ such that $g \circ h = b_\alpha$ since f_α is a strong epimorphism of $\mathcal{C}$. There is $c = h \circ p(Y)_\alpha : Y \to P$. But $g \circ c = g \circ h \circ p(Y)_\alpha = b_\alpha \circ p(Y)_\alpha = v \circ p(Y)^\alpha_\beta \circ p(Y)_\alpha = v \circ p(Y)_\beta = b$. Hence, f is a strong epimorphism of pro-$\mathcal{C}$. □

Lemma 3.3. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-$\mathcal{C}$. If f is a strong epimorphism of pro-$\mathcal{C}$ and each $p(Y)_\alpha$ is a strong epimorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$, then f_α is a strong epimorphism of $\mathcal{C}$ for each $\alpha \in I(X)$.*

Proof. Suppose that f is a strong epimorphism of pro-$\mathcal{C}$ and each $p(Y)_\alpha$ is a strong epimorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$. By Proposition 2.7, we have for each $\alpha \in I(X)$, $f_\alpha \circ u_\alpha = p(Y)_\alpha$ where $u_\alpha : Y \to X_\alpha$. Since $p(Y)_\alpha$ is a strong epimorphism, we have that f_α is a strong epimorphism of $\mathcal{C}$ by Lemma 2.4. □

Corollary 3.4. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-$\mathcal{C}$. If each $p(X)^\beta_\alpha$ is a strong monomorphism of $\mathcal{C}$ and f is a strong monomorphism of pro-$\mathcal{C}$, then f_β is a strong monomorphism of $\mathcal{C}$ for some $\beta \in I(X)$.*

Proof. Assume that each $p(X)^\beta_\alpha$ is a strong monomorphism of $\mathcal{C}$ and f is a strong monomorphism of pro-$\mathcal{C}$. By Proposition 2.7, we have $g_{\alpha,\beta} \circ f_\beta = p(X)^\beta_\alpha$ for some $\beta \in I(X)$ where $g_{\alpha,\beta} : Y_\beta \to X_\alpha$. Since $p(X)^\beta_\alpha$ is a strong monomorphism, we have that f_β is a strong monomorphism of $\mathcal{C}$ by Lemma 2.4. □

Proposition 3.5. *If $p(X)^\beta_\alpha$ is a strong monomorphism of $\mathcal{C}$ for each $\beta > \alpha$, then $p(X)_\alpha$ is a strong monomorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$.*

Proof. Assume that $p(X)^\beta_\alpha$ is a strong monomorphism of $\mathcal{C}$ for each $\beta > \alpha$. Assume that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{p(X)_\alpha} & X_\alpha \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative in pro-$\mathcal{C}$ with P, Q objects in $\mathcal{C}$. We may find $\beta \in I(X)$, $\beta > \alpha$, and representative $a_\beta : X_\beta \to P$ of a such that the following diagram

$$\begin{array}{ccc} X_\beta & \xrightarrow{p(X)^\beta_\alpha} & X_\alpha \\ {\scriptstyle a_\beta}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative. But $p(X)_\alpha^\beta$ is a strong monomorphism of $\mathcal{C}$. Thus, there is $h : X_\alpha \to P$ such that $h \circ p(X)_\alpha^\beta = a_\beta$. Therefore, $h \circ p(X)_\alpha^\beta \circ p(X)_\beta = a_\beta \circ p(X)_\beta$, that is, $h \circ p(X)_\alpha = a$. Hence, $p(X)_\alpha$ is a strong monomorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$. □

Proposition 3.6. *Let X be an object of pro-$\mathcal{C}$. Then the following conditions on X are equivalent.*

(i) *$p(X)_\alpha^\beta$ is a strong epimorphism of $\mathcal{C}$ for each $\beta > \alpha$.*

(ii) *$p(X)_\alpha$ is a strong epimorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$.*

Proof. (i)⇒(ii) Assume that $p(X)_\alpha^\beta$ is a strong epimorphism of $\mathcal{C}$ for each $\beta > \alpha$. Assume that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{p(X)_\alpha} & X_\alpha \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative in pro-$\mathcal{C}$ with P, Q objects in $\mathcal{C}$. We may find $\beta \in I(X)$, $\beta > \alpha$, and representative $a_\beta : X_\beta \to P$ of a such that the following diagram

$$\begin{array}{ccc} X_\beta & \xrightarrow{p(X)_\alpha^\beta} & X_\alpha \\ {\scriptstyle a_\beta}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

is commutative. But $p(X)_\alpha^\beta$ is a strong epimorphism of $\mathcal{C}$. Thus, there is $h : X_\alpha \to P$ such that $g \circ h = b$. Hence, $p(X)_\alpha$ is a strong epimorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$.

(ii)⇒(i) Assume that $p(X)_\alpha$ is a strong epimorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$. If $\beta > \alpha$, then $p(X)_\alpha = p(X)_\alpha^\beta \circ p(X)_\beta$. Hence, $p(X)_\alpha^\beta$ is a strong epimorphism of $\mathcal{C}$ by Lemma 2.4. □

Theorem 3.7. *Let $\mathcal{C}$ be a category with inverse limits. Let P be an object of $\mathcal{C}$ and let $f : X \to P$ be a morphism in pro-$\mathcal{C}$. If $\lim f$ is a strong epimorphism of $\mathcal{C}$, then f is a strong epimorphism of pro-$\mathcal{C}$.*

Proof. Suppose that $\lim f$ is a strong epimorphism. Suppose that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & P \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Q & \xrightarrow{g} & W \end{array}$$

is commutative in pro-$\mathcal{C}$ with Q, W objects in $\mathcal{C}$. Note that the following diagram

$$\begin{array}{ccc} \lim X & \xrightarrow{\lim f} & P \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{f} & P \end{array}$$

is commutative in pro-$\mathcal{C}$. Thus,

$$\begin{array}{ccc} \lim X & \xrightarrow{\lim f} & P \\ {\scriptstyle a\circ c}\downarrow & & \downarrow{\scriptstyle b} \\ Q & \xrightarrow{g} & W \end{array}$$

is commutative in $\mathcal{C}$. But $\lim f$ is a strong epimorphism. Therefore, there is $h : P \to Q$ such that $g \circ h = b$. Hence, f is a strong epimorphism of pro-$\mathcal{C}$. □

Remark 3.8. Note that if $f : X \to Y$ is a morphism in pro-$\mathcal{C}$, then we must assume that Y is uniformly movable for this theorem to hold and this result is Corollary 4.4 in [1].

Proposition 3.9. *Let X be an object of pro-$\mathcal{C}$. Then the following conditions on X are equivalent.*

(i) *There is a strong monomorphism $f : X \to P$, where P is an object of $\mathcal{C}$.*

(ii) *$p(X)_\alpha$ is a strong monomorphism of pro-$\mathcal{C}$ for some $\alpha \in I(X)$.*

(iii) *There is $\alpha \in I(X)$ such that $p(X)_\beta$ is a strong monomorphism of pro-$\mathcal{C}$ for all $\beta \geq \alpha$.*

Proof. (i)⇒(ii) Let $g : X_\alpha \to P$ be a representative of f. Thus, $f = g \circ p(X)_\alpha$. But f is a strong monomorphism. Hence, $p(X)_\alpha$ is a strong monomorphism of pro-$\mathcal{C}$ by Lemma 2.4.

(ii)⇒(iii) For all $\beta \geq \alpha$, we have $p(X)_\alpha = p(X)_\alpha^\beta \circ p(X)_\beta$. Hence, $p(X)_\beta$ is a strong monomorphism of pro-$\mathcal{C}$ by Lemma 2.4.

(iii)⇒(i) Put $f = p(X)_\beta$. Hence, the result holds. □

The following result is Proposition 4.2 in [1].

Proposition 3.10. *Let $\mathcal{C}$ be a category with inverse limits. Then if X is an object of pro-$\mathcal{C}$, then the following conditions on X are equivalent.*

(i) *There is a strong epimorphism $f : P \to X$, where P is an object of $\mathcal{C}$.*

(ii) *X is uniformly movable.*

Theorem 3.11. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha\in I(X)}$ is a level morphism of pro-$\mathcal{C}$ where $\mathcal{C}$ is a category with direct sums such that each $p(X)_\alpha^\beta$ is a strong monomorphism of $\mathcal{C}$ and each $p(Y)_\alpha$ is a strong epimorphism of $\mathcal{C}$. If f is an isomorphism of pro-$\mathcal{C}$, then there is $\alpha \in I(X)$ such that f_β is isomorphism of $\mathcal{C}$ for all $\beta \geq \alpha$.*

Proof. Assume that f is an isomorphism of pro-$\mathcal{C}$. By Corollary 3.4, f_α is a strong monomorphism of $\mathcal{C}$ for some $\alpha \in I(X)$. By Lemma 3.3, f_α is a strong epimorphism of $\mathcal{C}$ for each $\alpha \in I(X)$. Since f_α is a strong monomorphism of $\mathcal{C}$ for some $\alpha \in I(X)$, we have that f_α is a monomorphism by Lemma 2.3. Thus, there is $\alpha \in I(X)$ such that f_β is a monomorphism of $\mathcal{C}$ for all $\beta \geq \alpha$ by Corollary 2.9 in [3]. Therefore, f_β is a strong epimorphism and a monomorphism. Hence, f_β is isomorphism of $\mathcal{C}$ for each $\beta \geq \alpha$ by Theorem 2.6. □

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