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Research Article

A Semigroup Approach to the System with Primary and Secondary Failures

Abdukerim Haji

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China

Correspondence should be addressed to Abdukerim Haji, abdukerimhaji63@yahoo.com.cn

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We investigate the solution of a repairable parallel system with primary as well as secondary failures. By using the method of functional analysis, especially, the spectral theory of linear operators and the theory of C_0-semigroups, we prove well-posedness of the system and the existence of positive solution of the system. And then we show that the time-dependent solution strongly converges to steady-state solution, thus we obtain the asymptotic stability of the time-dependent solution.

1. Introduction

As science and technology develop, the theory of reliability has infiltrated into the basic sciences, technological sciences, applied sciences, and management sciences. It is well known that repairable parallel systems are the most essential and important systems in reliability theory. In practical applications, repairable parallel systems consisting of three units are often used. Since the strong practical background of such systems, many researchers have studied them extensively under varying assumptions on the failures and repairs; see [1–5] and their references.

The mathematical model of a repairable parallel system with primary as well as secondary failures was first put forward by Gupta; see [1]. This system is consisted of three independent identical units, which are connected in parallel. In the system, one of those units operates, the other two act as warm standby. If the operating unit fails, a warm standby unit is instantaneously switched into operation. The operating unit submits primary failures and secondary failures. The primary failures are the result of a deficiency in a unit while it is operating within the design limits. The secondary failures are the result of causes that stem from a unit operating in a conditions that are outside its design limits. Two important types of secondary failures are common cause failures and human error failures. A Common cause failure refers to the situation where multiple units fail due to a single cause such as fire, earthquake, flood, explosion, design flaw, and poor maintenance; see [2, 3]. A human error failure implies a failure of the system due to a mistake made by a human caused by such reasons as inadequate training, improper tools, and working in a poor lighting environment; see [4, 5]. There is one repairman available to repair these units. Once repaired, these units are as good as new. The failure rates of units and system are constant and independent. When the system is operating, the repairman can repair only one unit at a time. If all units fail, the entire system is repaired and checked before beginning further operation of these units. Unlike [4, 5], the repair times in this system are arbitrarily distributed.

The parallel repairable system with primary and secondary failures can be described by the following equations (see [1]):

$$\begin{aligned}
\frac{dp_0(t)}{dt} &= -(\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})p_0(t) + \mu p_1(t) + \sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i(x,t)dx, \\
\frac{dp_1(t)}{dt} &= (\lambda + 2\alpha)p_0(t) - \left(\mu + \lambda + \alpha + \lambda_{c_1} + \lambda_{h_1}\right)p_1(t) + \mu p_2(t), \\
\frac{dp_2(t)}{dt} &= (\lambda + \alpha)p_1(t) - \left(\mu + \lambda + \lambda_{c_2} + \lambda_{h_2}\right)p_2(t), \\
\frac{\partial p_i(x,t)}{\partial t} + \frac{\partial p_i(x,t)}{\partial x} &= -\mu_i(x)p_i(x,t), \quad i = 3,4,5.
\end{aligned} \tag{R}$$

For $x = 0$, the boundary conditions

$$\begin{aligned}
p_3(0,t) &= \lambda p_2(t), \\
p_4(0,t) &= \sum_{i=0}^{2}\lambda_{c_i}p_i(t), \\
p_5(0,t) &= \sum_{i=0}^{2}\lambda_{h_i}p_i(t)
\end{aligned} \tag{BC}$$

are prescribed, and we consider the usual initial condition

$$\begin{aligned}
&p_0(0) = c \in \mathbb{C}, \qquad p_i(0) = b_i \in \mathbb{C}, \quad i = 1,2, \\
&p_j(x,0) = f_j(x), \quad j = 3,4,5,
\end{aligned} \tag{IC}$$

where $f_j(x) \in L^1[0,\infty)$. The most interesting initial condition is

$$\begin{aligned}
&p_0(0) = 1, \qquad p_i(0) = 0, \quad i = 1,2, \\
&p_j(x,0) = 0, \quad j = 3,4,5.
\end{aligned} \tag{IC_0}$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$; $p_i(t)$ represents the probability that the system is in state i at time t, $i = 0,1,2$; $p_j(x,t)$ represents the probability that at time t the failed system is in state j and has an elapsed repair time of x, $j = 3,4,5$; λ represents failure rate of an operating unit; λ_{c_i} represents common-cause failure rates from state i to state 4, $i = 0,1,2$; λ_{h_i} represents human-error rates from state i to state 5, $i = 0,1,2$; α represents failure rate of standby unit; μ represents constant repair rate if the system is operating; $\mu_j(x)$ represents repair-rate when the failed system is in state j and has an elapsed repair time of x for $j = 3,4,5$ which satisfies $\mu_j(x) \geq 0$ $(j = 3,4,5)$; λ_{c_i} $(i = 0,1,2)$, λ_{h_i} $(i = 0,1,2)$, λ, μ, and α are positive constants.

In [1] the author analyzed the system using supplementary variable technique and obtained various expressions including the system availability, reliability, and mean time of the failure using the Laplace transform. And then he discovered that the time-dependent availability decreases as time increases for exponential repair-time distribution under the following hypotheses.

Hypothesis 1. The system has a unique positive time-dependent solution $p(x,t)$.

Hypothesis 2. The time-dependent solution $p(x,t)$ converges to the steady-state solution $p(x)$ as time tends to infinity, where

$$\begin{aligned} p(x,t) &= \big(p_0(t), p_1(t), p_2(t), p_3(x,t), p_4(x,t), p_5(x,t)\big), \\ p(x) &= \big(p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)\big). \end{aligned} \tag{1.1}$$

The availability and the reliability depend on the time-dependent solution of the system. In fact, the author used the time-dependent solution in calculating the availability and the reliability. But the author did not discuss the existence of the time-dependent solution and its asymptotic stability, that is, the author did not prove the correctness of the above hypotheses. It is well known that the above hypotheses do not always hold and it is necessary to prove the correctness. Motivated by this, we will show the well-posedness of the system and study the asymptotic stability of the time-dependent solution in this paper, by using the theory of strongly continuous operator semigroups, from [6–8]. First, we convert the model of the system into an abstract Cauchy problem in a Banach space. Next, we show that the operator corresponding to this model generates a positive contraction C_0-semigroup. Furthermore, we prove that the system is well-posed and there is a positive solution for given initial value. Finally, we prove that the time-dependent solution converging to its static solution in the sense of the norm through studying the spectrum of the operator and irreducibility of the corresponding semigroup, thus we obtain the asymptotic stability of the time-dependent solution of this system.

In this paper, we require the following assumption for the failure rate $\mu_j(x)$.

Assumption 1.1 (general assumption). The function $\mu_j : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded such that $\lim_{x\to\infty}\mu_j(x)$ exists and

$$\mu_\infty^{(j)} := \lim_{x\to\infty}\mu_j(x) > 0, \quad j = 3,4,5, \qquad \mu_\infty := \min\Big(\mu_\infty^{(3)}, \mu_\infty^{(4)}, \mu_\infty^{(5)}\Big). \tag{1.2}$$

2. The Problem as an Abstract Cauchy Problem

In this section, we rewrite the underlying problem as an abstract Cauchy problem on a suitable space X, see [6, Definition II.6.1], also see [7, Definition II.6.1]. As the state space for our problem we choose

$$X := \mathbb{C}^3 \times \left(L^1[0,\infty)\right)^3. \tag{2.1}$$

It is obvious that X is a Banach space endowed with the norm

$$\|p\| := \sum_{i=0}^{2} |p_i| + \sum_{n=3}^{5} \|p_n\|_{L^1[0,\infty)}, \tag{2.2}$$

where $p = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x))^t \in X$.

For simplicity, let

$$\begin{aligned} a_0 &:= \lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0}, \\ a_1 &:= \mu + \lambda + \alpha + \lambda_{c_1} + \lambda_{h_1}, \\ a_2 &:= \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}, \end{aligned} \tag{2.3}$$

and we denote by φ_j the linear functionals

$$\varphi_j : L^1[0,\infty) \longrightarrow \mathbb{C}, \quad f \longmapsto \varphi_j(f) := \int_0^\infty \mu_j(x) f(x)\,dx, \quad j = 3,4,5. \tag{2.4}$$

Moreover, we define the operators D_j on $W^{1,1}[0,\infty)$ as

$$D_j f := -\frac{d}{dx} f - \mu_j f, \quad f \in W^{1,1}[0,\infty),\ j = 3,4,5, \tag{2.5}$$

respectively. To define the appropriate operator $(A, D(A))$ we introduce a "maximal operator" $(A_m, D(A_m))$ on X given as

$$A_m := \begin{pmatrix} -a_0 & \mu & 0 & \varphi_3 & \varphi_4 & \varphi_5 \\ \lambda + 2\alpha & -a_1 & \mu & 0 & 0 & 0 \\ 0 & \lambda + \alpha & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_5 \end{pmatrix}, \tag{2.6}$$
$$D(A_m) := \mathbb{C}^3 \times \left(W^{1,1}[0,\infty)\right)^3.$$

To model the boundary conditions (BC) we use an abstract approach as in, for example, [9]. For this purpose we consider the "boundary space"

$$\partial X := \mathbb{C}^3, \tag{2.7}$$

and then define "boundary operators" L and Φ. As the operator L we take

$$L : D(A_m) \longrightarrow \partial X, \qquad \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} \longmapsto L \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} := \begin{pmatrix} p_3(0) \\ p_4(0) \\ p_5(0) \end{pmatrix}, \tag{2.8}$$

and the operator $\Phi \in \mathcal{L}(D(A_m), \partial X)$ is given by

$$\Phi p := \begin{pmatrix} 0 & 0 & \lambda & 0 & 0 & 0 \\ \lambda_{c_0} & \lambda_{c_1} & \lambda_{c_2} & 0 & 0 & 0 \\ \lambda_{h_0} & \lambda_{h_1} & \lambda_{h_2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix}, \tag{2.9}$$

where $p = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x))^t \in D(A_m)$.

The operator $(A, D(A))$ on X corresponding to our original problem is then defined as

$$Ap := A_m p, \qquad D(A) := \{p \in D(A_m) \mid Lp = \Phi p\}. \tag{2.10}$$

Let $p_j(0) = p_j(0,t)$, $j = 3, 4, 5$, $t \geq 0$, then the condition $Lp = \Phi p$ in $D(A)$ is equivalent to (BC). The system of integrodifferential equations (R) can be written as the following equation:

$$\begin{pmatrix} \dfrac{dp_0(t)}{dt} \\ \dfrac{dp_1(t)}{dt} \\ \dfrac{dp_2(t)}{dt} \\ \dfrac{dp_3(x,t)}{dt} \\ \dfrac{dp_4(x,t)}{dt} \\ \dfrac{dp_5(x,t)}{dt} \end{pmatrix} = \begin{pmatrix} -a_0 & \mu & 0 & \varphi_3 & \varphi_4 & \varphi_5 \\ \lambda + 2\alpha & -a_1 & \mu & 0 & 0 & 0 \\ 0 & \lambda + \alpha & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_5 \end{pmatrix} \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(x,t) \\ p_4(x,t) \\ p_5(x,t) \end{pmatrix}. \tag{2.11}$$

Let $p(t) = (p_0(t), p_1(t), p_2(t), p_3(\cdot,t), p_4(\cdot,t), p_5(\cdot,t))^t \in X$, then (2.11) is equivalent to the following operator equation:

$$\frac{dp(t)}{dt} = Ap(t), \quad t \in [0,\infty). \tag{2.12}$$

Thus, the above equations (R), (BC), and (IC) can be equivalently formulated as the abstract Cauchy problem

$$\begin{aligned} &\frac{dp(t)}{dt} = Ap(t), \quad t \in [0,\infty), \\ &p(0) = \left(c, b_1, b_2, f_1, f_2, f_3\right)^t \in X. \end{aligned} \tag{ACP}$$

If A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and the initial value in (IC) satisfies $p(0) = (c, b_1, b_2, f_1, f_2, f_3)^t \in D(A)$, then the unique solution of (R), (BC), and (IC) is given by

$$\begin{aligned} p_i(t) &= (T(t)p(0))_{i+1}, \quad 0 \leq i \leq 2, \\ p_j(x,t) &= (T(t)p(0))_{j+1}(x), \quad 3 \leq j \leq 5. \end{aligned} \tag{2.13}$$

For this reason it suffices to study (ACP).

3. Boundary Spectrum

In this section we investigate the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ of A. In order to characterise $\sigma(A)$ by the spectrum of a scalar 3×3-matrix, that is, or on the boundary space ∂X, we apply techniques and results from [10]. We start from the operator $(A_0, D(A_0))$ defined by

$$\begin{aligned} D(A_0) &:= \left\{p \in D(A_m) Lp = 0\right\}, \\ A_0 p &:= A_m p \end{aligned} \tag{3.1}$$

We give the the representation of the resolvent of the operator A_0 needed below to prove the irreducibility of the semigroup generated by the operator A.

Lemma 3.1. *Let*

$$\mathbb{A} := \begin{pmatrix} -a_0 & \mu & 0 \\ \lambda + 2\alpha & -a_1 & \mu \\ 0 & \lambda + \alpha & -a_2 \end{pmatrix} \tag{3.2}$$

and set $S := \{\gamma \in \mathbb{C} \mid \Re\gamma > -\mu_\infty\} \setminus \sigma(\mathbb{A})$. Then one has

$$S \subseteqq \rho(A_0). \tag{3.3}$$

Moreover, if $\gamma \in S$, then

$$R(\gamma, A_0) = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & r_{1,5} & r_{1,6} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} & r_{2,6} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} & r_{3,5} & r_{3,6} \\ 0 & 0 & 0 & r_{4,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{5,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{6,6} \end{pmatrix}, \tag{3.4}$$

where

$$r_{1,1} = \frac{(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{1,2} = \frac{\mu(\gamma + a_2)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{1,3} = \frac{\mu^2}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{1,4} = \frac{[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)]\psi_3 R(\gamma, D_3)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{1,5} = \frac{\mu(\gamma + a_2)\psi_4 R(\gamma, D_4)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{1,6} = \frac{\mu^2\psi_5 R(\gamma, D_5)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{2,1} = \frac{(\lambda + 2\alpha)(\gamma + a_2)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{2,2} = \frac{(\gamma + a_0)(\gamma + a_2)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{2,3} = \frac{\mu(\gamma + a_0)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{2,4} = \frac{(\lambda + 2\alpha)(\gamma + a_2)\psi_3 R(\gamma, D_3)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{2,5} = \frac{(\lambda + a_0)(\gamma + a_2)\psi_4 R(\gamma, D_4)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$r_{2,6} = \frac{\mu(\gamma + a_0)\psi_5 R(\gamma, D_5)}{(\gamma + a_0)(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)(\gamma + a_0) - \mu(\lambda + 2\alpha)(\gamma + a_2)},$$

$$
\begin{aligned}
r_{3,1} &= \frac{(\lambda+2\alpha)(\lambda+\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)},\\
r_{3,2} &= \frac{(\gamma+a_0)(\lambda+\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)},\\
r_{3,3} &= \frac{(\gamma+a_0)(\gamma+a_1)-\mu(\lambda+2\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)},\\
r_{3,4} &= \frac{(\lambda+2\alpha)(\lambda+\alpha)\psi_3 R(\gamma,D_3)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)},\\
r_{3,5} &= \frac{(\lambda+a_0)(\lambda+\alpha)\psi_4 R(\gamma,D_4)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)},\\
r_{3,6} &= \frac{\left[(\lambda+a_0)(\lambda+a_1)-\mu(\lambda+2\alpha)\right]\psi_5 R(\gamma,D_5)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)},\\
r_{4,4} &= R(\gamma,D_3),\\
r_{5,5} &= R(\gamma,D_4),\\
r_{6,6} &= R(\gamma,D_3).
\end{aligned} \tag{3.5}
$$

The resolvent operators of the differential operators D_j $(j=3,4,5)$ are given by

$$
(R(\gamma,D_j)p)(x) = e^{-\gamma x-\int_0^x \mu_j(\xi)d\xi}\int_0^x e^{\gamma x+\int_0^x \mu_j(\xi)d\xi}p(s)ds \tag{3.6}
$$

for $p \in L^1[0,\infty)$.

Proof. A combination of [11, Proposition 2.1] and [12, Theorem 2.4] yields that the resolvent set of A_0 satisfies

$$
\rho(A_0) \supseteq S. \tag{3.7}
$$

For $\gamma \in S$ we can compute the resolvent of A_0 explicitly applying the formula for the inverse of operator matrices; see [12, Theorem 2.4]. This leads to the representation (3.4) of the resolvent of A_0.

Clearly, knowing the operator matrix in (3.4), we can directly compute that it represents the resolvent of A_0. □

The following consequence is useful to compute the boundary spectrum of A.

Corollary 3.2. *The imaginary axis belongs to the resolvent set of A_0, that is,*

$$
i\mathbb{R} \subseteq \rho(A_0). \tag{3.8}
$$

The eigenvectors in $\ker(\gamma - A_m)$ can be computed as follows.

Lemma 3.3. *For $\gamma \in \mathbb{C}$, one has*

$$p \in \ker(\gamma - A_m) \Longleftrightarrow \tag{3.9}$$

$$p = (p_0, p_1, p_2, p_3(\cdot), p_4(\cdot), p_5(\cdot))^t \in D(A_m), \quad \text{with} \tag{3.10}$$

$$\begin{aligned} p_0 = {} & \frac{(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\ & \times \sum_{j=3}^{5} c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx, \end{aligned} \tag{3.11}$$

$$p_1 = \frac{(\lambda + 2\alpha)(\gamma + a_2) \sum_{j=3}^{5} c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \tag{3.12}$$

$$p_2 = \frac{(\lambda + \alpha)(\lambda + 2\alpha) \sum_{j=3}^{5} c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \tag{3.13}$$

$$p_j(x) = c_j e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi}, \quad j = 3, 4, 5, \tag{3.14}$$

where $c_3, c_4, c_5 \in \mathbb{C}$.

Proof. If for $p \in X$, (3.11)–(3.14) are fulfilled, then we can easily compute that $p \in \ker(\gamma - A_m)$. Conversely, condition (3.9) gives a system of differential equations. Solving these differential equations, we see that (3.11)–(3.14) are indeed satisfied. □

The domain $D(A_m)$ of the maximal operator A_m decomposes, using [10, Lemma 1.2], as

$$D(A_m) = D(A_0) \oplus \ker(\gamma - A_m). \tag{3.15}$$

Moreover, since L is surjective,

$$L|_{\ker(\gamma - A_m)} : \ker(\gamma - A_m) \longrightarrow \partial X \tag{3.16}$$

is invertible for each $\gamma \in \rho(A_0)$, see [10, Lemma 1.2]. We denote its inverse by

$$D_\gamma := \left(L|_{\ker(\gamma - A_m)}\right)^{-1} : \partial X \longrightarrow \ker(\gamma - A_m) \tag{3.17}$$

and call it "Dirichlet operator."

We can give the explicit form of D_γ as follows.

Lemma 3.4. *For each $\gamma \in \rho(A_0)$, the operator D_γ has the form*

$$D_\gamma = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \\ d_{4,1} & 0 & 0 \\ 0 & d_{5,2} & 0 \\ 0 & 0 & d_{6,3} \end{pmatrix}, \tag{3.18}$$

where

$$\begin{aligned}
d_{1,1} &= \frac{\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{1,2} &= \frac{\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{1,3} &= \frac{\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{2,1} &= \frac{(\lambda + 2\alpha)(\gamma + a_2) \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{2,2} &= \frac{(\lambda + 2\alpha)(\gamma + a_2) \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{2,3} &= \frac{(\lambda + 2\alpha)(\gamma + a_2) \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{3,1} &= \frac{(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{3,2} &= \frac{(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{3,3} &= \frac{(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{4,1} &= e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi}, \\
d_{5,2} &= e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi}, \\
d_{6,3} &= e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi}.
\end{aligned} \tag{3.19}$$

The operator ΦD_γ can be computed explicitly for $\gamma \in \rho(A_0)$.

Remark 3.5. For $\gamma \in \rho(A_0)$ the operator ΦD_γ can be represented by the 3×3-matrix

$$\Phi D_\gamma = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}, \tag{3.20}$$

where

$$\begin{aligned}
a_{1,1} &= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)\int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\
a_{1,2} &= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)\int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\
a_{1,3} &= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)\int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\
a_{2,1} &= \frac{\lambda_{c_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{c_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \\
&\quad \times \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx, \\
a_{2,2} &= \frac{\lambda_{c_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{c_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \\
&\quad \times \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx, \\
a_{2,3} &= \frac{\lambda_{c_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{c_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \\
&\quad \times \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx, \\
a_{3,1} &= \frac{\lambda_{h_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{h_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \\
&\quad \times \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx, \\
a_{3,2} &= \frac{\lambda_{h_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{h_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \\
&\quad \times \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx,
\end{aligned}$$

$$a_{3,3} = \frac{\lambda_{h_0}\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] + \lambda_{h_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)\left[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)\right] - \mu(\lambda + 2\alpha)(\gamma + a_2)}$$
$$\times \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx. \tag{3.21}$$

The operators D_γ and Φ allow to characterise the spectrum $\sigma(A)$ and the point spectrum $\sigma_p(A)$ of A. Before doing so we extend the given operators to the product $X \times \partial X$ as in [13, Section 3].

Definition 3.6. (i) $\mathcal{X} := X \times \partial X$.

(ii) $\mathcal{A}_0 := \begin{pmatrix} A_m & 0 \\ -L & 0 \end{pmatrix}$, $D(\mathcal{A}_0) := D(A_m) \times \{0\}$.

(iii) $\mathcal{X}_0 := X \times \{0\} = \overline{D(A_m) \times \{0\}} = \overline{D(\mathcal{A}_0)}$.

(iv) $\mathcal{B} := \begin{pmatrix} 0 & 0 \\ \Phi & 0 \end{pmatrix}$, $D(\mathcal{B}) := D(A_m) \times \partial X$.

(v) $\mathcal{A} := \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A_m & 0 \\ \Phi - L & 0 \end{pmatrix}$, $D(\mathcal{A}) := D(A_m) \times \{0\}$.

Remark 3.7. (i) Note that $\rho(\mathcal{A}_0) \supseteq \rho(A_0)$. For $\gamma \in \rho(A_0)$ the resolvent of $\mathcal{A}_0$ is

$$R(\gamma, \mathcal{A}_0) = \begin{pmatrix} R(\gamma, A_0) & D_\gamma \\ 0 & 0 \end{pmatrix}. \tag{3.22}$$

(ii) The part $\mathcal{A}|_{\mathcal{X}_0}$ of $\mathcal{A}$ in $\mathcal{X}_0$ is

$$D(\mathcal{A}|_{\mathcal{X}_0}) = D(A) \times \{0\}, \quad \mathcal{A}|_{\mathcal{X}_0} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.23}$$

Hence, $\mathcal{A}|_{\mathcal{X}_0}$ can be identified with the operator $(A, D(A))$.

The spectrum of A can be characterise by the spectrum of operators on the boundary space ∂X as follows.

Characteristic Equation 3.8.

Let $\gamma \in \rho(A_0)$. Then

(i)

$$\gamma \in \sigma_p(A) \Longleftrightarrow 1 \in \sigma_p(\Phi D_\gamma). \tag{3.24}$$

(ii) If, in addition, there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$, then

$$\gamma \in \sigma(A) \Longleftrightarrow 1 \in \sigma(\Phi D_{\gamma}). \tag{3.25}$$

Proof. Let us first show the equivalence

$$\gamma \in \sigma(\mathcal{A}) \Longleftrightarrow 1 \in \sigma(\Phi D_{\gamma}). \tag{3.26}$$

We can decompose $\gamma - \mathcal{A}$ as

$$\gamma - \mathcal{A} = \gamma - \mathcal{A}_0 - \mathcal{B} = \left(\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)\right)(\gamma - \mathcal{A}_0). \tag{3.27}$$

We conclude from this that the invertibility of $\gamma - \mathcal{A}$ is equivalent to the invertibility of $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$. From

$$\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0) = \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_{\gamma} \end{pmatrix}, \tag{3.28}$$

one can easily see that $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ is invertible if and only if $1 \notin \sigma(\Phi D_{\gamma})$. This proves (3.26). Since by our assumption $1 \notin \sigma(\Phi D_{\gamma_0})$, it follows that $\gamma_0 \in \rho(\mathcal{A})$. Therefore, $\rho(\mathcal{A})$ is not empty. Hence we obtain from [6, Proposition IV.2.17] that

$$\sigma(\mathcal{A}) = \sigma(A), \tag{3.29}$$

since A is the part of $\mathcal{A}$ in $\mathcal{X}_0$. This shows (ii).

To prove (i) observe first that $\mathcal{A}$ and A have the same point spectrum, that is,

$$\sigma_p(\mathcal{A}) = \sigma_p(A). \tag{3.30}$$

Suppose now that $1 \in \sigma_p(\Phi D_{\gamma})$. Then there exists $0 \neq f \in \partial X$ such that $(Id_{\partial X} - \Phi D_{\gamma})f = 0$. Since $0 \neq \binom{D_{\gamma} f}{0} \in D(\mathcal{A})$, we can compute

$$\begin{aligned}
(\gamma - \mathcal{A})\begin{pmatrix} D_{\gamma} f \\ 0 \end{pmatrix} &= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_{\gamma} \end{pmatrix}\begin{pmatrix} (\gamma - A_m) D_{\gamma} f \\ L D_{\gamma} f \end{pmatrix} \\
&= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_{\gamma} \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (Id_{\partial X} - \Phi D_{\gamma}) f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.31}$$

This shows that $\gamma \in \sigma_p(\mathcal{A})$.

Conversely, if we assume that $\gamma \in \sigma_p(\mathcal{A})$, then there exists $0 \neq f \in D(A_m)$ such that $(\gamma - \mathcal{A})\binom{f}{0} = 0$. From

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= (\gamma - \mathcal{A})\begin{pmatrix} f \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix}\begin{pmatrix} (\gamma - A_m)f \\ Lf \end{pmatrix} \\
&= \begin{pmatrix} (\gamma - A_m)f \\ -\Phi R(\gamma, A_0)(\gamma - A_m)f + (Id_{\partial X} - \Phi D_\gamma)Lf \end{pmatrix}
\end{aligned} \tag{3.32}$$

we conclude that $f \in \ker(\gamma - A_m)$ and thus

$$0 = -\Phi R(\gamma, A_0)(\gamma - A_m)f + (Id_{\partial X} - \Phi D_\gamma)Lf = (Id_{\partial X} - \Phi D_\gamma)Lf. \tag{3.33}$$

It follows from the decomposition (3.15) that $Lf \neq 0$ and hence $1 \in \sigma_p(\Phi D_\gamma)$. □

Using the Characteristic Equation 3.8 we can show that 0 is in the point spectrum of A.

Lemma 8.8. *For the operator $(A, D(A))$ one has $0 \in \sigma_p(A)$.*

Proof. By the Characteristic Equation 3.8 it suffices to prove that $1 \in \sigma_p(\Phi D_0)$. Since

$$\Phi D_0 = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}, \tag{3.34}$$

where

$$\begin{aligned}
b_{1,1} &= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}, \\
b_{1,2} &= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}, \\
b_{1,3} &= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}, \\
b_{2,1} &= \frac{\lambda_{c_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{c_1}(\lambda+2\alpha)a_2 + \lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2},
\end{aligned}$$

$$
\begin{aligned}
b_{2,2} &= \frac{\lambda_{c_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{c_1}(\lambda+2\alpha)a_2 + \lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2) - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)(\gamma+a_2)},\\
b_{2,3} &= \frac{\lambda_{c_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{c_1}(\lambda+2\alpha)a_2 + \lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2},\\
b_{3,1} &= \frac{\lambda_{h_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{h_1}(\lambda+2\alpha)a_2 + \lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2},\\
b_{3,2} &= \frac{\lambda_{h_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{h_1}(\lambda+2\alpha)a_2 + \lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2},\\
b_{3,3} &= \frac{\lambda_{h_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{h_1}(\lambda+2\alpha)a_2 + \lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}.
\end{aligned}
\tag{3.35}
$$

We can compute the jth column sum ($j = 1,2,3$) of the 3×3-matrix ΦD_0 as follows:

$$
\begin{aligned}
\sum_{i=1}^{3}(\Phi D_0)_{i,j} &= b_{1,j} + b_{2,j} + b_{3,j}\\
&= \frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&\quad + \frac{\lambda_{c_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{c_1}(\lambda+2\alpha)a_2 + \lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&\quad + \frac{\lambda_{h_0}\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \lambda_{h_1}(\lambda+2\alpha)a_2 + \lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&= \frac{(\lambda_{c_0}+\lambda_{h_0})\left[a_1a_2 - \mu(\lambda+\alpha)\right] + (\lambda_{c_1}+\lambda_{h_1})(\lambda+2\alpha)a_2}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&\quad + \frac{(\lambda+\lambda_{c_2}+\lambda_{c_2})(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&= \frac{\left[a_0-(\lambda+2\alpha)\right]\left[a_1a_2 - \mu(\lambda+\alpha)\right] + \left[a_1 - (\mu+\lambda+\alpha)\right](\lambda+2\alpha)a_2}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&\quad + \frac{(a_2-\mu)(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&= \frac{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - a_1a_2(\lambda+2\alpha) + \mu(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}\\
&\quad + \frac{a_1a_2(\lambda+2\alpha) - \mu(\lambda+2\alpha)a_2 - a_2(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2 - \mu(\lambda+\alpha)\right] - \mu(\lambda+2\alpha)a_2}
\end{aligned}
$$

$$+\frac{a_2(\lambda+\alpha)(\lambda+2\alpha)-\mu(\lambda+\alpha)(\lambda+2\alpha)}{a_0\left[a_1a_2-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)a_2}$$
$$=\frac{a_0\left[a_1a_2-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)a_2}{a_0\left[a_1a_2-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)a_2}=1. \tag{3.36}$$

This shows that ΦD_0 is column stochastic, its transpose $(\Phi D_0)^t$ is row stochastic, and hence $1 \in \sigma_p((\Phi D_0)^t)$. Since $\sigma_p(\Phi D_0) = \sigma_p((\Phi D_0)^t)$, also $1 \in \sigma_p(\Phi D_0)$ holds. Therefore, by the Characteristic Equation 3.8 we conclude that $0 \in \sigma_p(A)$. □

Indeed, 0 is even the only spectral value of A on the imaginary axis.

Lemma 8.9. *Under Assumption 1.1, the spectrum $\sigma(A)$ of A satisfies*

$$\sigma(A)\cap i\mathbb{R}=\{0\}. \tag{3.37}$$

Proof. For any $a\in\mathbb{R}$, $a\neq 0$, $\Psi=(\psi_0,\psi_1,\psi_2,\psi_3(x),\psi_4(x),\psi_5(x))\in X$, we consider the resolvent equation

$$(aiId-A)P=\Psi, \tag{3.38}$$

where $P=(p_0,p_1,p_2,p_3(x),p_4(x),p_5(x))$. This equation is equivalent to the following system of equations:

$$(ai-a_0)p_0+\mu p_1+\sum_{j=3}^{5}\int_0^\infty \mu_j(x)p_j(x)dx=\psi_0, \tag{3.39}$$

$$-(\lambda+2\alpha)p_0+(ai-a_1)p_1-\mu p_2=\psi_1, \tag{3.40}$$

$$-(\lambda+\alpha)p_1+(ai-a_2)p_2=\psi_2, \tag{3.41}$$

$$\frac{dp_j(x)}{dx}+\left(ai-\mu_j(x)\right)p_j(x)=\psi_j(x),\quad j=3,4,5, \tag{3.42}$$

$$p_3(0)=\lambda p_2,\quad t>0, \tag{3.43}$$

$$p_4(0)=\sum_{i=0}^{2}\lambda_{c_i}p_i, \tag{3.44}$$

$$p_5(0)=\sum_{i=0}^{2}\lambda_{h_i}p_i. \tag{3.45}$$

Solving (3.42)–(3.45), we get

$$p_j(x)=p_j(0)e^{-aix-\int_0^x\mu_j(\xi)d\xi}+e^{-aix-\int_0^x\mu_j(\xi)d\xi}\int_0^x\psi_j(u)e^{aiu+\int_0^u\mu_j(\xi)d\xi}du. \tag{3.46}$$

Since

$$\int_0^\infty |p_j(x)|dx = |p_j(0)|\int_0^\infty e^{-aix-\int_0^x \mu_j(\xi)d\xi}dx + \int_0^\infty \left[e^{-\int_0^x \mu_j(\xi)d\xi}\int_0^x |\varphi_j(u)|e^{\int_0^u \mu_j(\xi)d\xi}du\right]dx,$$
$$\int_0^\infty \left[e^{-\int_0^x \mu_j(\xi)d\xi}\int_0^x |\varphi_j(u)|e^{\int_0^u \mu_j(\xi)d\xi}du\right]dx = \int_0^\infty \left[|\varphi_j(u)|e^{\int_0^u \mu_j(\xi)d\xi}\int_u^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx\right]du. \tag{3.47}$$

By Assumption 1.1, we have

$$\begin{aligned}
\lim_{u\to+\infty} e^{\int_0^u \mu_j(\xi)d\xi}\int_u^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx &= \lim_{u\to+\infty}\frac{\int_u^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx}{e^{-\int_0^u \mu_j(\xi)d\xi}} \\
&= \lim_{u\to+\infty}\frac{e^{-\int_0^u \mu_j(\xi)d\xi}dx}{\mu_j(u)e^{-\int_0^u \mu_j(\xi)d\xi}} \\
&= \lim_{u\to+\infty}\frac{1}{\mu_j(u)} \\
&< +\infty.
\end{aligned} \tag{3.48}$$

It follows that $p_j(x) \in L^1[0,+\infty)$, $j = 3,4,5$. Let

$$I_j = \int_0^\infty \mu_j(x)e^{-aix-\int_0^x \mu_j(\xi)d\xi}dx,$$
$$K_j = \int_0^\infty \left[\mu_j(x)e^{-aix-\int_0^x \mu_j(\xi)d\xi}\int_0^x \varphi_j(u)e^{aiu+\int_0^u \mu_j(\xi)d\xi}du\right]dx. \tag{3.49}$$

Then

$$|I_j| \le \int_0^\infty \mu_j(x)e^{-\int_0^x \mu_j(\xi)d\xi}dx = 1,$$
$$|K_j| \le \int_0^\infty \left[\mu_j(x)e^{-\int_0^x \mu_j(\xi)d\xi}\int_0^x |\varphi_j(u)|e^{\int_0^u \mu_j(\xi)d\xi}du\right]dx. \tag{3.50}$$

Since

$$\int_0^\infty \mu_j(x)p_j(x)dx = p_j(0)I_j + K_j, \tag{3.51}$$

hence $\mu_j(x)p_j(x) \in L^1[0,+\infty)$, $j = 3,4,5$.

Substituting $p_j(x)$ into (3.39)–(3.41) we get the following system of equations:

$$
\begin{aligned}
&(ai + a_0 - \lambda_{c_0} I_4 - \lambda_{h_0} I_5)p_0 - (\mu + \lambda_{c_1} + \lambda_{h_1} I_5)p_1 \\
&\qquad - (\lambda I_3 + \lambda_{c_2} I_4 + \lambda_{h_2} I_5)p_2 = \psi_0 + K_3 + K_4 + K_5, \\
&(-\lambda - 2\alpha)p_0 + (ai + a_1)p_1 - \mu p_2 = \psi_1, \\
&(-\lambda - \alpha)p_1 + (ai + a_2)p_2 = \psi_2.
\end{aligned}
\tag{3.52}
$$

The matrix of the coefficient of the above system is denoted by

$$
D = \begin{pmatrix} ai + a_0 - \lambda_{c_0} I_4 - \lambda_{h_0} I_5 & -\mu - \lambda_{c_1} - \lambda_{h_1} I_5 & -\lambda I_3 - \lambda_{c_2} I_4 - \lambda_{h_2} I_5 \\ -\lambda - 2\alpha & ai + a_1 & -\mu \\ 0 & -\lambda - \alpha & ai + a_2 \end{pmatrix}. \tag{3.53}
$$

Since

$$
\begin{aligned}
|ai + a_0 - \lambda_{c_0} I_4 - \lambda_{h_0} I_5| &\geq |ai + a_0| - \lambda_{c_0}|I_4| - \lambda_{h_0}|I_5| \\
&> |ai + a_0| - \lambda_{c_0} - \lambda_{h_0} \\
&> a_0 - \lambda_{c_0} - \lambda_{h_0} = \lambda + 2\alpha, \\
\left|-\mu - \lambda_{c_1} - \lambda_{h_1} I_5\right| + |-\lambda - \alpha| &\leq \mu + \lambda_{c_1}|I_4| + \lambda_{h_1}|I_5| + \lambda + \alpha \\
&< \mu + \lambda_{c_1} + \lambda_{h_1} + \lambda + \alpha = a_1 \\
&< |ai + a_1|, \\
|-\lambda I_3 - \lambda_{c_2} I_4 - \lambda_{h_2} I_5| + \left|-\mu\right| &\leq \lambda|I_3| + \lambda_{c_2}|I_4| + \lambda_{h_2}|I_5| + \mu \\
&< \lambda + \lambda_{c_2} + \lambda_{h_2} + \mu = a_2 \\
&< |ai + a_2|.
\end{aligned}
\tag{3.54}
$$

This shows that the matrix D is a diagonally dominant matrix, it follows that the determinant of the matrix D is not equal to 0. Therefore, system (3.52) has a unique solution (p_0, p_1, p_2). Combining this with (3.46) we obtain that the equation $(aiId - A)P = \Psi$ has exactly one solution $(p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \in D(A)$, this yields $ai \in \rho(A)$. □

4. Well-Posedness of the System

The main gaol in this section is to prove the well-posedness of the system. In order to prove this, we will need some lemmas.

Lemma 8.1. $A : D(A) \to R(A) \subset X$ *is a closed linear operator and* $D(A)$ *is dense in* X.

Proof. We will prove the assertion in two steps.

We first prove that A is closed. For any given

$$P_n = \left(p_0^{(n)}, p_1^{(n)}, p_2^{(n)}, p_3^{(n)}(x), p_4^{(n)}(x), p_5^{(n)}(x)\right)^t \in D(A),$$
$$P_0 = \left(p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_3^{(0)}(x), p_4^{(0)}(x), p_5^{(0)}(x)\right)^t \in \mathrm{X}. \tag{4.1}$$

We suppose that

$$\lim_{n\to\infty} P_n = P_0,$$
$$\lim_{n\to\infty} AP_n = F, \tag{4.2}$$

where $F = (f_0, f_1, f_2, f_3(x), f_4(x), f_5(x))^t \in \mathrm{X}$. That is,

$$\lim_{n\to\infty} p_i^{(n)} = p_i^{(0)} \quad (i = 0,1,2),$$
$$\lim_{n\to\infty} \int_0^\infty \left|p_j^{(n)}(x) - p_j^{(0)}(x)\right| dx = 0 \quad (j = 3,4,5). \tag{4.3}$$

Then we obtain from Assumption 1.1 that

$$\lim_{n\to\infty} \int_0^\infty p_j^{(n)}(x)\mu_j(x)dx = \int_0^\infty p_j^{(0)}(x)\mu_j(x) \quad j = 3,4,5. \tag{4.4}$$

Furthermore,

$$\lim_{n\to\infty} AP_n = \lim_{n\to\infty} \begin{pmatrix} -a_0 p_0^{(n)} + \mu p_1^{(n)} + \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i^{(n)}(x)dx \\ (\lambda + 2\alpha) p_0^{(n)} - a_1 p_1^{(n)} + \mu p_2^{(n)} \\ (\lambda + \alpha) p_1^{(n)} - a_2 p_2^{(n)} \\ -\dfrac{dp_3^{(n)}(x)}{dx} - \mu_3(x) p_3^{(n)}(x) \\ -\dfrac{dp_4^{(n)}(x)}{dx} - \mu_4(x) p_4^{(n)}(x) \\ -\dfrac{dp_5^{(n)}(x)}{dx} - \mu_5(x) p_5^{(n)}(x) \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{pmatrix}. \tag{4.5}$$

This is equivalent to the following system of equations:

$$\begin{aligned}
&\lim_{n\to\infty}\left[-a_0p_0^{(n)}+\mu p_1^{(n)}+\sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i^{(n)}(x)dx\right]=f_0,\\
&\lim_{n\to\infty}\left[(\lambda+2\alpha)p_0^{(n)}-a_1p_1^{(n)}+\mu p_2^{(n)}\right]=f_1,\\
&\lim_{n\to\infty}\left[(\lambda+\alpha)p_1^{(n)}-a_2p_2^{(n)}\right]=f_2,\\
&\lim_{n\to\infty}\left[-\frac{dp_3^{(n)}(x)}{dx}-\mu_3(x)p_3^{(n)}(x)\right]=f_3(x),\\
&\lim_{n\to\infty}\left[-\frac{dp_4^{(n)}(x)}{dx}-\mu_4(x)p_4^{(n)}(x)\right]=f_4(x),\\
&\lim_{n\to\infty}\left[-\frac{dp_5^{(n)}(x)}{dx}-\mu_5(x)p_5^{(n)}(x)\right]=f_5(x).
\end{aligned}\tag{4.6}$$

Integrating both sides of last three equations from 0 to $\beta > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\beta\left[-\frac{dp_j^{(n)}(x)}{dx}-\mu_j(x)p_j^{(n)}(x)\right]&=\int_0^\beta\lim_{n\to\infty}\left[-\frac{dp_j^{(n)}(x)}{dx}-\mu_j(x)p_j^{(n)}(x)\right]\\
&=\int_0^\beta f_j(x),\quad j=3,4,5.
\end{aligned}\tag{4.7}$$

This yields

$$\begin{aligned}
\lim_{n\to\infty}\left[-p_j^{(n)}(\beta)-p_j^{(n)}(0)-\int_0^\beta\mu_j(x)p_j^{(n)}(x)dx\right]&=-p_j^{(0)}(\beta)-p_j^{(0)}(0)-\int_0^\beta\mu_j(x)p_j^{(0)}(x)dx\\
&=\int_0^\beta f_j(x),\quad j=3,4,5.
\end{aligned}\tag{4.8}$$

We know from the boundedness of $\mu_j(x)$ that $\int_0^\infty|\mu_j(x)p_j^{(0)}(x)|dx<\infty$. Furthermore, we have $\int_0^\infty|f_j(x)|dx<\infty$. It follows from (4.8) that $p_j^{(0)}(\beta)$ is absolutely continuous and

$$p_j'^{(0)}(\beta)=-\mu_j(\beta)p_j^{(0)}(\beta)-f_j(x)\in L^1[0,\infty).\tag{4.9}$$

Therefore, $P_0\in D(A)$ and

$$\lim_{n\to\infty}p_j'^{(n)}(\beta)=\lim_{n\to\infty}\left[-\mu_j(\beta)p_j^{(n)}(\beta)\right]-f_j(x)=p_j'^{(0)}(\beta).\tag{4.10}$$

From the above deduction we have

$$\begin{aligned}
-a_0p_0^{(0)} + \mu p_1^{(0)} + \sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i^{(0)}(x)dx &= f_0,\\
(\lambda+2\alpha)p_0^{(0)} - a_1p_1^{(0)} + \mu p_2^{(0)} &= f_1,\\
(\lambda+\alpha)p_1^{(0)} - a_2p_2^{(0)} &= f_2,\\
-\frac{dp_3^{(0)}(x)}{dx} - \mu_3(x)p_3^{(0)}(x) &= f_3(x),\\
-\frac{dp_4^{(0)}(x)}{dx} - \mu_4(x)p_4^{(0)}(x) &= f_4(x),\\
-\frac{dp_5^{(0)}(x)}{dx} - \mu_5(x)p_5^{(0)}(x) &= f_5(x).
\end{aligned} \tag{4.11}$$

This shows that $A(P_0)^t = (F)^t$, hence $(A, D(A))$ is closed.

We now prove that $D(A)$ is dense in X. We define

$$E = \left\{ p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \;\middle|\; \begin{array}{c} p_i \in \mathbb{C},\ i = 0,1,2, \\ p_i(x) \in C_0^\infty[0,\infty),\ i = 3,4,5 \end{array} \right\}. \tag{4.12}$$

Then by [14] E is dense in X. If we define

$$H = \left\{ p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \;\middle|\; \begin{array}{c} p_i(x) \in C^\infty[0,\infty) \text{ and} \\ \text{there exists a number} \\ \alpha_i \text{ such that } p_i(x) = 0, \\ \text{for } x \in [0,\alpha_i],\ i = 3,4,5 \end{array} \right\}, \tag{4.13}$$

then H is dense in E. Therefore, in order to prove that $D(A)$ is dense in X, it suffices to prove that $D(A)$ is dense in H. Take any

$$p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \in H, \tag{4.14}$$

then there exist numbers α_i such that $p_i(x) = 0$, for all $x \in [0,\alpha_i]$ $(i = 3,4,5)$; that is, $p_i(x) = 0$ for $x \in [0,s]$, here $0 < s = \min\{\alpha_3, \alpha_4, \alpha_5\}$. We introduce a function

$$\begin{aligned}
\varphi^s(0) &= (\varphi_0^s, \varphi_1^s, \varphi_2^s, \varphi_3^s(0), \varphi_4^s(0), \varphi_5^s(0)) \\
&= \left(p_0, p_1, p_2, \lambda p_2, \sum_{i=0}^{2}\lambda_{c_i}p_i, \sum_{i=0}^{2}\lambda_{h_i}p_i \right)
\end{aligned} \tag{4.15}$$

$$\varphi^s(x) = (\varphi_0^s, \varphi_1^s, \varphi_2^s, \varphi_3^s(x), \varphi_4^s(x), \varphi_5^s(x)),$$

where

$$\varphi_i^s(x) = \begin{cases} \varphi_i^s(0)\left(1 - \dfrac{x}{s}\right)^2 & \text{if } x \in [0, s), \\ p_i(x) & \text{if } x \in [s, \infty), \end{cases} \quad i = 3, 4, 5. \tag{4.16}$$

It is easy to verify that $\varphi^s(x) \in D(A)$. Moreover

$$\|p - \varphi^s\| = \sum_{i=3}^{5} \int_0^s |\varphi_i^s(0)| \left(1 - \frac{x}{s}\right)^2 dx = \sum_{i=3}^{5} |\varphi_i^s(0)| \frac{s}{3} \longrightarrow 0 \quad \text{as } s \longrightarrow 0. \tag{4.17}$$

This shows that $D(A)$ is dense in H. □

Lemma 8.2. $(A, D(A))$ *is a dispersive operator.*

Proof. For $p \in D(A)$, we may choose

$$\phi(x) = \left(\frac{[p_0]^+}{p_0}, \frac{[p_1]^+}{p_1}, \frac{[p_2]^+}{p_2}, \frac{[p_3(x)]^+}{p_3(x)}, \frac{[p_4(x)]^+}{p_4(x)}, \frac{[p_5(x)]^+}{p_5(x)} \right), \tag{4.18}$$

where

$$[p_i]^+ = \begin{cases} p_i & \text{if } p_i > 0, \\ 0 & \text{if } p_i \le 0, \end{cases} \quad i = 0, 1, 2, \qquad [p_i(x)]^+ = \begin{cases} p_i(x) & \text{if } p_i(x) > 0, \\ 0 & \text{if } p_i(x) \le 0, \end{cases} \quad i = 3, 4, 5. \tag{4.19}$$

If we define $W_i = \{x \in [0, \infty) \mid p_i(x) > 0\}$ and $Q_i = \{x \in [0, \infty) \mid p_i(x) \le 0\}$ for $i = 3, 4, 5$, then we have

$$\begin{aligned} \int_0^\infty \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx &= \int_{W_i} \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx + \int_{Q_i} \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx \\ &= \int_{W_i} \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx = \int_{W_i} \frac{dp_i(x)}{dx} dx \\ &= \int_0^\infty \frac{d[p_i(x)]^+}{dx} dx = -[p_i(0)]^+, \quad i = 3, 4, 5, \\ \int_0^\infty \mu_i(x) p_i(x) dx &\le \int_0^\infty \mu_i(x) [p_i(x)]^+ dx, \quad i = 3, 4, 5. \end{aligned} \tag{4.20}$$

By (4.20) and the boundary conditions on $p \in D(A)$ we obtain that

$$\begin{aligned}
\langle Ap, \phi \rangle &= \left\{ -a_0 p_0 + \mu p_1 + \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x) dx \right\} \frac{[p_0]^+}{p_0} \\
&\quad + \left\{ (\lambda + 2\alpha) p_0 - a_1 p_1 + \mu p_2 \right\} \frac{[p_1]^+}{p_1} + \left\{ (\lambda + \alpha) p_1 - a_2 p_2 \right\} \frac{[p_2]^+}{p_2} \\
&\quad + \sum_{i=3}^{5} \int_0^\infty \left\{ -\frac{dp_i(x)}{dx} - \mu_i(x) p_i(x) \right\} \frac{[p_i(x)]^+}{p_i(x)} dx \\
&= -a_0 [p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x) dx \\
&\quad + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1 [p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
&\quad - a_2 [p_2]^+ - \sum_{i=3}^{5} \int_0^\infty \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx - \sum_{i=3}^{5} \int_0^\infty \mu_i(x) [p_i(x)]^+ dx \\
&= -a_0 [p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x) dx \\
&\quad + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1 [p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
&\quad - a_2 [p_2]^+ + \sum_{i=3}^{5} [p_i(0)]^+ - \sum_{i=3}^{5} \int_0^\infty \mu_i(x) [p_i(x)]^+ dx \\
&= -a_0 [p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x) dx \\
&\quad + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1 [p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
&\quad - a_2 [p_2]^+ + \left\{ [\lambda p_2]^+ + [\lambda_{c_0} p_0 + \lambda_{c_1} p_1 + \lambda_{c_2} p_2]^+ + [\lambda_{h_0} p_0 + \lambda_{h_1} p_1 + \lambda_{h_2} p_2]^+ \right\} \\
&\quad - \sum_{i=3}^{5} \int_0^\infty \mu_i(x) [p_i(x)]^+ dx \\
&\leq -a_0 [p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x) dx \\
&\quad + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1 [p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
&\quad - a_2 [p_2]^+ + \left\{ \lambda [p_2]^+ + \left\{ \lambda_{c_0} [p_0]^+ + \lambda_{c_1} [p_1]^+ + \lambda_{c_2} [p_2]^+ \right\} \right.
\end{aligned}$$

$$
\begin{aligned}
&+\left\{\lambda_{h_0}[p_0]^+ + \lambda_{h_1}[p_1]^+ + \lambda_{h_2}[p_2]^+\right\} - \sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx \\
&= (\lambda_{c_0} + \lambda_{h_0} - a_0)[p_0]^+ + (\lambda_{c_1} + \lambda_{h_1} - a_0)[p_1]^+ + (\lambda + \lambda_{c_2} + \lambda_{h_2} - a_0)[p_2]^+ \\
&\quad + \mu p_1\frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0}\sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i(x)dx + (\lambda+2\alpha)\frac{[p_1]^+}{p_1}p_0 \\
&\quad + \mu\frac{[p_1]^+}{p_1}p_1 + (\lambda+\alpha)\frac{[p_2]^+}{p_2}p_1 - \sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx \\
&= -(\lambda+2\alpha)[p_0]^+ - (\mu+\lambda+\alpha)[p_1]^+ - \mu[p_2]^+ + \mu p_1\frac{[p_0]^+}{p_0} \\
&\quad + \frac{[p_0]^+}{p_0}\sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i(x)dx + (\lambda+2\alpha)\frac{[p_1]^+}{p_1}p_0 + \mu\frac{[p_1]^+}{p_1}p_2 \\
&\quad + (\lambda+\alpha)\frac{[p_2]^+}{p_2}p_1 - \sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx \\
&\le -(\lambda+2\alpha)[p_0]^+ - (\mu+\lambda+\alpha)[p_1]^+ - \mu[p_2]^+ + \mu p_1\frac{[p_0]^+}{p_0} \\
&\quad + \frac{[p_0]^+}{p_0}\sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx + (\lambda+2\alpha)\frac{[p_1]^+}{p_1}p_0 + \mu\frac{[p_1]^+}{p_1}p_2 \\
&\quad + (\lambda+\alpha)\frac{[p_2]^+}{p_2}p_1 - \sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx \\
&\le -(\lambda+2\alpha)[p_0]^+ - (\mu+\lambda+\alpha)[p_1]^+ - \mu[p_2]^+ + \mu[p_1]^+\frac{[p_0]^+}{p_0} \\
&\quad + \left(\frac{[p_0]^+}{p_0} - 1\right)\sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx + (\lambda+2\alpha)\frac{[p_1]^+}{p_1}[p_0]^+ \\
&\quad + \mu\frac{[p_1]^+}{p_1}[p_2]^+ + (\lambda+\alpha)\frac{[p_2]^+}{p_2}[p_1]^+ \\
&\le -(\lambda+2\alpha)[p_0]^+ - (\mu+\lambda+\alpha)[p_1]^+ - \mu[p_2]^+ + \mu[p_1]^+ \\
&\quad + \left(\frac{[p_0]^+}{p_0} - 1\right)\sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx + (\lambda+2\alpha)[p_0]^+ + \mu[p_2]^+ + (\lambda+\alpha)[p_1]^+ \\
&= \left(\frac{[p_0]^+}{p_0} - 1\right)\sum_{i=3}^{5}\int_0^\infty \mu_i(x)[p_i(x)]^+dx \le 0.
\end{aligned}
\tag{4.21}
$$

This shows that $(A, D(A))$ is a dispersive operator. □

Lemma 8.3. *If $\gamma \in \mathbb{R}$, $\gamma > 0$, then $\gamma \in \rho(A)$.*

Proof. Let $\gamma \in \mathbb{R}$, $\gamma > 0$, then all the entries of ΦD_γ are positive and we have

$$\begin{aligned}
&a_0\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)-\gamma(\lambda+2\alpha)(\gamma+a_2)\\
&\quad=(\lambda+2\alpha+\lambda_{c_0}+\lambda_{h_0})(\gamma+\mu+\lambda+\alpha+\lambda_{c_1}+\lambda_{h_1})(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\\
&\qquad-(\lambda+2\alpha+\lambda_{c_0}+\lambda_{h_0})\mu(\lambda+\alpha)-\mu(\lambda+2\alpha)(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\\
&\qquad-\gamma(\lambda+2\alpha)(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\\
&\quad=\left[(\lambda+2\alpha+\lambda_{c_0}+\lambda_{h_0})\gamma(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\right.\\
&\qquad\quad\left.-\gamma(\lambda+2\alpha)(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\right]\\
&\qquad+\left[(\lambda+2\alpha+\lambda_{c_0}+\lambda_{h_0})\mu(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\right.\\
&\qquad\quad\left.-\mu(\lambda+2\alpha)(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\right]\\
&\qquad+\left[(\lambda+2\alpha+\lambda_{c_0}+\lambda_{h_0})(\lambda+\alpha+\lambda_{c_1}+\lambda_{h_1})(\gamma+\mu+\lambda+\lambda_{c_2}+\lambda_{h_2})\right.\\
&\qquad\quad\left.-(\lambda+2\alpha+\lambda_{c_0}+\lambda_{h_0})\mu(\lambda+\alpha)\right]>0\\
&\quad\Longrightarrow a_0\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)>\gamma(\lambda+2\alpha)(\gamma+a_2).
\end{aligned}\tag{4.22}$$

We also have

$$\int_0^\infty \mu_j(x)e^{-\gamma s-\int_0^s \mu_j(\xi)d\xi}ds<\int_0^\infty \mu_j(x)e^{-\int_0^s \mu_j(\xi)d\xi}ds=1.\tag{4.23}$$

Using (4.22) and (4.23) we can estimate the jth column sum as

$$\begin{aligned}
\sum_{i=1}^{3}(\Phi D_\gamma)_{i,j}&=\frac{\lambda(\lambda+\alpha)(\lambda+2\alpha)\int_0^\infty \mu_j(x)e^{-\gamma x-\int_0^x \mu_j(\xi)d\xi}dx}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)}\\
&\quad+\frac{\lambda_{c_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{c_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{c_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)}\\
&\quad\times\int_0^\infty \mu_j(x)e^{-\gamma x-\int_0^x \mu_j(\xi)d\xi}dx\\
&\quad+\frac{\lambda_{h_0}\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\lambda_{h_1}(\lambda+2\alpha)(\gamma+a_2)+\lambda_{h_2}(\lambda+\alpha)(\lambda+2\alpha)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)}\\
&\quad\times\int_0^\infty \mu_j(x)e^{-\gamma x-\int_0^x \mu_j(\xi)d\xi}dx
\end{aligned}$$

$$= \left\{ 1 - \frac{\gamma\left[(\lambda+2\alpha)(\gamma+a_2)+(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \right\}$$
$$\times \int_0^\infty \mu_j(x) e^{-\gamma x-\int_0^x \mu_j(\xi)d\xi} dx$$
$$= \left\{ 1 - \frac{\gamma\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]+\gamma(\lambda+2\alpha)(\gamma+a_2)}{(\gamma+a_0)\left[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)\right]-\mu(\lambda+2\alpha)(\gamma+a_2)} \right\}$$
$$\times \int_0^\infty \mu_j(x) e^{-\gamma x-\int_0^x \mu_j(\xi)d\xi} dx < 1. \tag{4.24}$$

It follows from this that $\|\Phi D_\gamma\| < 1$, and thus also

$$r(\Phi D_\gamma) \le \|\Phi D_\gamma\| < 1. \tag{4.25}$$

Therefore, $1 \notin \sigma(\Phi D_\gamma)$. Using the Characteristic Equation 3.8 we conclude that $\gamma \in \rho(A)$ for $\gamma \in \mathbb{R}$, $\gamma > 0$. □

From Lemmas 8.1, 8.2, and 8.3 and Phillips theorem (see [8, Theorem C-II 1.2]), we immediately obtain the following result.

Theorem 8.4. *The operator $(A, D(A))$ generates a positive contraction C_0-semigroup $(T(t))_{t\ge 0}$.*

We now characterize the well-posedness of (ACP) as follows; see [6, Corollary II.6.9].

Theorem 8.5. *For a closed operator $(A, D(A))$ on X the associated abstract Cauchy problem (ACP) is well-posed if and only if $(A, D(A))$ generates a strongly continuous semigroup on X.*

From Theorem 8.5 and [6, Proposition II.6.2] we can state our main result.

Theorem 8.6. *The system (R), (BC), and (IC_0) has a unique positive solution $p(x,t)$ which satisfies $\|p(\cdot,t)\| = 1$, $t \in [0,\infty)$.*

Proof. From Theorems 8.4, 8.5, and [6, Proposition II.6.2] we obtain that the associated abstract Cauchy problem (ACP) has a unique positive time-dependent solution $p(x,t)$ which can be expressed as

$$p(x,t) = T(t)p(0) = T(t)(1,0,0,0,0,0). \tag{4.26}$$

Let $P(t) = p(x,t) = (p_0(t), p_1, p_2, p_3(x,t), p_4(x,t), p_5(x,t))$, then $P(t)$ satisfies the system of equations

$$
\begin{aligned}
\frac{dp_0(t)}{dt} &= -a_0 p_0(t) + \mu p_1(t) + \sum_{i=3}^{5}\int_0^\infty \mu_i(x) p_i(x,t)\,dx,\\
\frac{dp_1(t)}{dt} &= (\lambda + 2\alpha) p_0(t) - a_1 p_1(t) + \mu p_2(t),\\
\frac{dp_2(t)}{dt} &= (\lambda + \alpha) p_1(t) - a_2 p_2(t),\\
\frac{\partial p_3(x,t)}{\partial t} &= -\frac{\partial p_3(x,t)}{\partial x} - \mu_3(x) p_3(x,t),\\
\frac{\partial p_4(x,t)}{\partial t} &= -\frac{\partial p_4(x,t)}{\partial x} - \mu_4(x) p_4(x,t),\\
\frac{\partial p_5(x,t)}{\partial t} &= -\frac{\partial p_5(x,t)}{\partial x} - \mu_5(x) p_5(x,t),\\
p_3(0,t) &= \lambda p_2(t), \quad t > 0,\\
p_4(0,t) &= \sum_{i=0}^{2} \lambda_{c_i} p_i(t), \quad t > 0,\\
p_5(0,t) &= \sum_{i=0}^{2} \lambda_{h_i} p_i(t), \quad t > 0,\\
P(0) &= (1,0,0,0,0,0).
\end{aligned}
\tag{4.27}
$$

Since

$$
\int_0^\infty \frac{\partial p_j(x,t)}{\partial x} dx = p_j(\infty,t) - p_j(0,t) = -p_j(0,t), \quad j = 3,4,5. \tag{4.28}
$$

Using (4.27)-(4.28) we compute

$$
\begin{aligned}
\frac{d\|P(t)\|}{dt} &= \sum_{i=0}^{2} \frac{dp_i(t)}{dt} + \sum_{j=3}^{5}\int_0^\infty \frac{\partial p_j(x,t)}{\partial t} dx\\
&= -a_0 p_0(t) + \mu p_1(t) + \sum_{j=3}^{5}\int_0^\infty \mu_i(x) p_i(x,t)\,dx,\\
&\quad + (\lambda + 2\alpha) p_0(t) - a_1 p_1(t) + \mu p_2(t) + (\lambda + \alpha) p_1(t) - a_2 p_2(t),\\
&\quad + \sum_{j=3}^{5}\int_0^\infty \left[-\frac{\partial p_j(x,t)}{\partial x} - \mu_j(x) p_j(x,t)\right] dx
\end{aligned}
$$

$$= (-a_0 + \lambda + 2\alpha)p_0(t) + \left(\mu - a_1 + \lambda + \alpha\right) + \left(\mu - a_2\right)p_2(t) + \sum_{j=3}^{5} p_j(0,t)$$
$$= -\sum_{j=3}^{5} p_j(0,t) + \sum_{j=3}^{5} p_j(0,t) = 0. \tag{4.29}$$

By (4.26) and (4.29) we obtain

$$\frac{d\|P(t)\|}{dt} = \frac{d\|T(t)P(0)\|}{dt} = 0. \tag{4.30}$$

Therefore,

$$\|T(t)P(0)\| = \|P(t)\| = \|P(0)\| = 1. \tag{4.31}$$

This shows $\|p(\cdot,t)\| = 1$, for all $t \in [0,\infty)$. □

5. Asymptotic Stability of the Solution

In this section, we prove the asymptotic stability of the system by using C_0-semigroup theory. First we express the resolvent of A in terms of the resolvent of A_0, the Dirichlet operator D_γ and the boundary operator Φ, compare with [10].

Lemma 8.1. *Let $\gamma \in \rho(A_0) \cap \rho(A)$. Then*

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma (Id - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0). \tag{5.1}$$

Proof. Under our assumption, we see from the Characteristic Equation 3.8 that $1 \notin \sigma(\Phi D_\gamma)$ and it follows from theProof that $\gamma - \mathcal{A}$ is invertible with inverse

$$R(\gamma, \mathcal{A}) = (\gamma - \mathcal{A}_0)^{-1} (\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))^{-1}. \tag{5.2}$$

Using the explicit representation (3.28) for $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ we compute

$$(\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))^{-1} = \begin{pmatrix} Id_X & 0 \\ (Id_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0) & (Id_{\partial X} - \Phi D_\gamma)^{-1} \end{pmatrix}. \tag{5.3}$$

Define $R(\gamma) := (Id_X + D_\gamma (Id_{\partial X} - \Phi D_\gamma)^{-1} \Phi) R(\gamma, A_0)$. Then

$$R(\gamma, \mathcal{A}) = \begin{pmatrix} R(\gamma) & D_\gamma (Id_{\partial X} - \Phi D_\gamma)^{-1} \\ 0 & 0 \end{pmatrix}. \tag{5.4}$$

Since

$$\begin{pmatrix} R(\gamma) & 0 \\ 0 & 0 \end{pmatrix} = R(\gamma, \mathcal{A})|_{\mathcal{X}_0} = R(\gamma, \mathcal{A}|_{\mathcal{X}_0}) \tag{5.5}$$

and since $A \cong \mathcal{A}|_{\mathcal{X}_0}$, it follows that

$$R(\gamma, A) = R(\gamma). \tag{5.6}$$

□

The above representation for the resolvent of A_0 shows that it is a positive operator for $\gamma > 0$. This property is very useful in the following lemma to prove the irreducibility of the semigroup generated by A. For the notation and terminology concerning positive operators we refer to the books [8, 15].

Lemma 8.2. *The semigroup $(T(t))_{t\geq 0}$ generated by $(A, D(A))$ is irreducible.*

Proof. We know from [8, Definition C-III 3.1] that the irreducibility of $(T(t))_{t\geq 0}$ is equivalent to the existence of $\gamma > 0$ such that $0 < p \in X$ implies $R(\gamma, A)p \gg 0$. We now suppose that $\gamma > 0$ and $0 < p \in \mathrm{X}$. Then also $R(\gamma, A_0)p > 0$ and $\Phi R(\gamma, A_0)p > 0$. It follows from theProof of Lemma 8.3 that $\|\Phi D_\gamma\| < 1$ for all $\gamma > 0$. Hence the inverse of $Id_{\partial X} - \Phi D_\gamma$ can be computed via the Neumann series

$$(Id_{\partial X} - \Phi D_\gamma)^{-1} = \sum_{n=0}^{\infty} (\Phi D_\gamma)^n. \tag{5.7}$$

We know from the form of ΦD_γ that for every $i \in \{1, 2, 3\}$ there exists $k \in \mathbb{N}$ such that the real number $((\Phi D_\gamma)^k \Phi R(\gamma, A_0)p)_i > 0$. Therefore,

$$(Id_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0)p \gg 0, \tag{5.8}$$

and by the form of D_γ we have

$$D_\gamma (Id_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0)p \gg 0. \tag{5.9}$$

This implies

$$R(\gamma, A)p \gg 0, \tag{5.10}$$

and hence $(T(t))_{t\geq 0}$ is irreducible. □

We now use the information obtained on $\sigma(A) \cap i\mathbb{R}$ and on $(T(t))_{t\geq 0}$ to prove our main result on the asymptotic behaviour of the solutions of (ACP). We first show that the semigroup is relatively weakly compact, see [6, Section V.2.b], and then we argue as in [16, 17]. Denote by

$$\operatorname{fix}(T(t))_{t\geq 0} := \bigcap_{t\geq 0} \operatorname{fix}(T(t)) = \{p \in X : T(t)p = p \, \forall t \geq 0\}. \tag{5.11}$$

According to [6, Corollary IV.3.8(i)] we have the equality

$$\operatorname{fix}(T(t))_{t\geq 0} = \ker A. \tag{5.12}$$

To study the asymptotic behaviour of the semigroup $(T(t))_{t\geq 0}$ the following compactness property is useful.

Lemma 8.3. *The set $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively compact for the weak operator topology. In particular, it is mean ergodic, that is,*

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r T(s)p \, ds \tag{5.13}$$

exists for all $p \in X$.

Proof. From $0 \in \sigma_p(B)$ and (5.12) it follows that there exists $0 \neq p \in \operatorname{fix}(T(t))_{t\geq 0}$. By the positivity of the semigroup we have

$$|p| = |T(t)p| \leq T(t)|p| \quad \forall t \geq 0. \tag{5.14}$$

Suppose that $|p| < T(t)|p|$. Since $(T(t))_{t\geq 0}$ is a contraction semigroup and the norm on X is strictly monotone, we obtain that

$$\|p\| < \|T(t)|p|\| \leq \|p\|, \tag{5.15}$$

which is a contradiction. Thus

$$|p| = T(t)|p| \tag{5.16}$$

holds, and we can already assume that $p > 0$. Since $(T(t))_{t\geq 0}$ is irreducible, we obtain from [8, Proposition C-III 3.5(a)] that p is a quasi-interior point of X which implies that

$$X_p := \bigcup_{n\geq 1} [-np, np] \tag{5.17}$$

is dense in X. Let $n \in \mathbb{N}$ and take $w \in [-np, np]$, that is, $-np \leq w \leq np$. Then

$$-np = -nT(t)p \leq T(t)w \leq nT(t)p = np \quad \forall t \geq 0. \tag{5.18}$$

Since the order interval $[-np, np]$ is weakly compact in X, see [15, page 92], the orbit $\{T(t)w : t \geq 0\}$ is relatively weakly compact in X. So far, we have shown that the orbits of elements $w \in X_p$ are relatively weakly compact. Since the semigroup $(T(t))_{t\geq 0}$ is bounded and X_p is dense in X, we know from [6, Lemma V.2.7] that $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively weakly compact. By [6, Lemma V.2.7] we obtain that the semigroup $(T(t))_{t\geq 0}$ is mean ergodic. □

We can now show the convergence of the semigroup to a one-dimensional equilibrium point.

Theorem 8.4. *The space X can be decomposed into the direct sum*

$$X = X_1 \oplus X_2, \tag{5.19}$$

where $X_1 = \operatorname{fix}(T(t))_{t\geq 0} = \ker A$ is one dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A$ of A. In addition, the restriction $(T(t)|_{X_2})_{t\geq 0}$ is strongly stable.

Proof. Since by Lemma 8.3 every $p \in X$ has a relatively weakly compact orbit, $(T(t))_{t\geq 0}$ is totally ergodic; see [18, Proposition 4.3.12]. This implies that X can be decomposed into

$$X = \ker A \oplus \overline{\operatorname{rg}(A)} =: X_1 \oplus X_2, \tag{5.20}$$

where $\ker A = \operatorname{fix}(T(t))_{t\geq 0}$ and X_1 and X_2 are invariant under $(T(t))_{t\geq 0}$; see [6, Lemma V.4.4]. There exists $\tilde{p} \in \ker A$ such that $\tilde{p} > 0$; see theProof of Lemma 8.3. Moreover, by the same construction as in theProof of [6, Lemma V.2.20(i)], we find $p' \in X'$ such that $p' > 0$ and $A'p' = 0$. Hence we obtain that

$$\dim \ker A = 1, \tag{5.21}$$

and that $\tilde{p}$ is strictly positive, that is, $\tilde{p} \gg 0$; see [8, Proposition C-III 3.5].

We now consider the generator $(A_2, D(A_2))$ of the restricted semigroup $(T_2(t))_{t\geq 0}$, where

$$A_2 v = Av, \qquad D(A_2) = D(A) \cap X_2, \tag{5.22}$$

and $T_2(t) = T(t)|_{X_2}$. Clearly, $(T_2(t))_{t\geq 0}$ is bounded and totally ergodic on X_2; that is, $(e^{-iat}T(t))_{t\geq 0}$ is mean ergodic for all $a \in \mathbb{R}$. This implies that $\ker(A_2 - iat)$ separates $\ker(A_2' - iat)$ for all $a \in \mathbb{R}$; see [6, Theorem V.4.5]. By Lemma 8.9 $\ker(A_2 - iat) = \{0\}$, thus $\ker(A_2' - iat) = \{0\}$ for all $a \in \mathbb{R}$. Hence it follows that $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. Applying the Arendt-Batty-Lyubich-Vũ Theorem, see [18, Theorem 5.5.5], we obtain the strong stability of $(T_2(t))_{t\geq 0}$. □

Combining Lemmas 8.8, 8.9, and 8.3 with Theorem 8.4 we obtain the following main result.

Corollary 8.5. *There exists $p' \in X'$, $p' \gg 0$, such that for all $p \in X$*

$$\lim_{t\to\infty} T(t)p = \langle p', p\rangle \tilde{p}, \tag{5.23}$$

where $\ker A = \langle \tilde{p}\rangle$, $\tilde{p} \gg 0$.

Since the semigroup gives the solutions of the original system, we obtain our final result.

Corollary 8.6. *The time-dependent solution of the system (R), (BC), and (IC_0) converges strongly to the steady-state solution as time tends to infinite, that is,* $\lim_{t\to\infty} p(\cdot, t) = \alpha\tilde{p}$*, where $\alpha > 0$ and $\tilde{p}$ as in Corollary 8.5.*

6. Conclusions

In this paper, we considered a repairable system involving primary as well as secondary failures. By using the C_0-semigroup theory of bounded linear operator on Banach space, we proved that the corresponding dynamic operator generates positive contractive C_0-semigroup and the system is well-posed. Furthermore, we proved the existence of positive solution of the system. Moreover, we obtained the result on the asymptotic stability of the solution of this system, that is, the convergence to a one-dimensional equilibrium. TheProof is based on the Arendt-Batty-Lyubich-Vũ Theorem [18, Theorem 5.5.5].

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