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Research Article

Existence of Multiple Solutions for a Class of n-Dimensional Discrete Boundary Value Problems

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By using critical point theory, we obtain some new results on the existence of multiple solutions for a class of n-dimensional discrete boundary value problems. Results obtained extend or improve existing ones.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of all natural numbers, integers, and real numbers, respectively. For any $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, a+2, \dots\}$, $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a < b$.

In this paper, we consider the existence of multiple solutions for the following n-dimensional discrete nonlinear boundary value problem:

$$\begin{gathered} \Delta\left[p(k)\Delta X(k-1)\right] + q(k)X(k) = f(k, X(k)), \quad k \in \mathbb{Z}(a+1, b), \\ X(a) + \alpha X(a+1) = A, \qquad X(b+1) + \beta X(b) = B, \end{gathered} \tag{1.1}$$

where $n \in \mathbb{N}$, $X(k) = (x_1(k), x_2(k), \dots, x_n(k))^T \in \mathbb{R}^n$, Δ is the forward difference operator defined by $\Delta X(k) = X(k+1) - X(k)$, $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(k, U) = (f_1(k,U), f_2(k,U), \dots, f_n(k,U))^T$, and is continuous for U. $a, b \in \mathbb{Z}$ with $a < b$, α and β are constants, $A = (a_1, a_2, \dots, a_n)^T$ and $B = (b_1, b_2, \dots, b_n)^T$ are n-dimensional vectors, $p(k)$ and $q(k)$ are real value functions defined on $\mathbb{Z}(a+1, b+1)$ and $\mathbb{Z}(a+1, b)$, respectively, and $p(k) \neq 0$.

Existence of solutions of discrete boundary value problems has been the subject of many investigations. Motivated by Hartman's landmark paper [1], Henderson [2] showed that the uniqueness of solutions implies the existence of solutions for some conjugate boundary value problems. In recent years, by using various methods and techniques, such as nonlinear alternative of Leray-Schauder type, the cone theoretic fixed point theorem, and the method of upper and lower solution, a series of existence results for the solutions of the BVP (1.1) in some special cases, for example, $n = 1, \alpha = \beta = 0, A = B = 0$; $n = 1, \alpha = 0, \beta = -1, A = B = 0$ have been obtained in literatures. We refer to [3–7].

The critical point theory has been an important tool for investigating the periodic solutions and boundary value problems of differential equations [8–10]. In recent years, it is applied to the study of periodic solutions [11–13] and boundary value problems [14–17] of difference equations.

For the case when $n = 1$, the scalar BVP (1.1) was studied by Yu and Guo in [17]. By using critical point theorem, they obtained various conditions to guarantee the existence of one solution, but they did not obtain the existence conditions of multiple solutions. In this scalar case, the BVP can be viewed as the discrete analogue of the following self-adjoint differential equation:

$$\frac{d}{dt}\left(p(t)\frac{dy}{dt}\right) + q(t)y = f(t,y), \tag{1.2}$$

which is a generalization of Emden-Fowler equation:

$$\frac{d}{dt}\left(t^{\rho}\frac{dy}{dt}\right) + t^{\delta}y^{\gamma} = 0. \tag{1.3}$$

The Emden-Fowler equation originated from earlier theories of gaseous dynamics in astrophysics [18], and later, found applications in the study of fluid mechanics, relative mechanics, nuclear physics, and in the study of chemical reaction system [19].

For the case where $p(k) \equiv -1$, $q(k) \equiv 0$, $A = B = \theta$ (the zero vector of $\mathbb{R}^n$), BVP (1.1) are reduced to

$$\begin{aligned} &-\Delta^2 X(k-1) = f(k, X(k)), \quad k \in \mathbb{Z}(a+1, b), \\ &X(a) = -\alpha X(a+1), \qquad X(b+1) = -\beta X(b), \end{aligned} \tag{1.4}$$

which were studied by Jiang and Zhou in [15]. They obtained the existence results of multiple solutions by using critical point theory again.

In the aforementioned references, most of the difference equations involved are scalar. The purpose of this paper is further to demonstrate the powerfulness of critical point theory in the study of existence of discrete boundary value problems and obtain various conditions for the existence of at least two nontrivial solutions for the BVP (1.1).

The remaining of this paper is organized as follows. First, in Section 2, we will establish the variational framework associated with (1.1) and transfer the problem of the existence of solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some basic results will also be recalled. Then, in Section 3, we present various new conditions on the existence of at least two nontrivial solutions for the BVP (1.1). Some examples are given to illustrate the conclusions. We mention that our results generalize the ones in [15] and improve the ones in [20].

To conclude the introduction, we refer to [21, 22] for the general background on difference equations.

2. Preliminary and Variational Framework

Let E be a real Hilbert space, $J \in C^1(E,\mathbb{R})$, which means that J is a continuously Frećhet-differentiable functional defined on E. J is said to satisfy Palais-Smale condition (P-S condition for short), if any sequence $\{x_n\}_{n=1}^{\infty} \subset E$ for which $\{J(x_n)\}_{n=1}^{\infty}$ is bounded and $J'(x_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in E.

Let B_ρ be the open ball in E with radius ρ and centered at 0 and let ∂B_ρ denote its boundary. The following lemmas will be useful in the proofs of our main results.

Lemma 2.1. *(Mountain Pass Lemma [10]). Let E be a real Hilbert space, and assume that $J \in C^1(E,\mathbb{R})$ satisfies the P-S condition and the following conditions.*

(1) *There exist constant $\rho > 0$ and $a > 0$ such that $J(x) \geq a, \forall x \in \partial B_\rho$, where $B_\rho = \{x \in E : \|x\|_E < \rho\}$.*

(2) *$J(0) \leq 0$ and there exists $x_0 \notin B_\rho$ such that $J(x_0) \leq 0$.*

Then J possesses a critical value $c \geq a$. Moreover, $c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s))$, where

$$\Gamma = \{h \in C([0,1],E) \mid h(0) = 0, h(1) = x_0\}. \tag{2.1}$$

Lemma 2.2. *Let $a_i \geq 0$, $\gamma > 2$, $i \in \mathbb{Z}(1,m)$, then*

$$\frac{1}{\sqrt{m}}\sum_{i=1}^{m} a_i \leq \sqrt{\sum_{i=1}^{m} a_i^2} \leq \sum_{i=1}^{m} a_i. \tag{2.2}$$

Without loss of generality, we assume that $a = 0$, $b = m$ where $m \in \mathbb{N}$, and there exists a function $P(k,U) \in C^1(\mathbb{Z}(1,m) \times \mathbb{R}^n,\mathbb{R})$, such that for any $(k,U) \in \mathbb{Z}(1,m) \times \mathbb{R}^n$, $\nabla_U P(k,U) = f(k,U)$, where $\nabla_U P(k,U) = (\partial P/\partial u_1,\ldots,\partial P/\partial u_n)^T$ is the gradient of $P(k,U)$ in U, and $P(k,\theta) \geq 0$ where θ is the zero vector in $\mathbb{R}^n$.

Let $c(k) = q(k) - p(k) - p(k+1)$, then BVP (1.1) reduces to

$$p(k+1)X(k+1) + c(k)X(k) + p(k)X(k-1) = f(k,X(k)), \quad k \in \mathbb{Z}(1,m), \tag{2.3}$$

$$X(0) + \alpha X(1) = A, \qquad X(m+1) + \beta X(m) = B, \tag{2.4}$$

where $X(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T \in \mathbb{R}^n$, $\forall k \in \mathbb{Z}(0,m+1)$.

Define

$$H = \left\{ X = (X(1), \dots, X(m))^T \mid X(k) = (x_1(k), \dots, x_n(k)) \in \mathbb{R}^n, \forall k \in \mathbb{Z}(1, m) \right\}. \tag{2.5}$$

Then, H can be equipped with the inner product

$$\langle X, Y \rangle_H = \sum_{k=1}^{m} (X(k), Y(k)) = \sum_{k=1}^{m} \sum_{i=1}^{n} x_i(k) y_i(k), \tag{2.6}$$

by which the norm $\| \cdot \|_H$ can be induced by

$$\|X\|_H = \sqrt{\sum_{k=1}^{m} \|X(k)\|_n^2}, \quad \forall X \in H, \tag{2.7}$$

where $X(k) = (x_1(k), x_2(k), \dots, x_n(k))^T$, $Y(k) = (y_1(k), y_2(k), \dots, y_n(k))^T \in \mathbb{R}^n$, and $\| \cdot \|_n$ and $(\cdot, \cdot)$ are the norm and the inner product in $\mathbb{R}^n$, respectively.

Define a linear mapping $L : H \to \mathbb{R}^{mn}$ by

$$L(X) = (x_1(1), \dots, x_1(m), x_2(1), \dots, x_2(m), \dots, x_n(1), \dots, x_n(m))^T. \tag{2.8}$$

Then L is a linear and one to one mapping. Clearly, $\|LX\|_{mn} = \|X\|_H$.

With this mapping, BVP (2.3) and (2.4) can be represented by the matrix equation:

$$ML(X) + N = L(F(X)), \tag{2.9}$$

where $X \in H$, $F(X) = (f(1, X(1)), f(2, X(2)), \dots, f(m, X(m))) \in H$ and

$$M = \begin{pmatrix} Q & & \cdots & 0 \\ & Q & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & Q \end{pmatrix}_{mn \times mn} \tag{2.10}$$

with

$$Q = \begin{pmatrix} c(1)-\alpha p(1) & p(2) & 0 & \cdots & 0 & 0 \\ p(2) & c(2) & p(3) & \cdots & 0 & 0 \\ 0 & p(3) & c(3) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c(m-1) & p(m) \\ 0 & 0 & 0 & \cdots & p(m) & c(m)-\beta p(m+1) \end{pmatrix}_{m\times m}, \tag{2.11}$$

$$N = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}_{mn\times 1}, \qquad \eta_j = \begin{pmatrix} p(1)a_j \\ 0 \\ \vdots \\ 0 \\ p(m+1)b_j \end{pmatrix}_{m\times 1}, \qquad j = 1, 2, \ldots, n.$$

Define a functional J on H as

$$J(X) = \frac{1}{2}(MLX, LX) + (N, LX) - W(X), \tag{2.12}$$

where $W(X) = \sum_{k=1}^{m} P(k, X(k))$. Then $J \in C^1(H, \mathbb{R})$.

For any $X = (X(1), X(2), \ldots, X(m)) \in H$, denote

$$\overline{X} = (X(0), X(1), \ldots, X(m), X(m+1)), \tag{2.13}$$

where $X(0), X(1), X(m), X(m+1)$ satisfying (2.4). Thus, there is a one to one correspondence from H to

$$\overline{H} = \left\{(X(0), X(1), \ldots, X(m+1)) \mid X(0) + \alpha X(1) = A, X(m+1) + \beta X(m) = B\right\}. \tag{2.14}$$

Clearly, $J'(X) = 0$ if and only if $\overline{X}$ satisfies (2.3) and (2.4). Therefore, the existence of solutions to BVP (2.3) and (2.4) is transferred to the existence of the critical point of the functional J on H.

By *a solution* $\{X(k)\}_{k=0}^{m+1}$ of (2.3) and (2.4), we mean that $\{X(k)\}_{k=1}^{m}$ satisfies (2.3), and (2.4) holds. $\{X(k)\}_{k=0}^{m+1}$ is nontrivial if $\|(X(1), \ldots, X(m))\|_H \neq 0$.

3. Main Results

In this section, we will suppose that the matrix Q defined in (2.11) is positive definite, $\lambda_{\min}$, $\lambda_{\max}$ are the minimal eigenvalue and maximal eigenvalue of Q, respectively. It is clear that $\lambda_{\min}$, $\lambda_{\max}$ are also the minimal eigenvalue and maximal eigenvalue of M, respectively. The first result is as follows.

Theorem 3.1. *If there exist constants* $\alpha_1, \alpha_2 : 0 < \alpha_1 < \lambda_{\min} \leq \lambda_{\max} < \alpha_2$, *and* $\rho > 0$, $\gamma \geq 0$, $\delta > d = 2\|N\|_{mn}/(\lambda_{\min} - \alpha_1)$ *satisfying*

$$W(X) \leq \frac{1}{2}\alpha_1 \|X\|_H^2, \quad \forall X \in H, \|X\|_H < \delta, \tag{3.1}$$

$$W(X) \geq \frac{1}{2}\alpha_2 \|X\|_H^2 - \gamma, \quad \forall X \in H, \|X\|_H > \rho, \tag{3.2}$$

then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Proof. Since $P(k,\theta) \geq 0$, we know that $W(\overline{\theta}) \geq 0$ where $\overline{\theta}$ denotes the zero element of H. By (3.1), we get $W(\overline{\theta}) = 0$ and $J(\overline{\theta}) = 0$. Let

$$\omega = \max\left\{ \left| W(X) - \frac{1}{2}\alpha_2 \|X\|_H^2 + \gamma \right| : X \in H, \|X\|_H \leq \rho \right\}. \tag{3.3}$$

Then by (3.2), we get

$$W(X) \geq \frac{1}{2}\alpha_2 \|X\|_H^2 - (\gamma + \omega) = \frac{1}{2}\alpha_2 \|X\|_H^2 - \gamma', \quad \forall X \in H, \tag{3.4}$$

where $\gamma' = \gamma + \omega \geq 0$. Then for all $X \in H$, we have

$$\begin{aligned} J(X) &= \frac{1}{2}(MLX, LX) + (N, LX) - W(X) \\ &\leq \frac{1}{2}\lambda_{\max}\|X\|_H^2 + \|N\|_{mn}\|X\|_H - \left(\frac{1}{2}\alpha_2\|X\|_H^2 - \gamma'\right) \\ &= \frac{\lambda_{\max} - \alpha_2}{2}\|X\|_H^2 + \|N\|_{mn}\|X\|_H + \gamma'. \end{aligned} \tag{3.5}$$

Since $\lambda_{\max} < \alpha_2$, we see that $J(X) \to -\infty$ as $\|X\|_H \to +\infty$. Thus $J(X)$ is bounded from above on H, and $J(X)$ can achieve its maximum on H. In other words, there exists $X_1 \in H$, such that $J(X_1) = \sup_{X \in H} J(X)$. So X_1 is a critical point of $J(X)$, and $\overline{X}_1$ is a solution of BVP (2.3) and (2.4).

For any $X \in H$ with $d < \|X\|_H < \delta$, we have

$$J(X) \geq \frac{\lambda_{\min} - \alpha_1}{2}\|X\|_H^2 - \|N\|_{mn}\|X\|_H > 0. \tag{3.6}$$

So $J(X_1) = \sup_{X \in H} J(X) > 0$ and $\overline{X}_1$ is a nontrivial solution to BVP (2.3) and (2.4).

To obtain another nontrivial solution of BVP (2.3) and (2.4), we will use Mountain Pass Lemma. We first show that $J(X)$ satisfies P-S condition.

In fact, for any sequence $\{X^{(k)}\}_{k=1}^{\infty}$ in H, $\{J(X^{(k)})\}_{k=1}^{\infty}$ is bounded and $\lim_{k\to\infty} J'(X^{(k)}) = 0$, then there exists $K > 0$ such that $J(X^{(k)}) > -K$, and it follows from (3.5) that

$$-K \le J\left(X^{(k)}\right) \le \frac{\lambda_{\max} - \alpha_2}{2} \left\|X^{(k)}\right\|_H^2 + \|N\|_{mn} \left\|X^{(k)}\right\|_H + \gamma', \tag{3.7}$$

that is,

$$\frac{\lambda_{\max} - \alpha_2}{2} \|X_k\|_H^2 + \|N\|_{mn} \|X_k\|_H + \gamma' + K \ge 0. \tag{3.8}$$

Since $\lambda_{\max} < \alpha_2$, this implies that $\{X^{(k)}\}_{k=1}^{\infty}$ is bounded and possesses a convergent subsequence. So $J(X)$ satisfies P-S condition on H.

Choosing $r \in (d, \delta)$, then for $X \in \partial B_r = \{X \in H : \|X\|_H = r\}$, from (3.6), we get

$$J(X) \ge \frac{\lambda_{\min} - \alpha_1}{2} r^2 - \|N\|_{mn} r = a > 0. \tag{3.9}$$

This shows that J satisfies condition (1) of the Mountain Pass Lemma.

On the other hand, from (3.5), $J(X) \to -\infty$ as $\|X\|_H \to +\infty$, so there exists $P > r$ such that $J(X) < 0$ for $\|X\|_H > P$. Pick $X_0 \in H$ such that $\|X_0\|_H > P > r$, then $X_0 \in H \setminus B_r$, and $J(X_0) < 0$. So the condition (2) of the Mountain Pass Lemma is satisfied. Therefore, J possesses a critical value $c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s))$, where

$$\Gamma = \left\{h \in C([0,1], H) : h(0) = \overline{\theta}, h(1) = X_0\right\}. \tag{3.10}$$

A critical point corresponding to c is nontrivial as $c \ge a > 0$. Let X_2 be a critical point corresponding to the critical value c of J. If $X_2 \ne X_1$, then we are done. Otherwise, $X_2 = X_1$, which gives

$$\max_{X\in H} J(X) = J(X_1) = J(X_2) = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)). \tag{3.11}$$

Pick $h(s) = sX_0, s \in [0,1]$, then $h \in \Gamma$, and we have

$$\max_{X\in H} J(X) = J(X_1) = J(X_2) = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)) \le \max_{s\in[0,1]} J(sX_0). \tag{3.12}$$

Thus, there exists $s_0 \in (0,1)$ such that $J(X_2) = J(s_0X_0)$, and $X_3 = s_0X_0$ is also a critical point of $J(X)$ in H.

If $X_3 \ne X_1$, then Theorem 3.1 holds. Otherwise, $X_3 = X_1 = s_0X_0$. In this situation, we replace X_0 with $-X_0$ in the above arguments; then J possesses a critical value $c^* \ge a$ and $c^* = \inf_{h\in\Gamma^*} \max_{s\in[0,1]} J(h(s))$, where

$$\Gamma^* = \left\{h \in C([0,1], H) : h(0) = \overline{\theta}, h(1) = -X_0\right\}. \tag{3.13}$$

Assume that X_4 is a critical point corresponding to c^*. If $X_4 \neq X_1$, then the proof is complete. Otherwise, $X_4 = X_1$. Similarly, we can find a critical point X_5 of J such that $X_5 = -s_1 X_0$ holds for some $s_1 \in (0,1)$. Clearly, $X_5 \neq X_3$. The proof of Theorem 3.1 is now complete. □

From Theorem 3.1, we have the following corollaries.

Corollary 3.2. *If there exist constants α_1, α_2 : $0 < \alpha_1 < \lambda_{\min} \leq \lambda_{\max} < \alpha_2$, and $\rho > 0$, $\gamma \geq 0$, $\delta > d = 2\|N\|_{mn}/(\lambda_{\min} - \alpha_1)$ satisfying*

$$\begin{gathered} P(k,U) \leq \frac{1}{2}\alpha_1 \|U\|_n^2, \quad \|U\|_n < \delta, k \in \mathbb{Z}(1,m), \\ P(k,U) \geq \frac{1}{2}\alpha_2 \|U\|_n^2 - \gamma, \quad \|U\|_n > \rho, k \in \mathbb{Z}(1,m). \end{gathered} \tag{3.14}$$

Then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Proof. For any $X \in H$, $\|X\|_H < \delta$, then $\|X(k)\|_n \leq \|X\|_H < \delta$, $k \in \mathbb{Z}(1,m)$. So

$$W(X) = \sum_{k=1}^{m} P(k, X(k)) \leq \sum_{k=1}^{m} \frac{1}{2}\alpha_1 \|X(k)\|_n^2 = \frac{1}{2}\alpha_1 \|X\|_H^2, \tag{3.15}$$

and condition (3.1) of Theorem 3.1 is satisfied. Let

$$\gamma' = \max\left\{ \left| P(k,U) - \frac{1}{2}\alpha_2 \|U\|_n^2 + \gamma \right| : \|U\|_n \leq \rho, k \in \mathbb{Z}(1,m) \right\}. \tag{3.16}$$

Then

$$P(k,U) \geq \frac{1}{2}\alpha_2 \|U\|_n^2 - \gamma_1, \quad k \in \mathbb{Z}(1,m), U \in \mathbb{R}^n, \tag{3.17}$$

where $\gamma_1 = \gamma + \gamma'$. So, for any $X \in H$,

$$W(X) = \sum_{k=1}^{m} P(k, X(k)) \geq \sum_{k=1}^{m} \frac{1}{2}\left[\alpha_2 \|X(k)\|_n^2 - \gamma_1\right] = \frac{1}{2}\alpha_2 \|X\|_H^2 - m\gamma_1, \tag{3.18}$$

and condition (3.2) of Theorem 3.1 is satisfied. The conclusion follows from Theorem 3.1. □

Remark 3.3. For the special case where $p(k) \equiv -1, q(\mathrm{k}) \equiv 0, A = B = \theta$, the BVP (2.3) and (2.4) was studied in [15]. Here, the corresponding matrix Q becomes

$$\begin{pmatrix} 2+\alpha & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2+\beta \end{pmatrix}_{m\times m}, \tag{3.19}$$

which is positive definite when $\alpha > -1, \beta \geq -1$ and $N = (0, \dots, 0)^T$. So, $d = 0$. In this case, Corollary 3.2 reduces to Theorem 3.1 of [15]. Therefore, our results extend the ones in [15]. Corollary 3.2 also improves the conclusion of Theorem 1 in [20].

Corollary 3.4. *If there exist constants $\alpha_1 < \lambda_{\min}$, $\alpha_2 > 0$, $\rho > 0$, $\gamma > 0$, $\delta > d = 2\|N\|_{mn}/(\lambda_{\min} - \alpha_1)$ and $\tau > 2$ such that* (3.1) *holds and*

$$W(X) \geq \frac{1}{2}\alpha_2\|X\|_H^{\tau} - \gamma, \quad \forall X \in H, \|X\|_H > \rho, \tag{3.20}$$

then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Proof. It suffices to prove that (3.20) implies (3.2). In fact, pick $\rho' = \max\{\rho, ((\lambda_{\max} + 1)/\alpha_2)^{1/(\tau-2)}\}$, then

$$W(X) \geq \frac{1}{2}\alpha_2\|X\|_H^{\tau-2} \cdot \|X\|_H^2 - \gamma \geq \frac{1}{2}(\lambda_{\max} + 1)\|X\|_H^2 - \gamma \tag{3.21}$$

for $\|X\|_H > \rho'$, so condition (3.2) of Theorem 3.1 is satisfied, and the proof are complete. □

The following corollaries is obvious.

Corollary 3.5. *If there exist $\alpha_1 : 0 < \alpha_1 < \lambda_{\min}$, $\delta > d = 2\|N\|_{mn}/(\lambda_{\min} - \alpha_1)$ such that* (3.1) *holds and*

$$\lim_{\|X\|_H \to \infty} \frac{W(X)}{\|X\|_H^2} = \beta > \frac{\lambda_{\max}}{2}. \tag{3.22}$$

Then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Corollary 3.6. *If there exists constants $\alpha_1 : 0 < \alpha_1 < \lambda_{\min}$, $\delta > d = 2\|N\|_{mn}/(\lambda_{\min} - \alpha_1)$ such that* (3.14) *holds and*

$$\lim_{\|U\|_n \to \infty} \frac{P(k, U)}{\|U\|_n^2} > \frac{\lambda_{\max}}{2}, \quad \forall k \in \mathbb{Z}(1, m). \tag{3.23}$$

Then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Now we will give an example to illustrate our results.

Example 3.7. Consider the discrete boundary problem:

$$2X(k+1) + 10X(k) + 2X(k-1) = f(k, X(k)), \quad k \in \mathbb{Z}(1,5) \tag{3.24}$$

with

$$X(0) + X(1) = A, \qquad X(6) + 2X(5) = B, \tag{3.25}$$

where

$$f(k,U) = \frac{9}{400k} U\left(4\|U\|_2^2 - 2\sin\|U\|_2^2\right) \tag{3.26}$$

for $k \in \mathbb{Z}(1,5)$ and $A = (1,0)^T$, $B = (0,0)^T$.

In this example, $m = 5, n = 2$, $p(k) \equiv 2$, $c(k) \equiv 10$, $\alpha = 1$, $\beta = 2$, $\eta_1 = (2,0,0,0,0)^T$, $\eta_2 = (0,0,0,0,0)^T$, $N = (2,0,0,0,0,0,0,0,0,0)^T$,

$$M = \begin{pmatrix} Q & \\ & Q \end{pmatrix}, \qquad Q = \begin{pmatrix} 8 & 2 & 0 & 0 & 0 \\ 2 & 10 & 2 & 0 & 0 \\ 0 & 2 & 10 & 2 & 0 \\ 0 & 0 & 2 & 10 & 2 \\ 0 & 0 & 0 & 2 & 6 \end{pmatrix},$$
$$P(k,U) = \frac{9}{400k}\left[\|U\|_2^4 - \left(1 - \cos\|U\|_2^2\right)\right]. \tag{3.27}$$

So, $\|N\|_{10} = 2$. With the Matlab software, we can get the approximate eigenvalue of matrix Q:

$$\lambda_1 = 4.9957, \qquad \lambda_2 = 6.3645, \qquad \lambda_3 = 8.4127, \qquad \lambda_4 = 11.0445, \qquad \lambda_5 = 13.1826. \tag{3.28}$$

Thus Q is positive definite and $4.9 < \lambda_{\min} < 5$, $13.1 < \lambda_{\max} < 13.2$. Take $\alpha_1 = 4.5$, we find that

$$8 = \frac{4}{5 - 4.5} < d = \frac{2\|N\|_{10}}{\lambda_{\min} - \alpha_1} < \frac{4}{4.9 - 4.5} = 10. \tag{3.29}$$

Pick $\alpha_2 = 14$, $\delta = 10 > d$, $\rho = 100$, then for any $U \in \mathbb{R}^2 : \|U\|_2 < \delta = 10$, $k \in \mathbb{Z}(1,5)$, we have

$$P(k,U) \le \frac{9}{400}\|U\|_2^4 < \frac{9}{4}\|U\|_2^2 = \frac{\alpha_1}{2}\|U\|_2^2. \tag{3.30}$$

For any $U \in \mathbb{R}^2$, $\|U\|_2 > \rho = 100$, $k \in \mathbb{Z}(1,5)$,

$$P(k,U) = \frac{9}{400k}\left[\|U\|_2^4 - \left(1 - \cos\|U\|_2^2\right)\right] \geq \frac{9}{2000}\left(\|U\|_2^4 - 2\right) > \frac{\alpha_2}{2}\|U\|_2^2 - \frac{9}{1000}. \tag{3.31}$$

In view of Corollary 3.2, we see that the boundary value problem (3.24) and (3.25) has at least two nontrivial solutions.

By a similar method, we can obtain the following results.

Theorem 3.8. *Assume that $P(k,\theta) = 0$ for any $k \in \mathbb{Z}(1,m)$. If there exist constants α_1, $\alpha_2 : 0 < \alpha_2 < \lambda_{\min} \leq \lambda_{\max} < \alpha_1$, $\rho > 0$, $\gamma \geq 0$, and $\delta > d = 2\|N\|_{mn}/(\alpha_1 - \lambda_{\max})$ satisfying*

$$W(X) \geq \frac{1}{2}\alpha_1\|X\|_H^2, \quad \forall X \in H, \|X\|_H < \delta, \tag{3.32}$$

$$W(X) \leq \frac{1}{2}\alpha_2\|X\|_H^2 + \gamma, \quad \forall X \in H, \|X\|_H > \rho. \tag{3.33}$$

Then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Corollary 3.9. *Assume that $P(k,\theta) = 0$ for any $k \in \mathbb{Z}(1,m)$. If there exist constants α_1, $\alpha_2 : 0 < \alpha_2 < \lambda_{\min} \leq \lambda_{\max} < \alpha_1$, $\rho > 0$, $\gamma \geq 0$, and $\delta > d = 2\|N\|_{mn}/(\alpha_1 - \lambda_{\max})$ satisfying*

$$P(k,U) \geq \frac{1}{2}\alpha_1\|U\|_n^2, \quad \|U\|_n < \delta, k \in \mathbb{Z}(1,m), \tag{3.34}$$

$$P(k,U) \leq \frac{1}{2}\alpha_2\|U\|_n^2 + \gamma, \quad \|U\|_n > \rho, k \in \mathbb{Z}(1,m), \tag{3.35}$$

then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Corollary 3.10. *Assume that $P(k,\theta) = 0$ for any $k \in \mathbb{Z}(1,m)$. If there exist constants $\alpha_1 > \lambda_{\max}$, $\alpha_2 > 0$, $\delta > d = 2\|N\|_{mn}/(\alpha_1 - \lambda_{\max})$ and $1 < \tau < 2$ such that* (3.32) *holds and*

$$W(X) \leq \frac{1}{2}\alpha_2\|X\|_H^\tau + \gamma, \quad \forall X \in H, \|X\|_H > \rho, \tag{3.36}$$

then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Corollary 3.11. *Assume that $P(k,\theta) = 0$ for any $k \in \mathbb{Z}(1,m)$. If there exist constants $\alpha_1 > \lambda_{\max}$, and $\delta > d = 2\|N\|_{mn}/(\alpha_1 - \lambda_{\max})$ such that* (3.32) *holds and*

$$\lim_{\|X\|_H \to \infty} \frac{W(X)}{\|X\|_H} = \beta < \frac{\lambda_{\min}}{2}, \tag{3.37}$$

then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

Corollary 3.12. *Assume that* $P(k,\theta) = 0$ *for any* $k \in \mathbb{Z}(1,m)$. *If there exist* $\alpha_1 > \lambda_{\max}$, *and* $\delta > d = 2\|N\|_{mn}/(\alpha_1 - \lambda_{\max})$ *such that* (3.34) *holds and*

$$\lim_{\|U\|_n \to \infty} \frac{P(k,U)}{\|U\|_n^2} < \frac{\lambda_{\min}}{2}, \quad k \in \mathbb{Z}(1,m), \tag{3.38}$$

then BVP (2.3) *and* (2.4) *has at least two nontrivial solutions.*

At last, we will give another example.

Example 3.13. *Consider the boundary value problem* (3.24) *and* (3.25) *where* $f(k,u)$ *is replaced by*

$$f(k,U) = \left(\frac{800(4 + \|U\|_2)}{k(2 + \|U\|_2)^2} + 2\sin\|U\|_2^2\right)U. \tag{3.39}$$

Clearly,

$$P(k,U) = \frac{800\|U\|_2^2}{k(2 + \|U\|_2)} + \left(1 - \cos\|U\|_2^2\right), \tag{3.40}$$

and $P(k,\theta) = 0$. Pick $\alpha_1 = 13.4$, $\alpha_2 = 4.8$, then we have $0 < \alpha_2 < \lambda_{\min} \le \lambda_{\max} < \alpha_1$, and

$$\frac{40}{3} = \frac{4}{13.4 - 13.1} < d = \frac{2\|N\|_{mn}}{\alpha_1 - \lambda_{\max}} < \frac{4}{13.4 - 13.2} = 20. \tag{3.41}$$

Pick $\delta = 20 > d$, $\rho = 200$, then for any $U \in \mathbb{R}^2 : \|U\|_2 < \delta = 20$, we can see that

$$P(k,U) = \frac{800\|U\|_2^2}{k(2 + \|U\|_2)} + \left(1 - \cos\|U\|_2^2\right) \ge \frac{800\|U\|_2^2}{5(2 + 20)} = \frac{80\|U\|_2^2}{11} > \frac{\alpha_1}{2}\|U\|_2^2. \tag{3.42}$$

For any $U \in \mathbb{R}^2 : \|U\|_2 > \rho = 200$, we can get

$$P(k,U) = \frac{800\|U\|_2^2}{k(2 + \|U\|_2)} + \left(1 - \cos\|U\|_2^2\right) \le \frac{800\|U\|_2^2}{2 + 200} + 2 = \frac{80\|U\|_2^2}{202} + 2 < \frac{\alpha_2}{2}\|U\|_2^2 + 2. \tag{3.43}$$

According to Corollary 3.9, we know that the given BVP in this example has at least two nontrivial solutions.

Remark 3.14. When Q is negative definite, we can get similar conclusions. We do not repeat here.

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