Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 162464, 10 pages
doi:10.1155/2010/162464

Research Article

Properties of Slender Rings

E. F. Cornelius Jr.

College of Engineering and Science, University of Detroit Mercy, Detroit, MI 48221-3038, USA

Correspondence should be addressed to E. F. Cornelius Jr., efcornelius@comcast.net

Received 28 January 2010; Accepted 18 August 2010

Academic Editor: Howard Bell

Copyright © 2010 E. F. Cornelius Jr. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Slender rings and their products are characterized in terms of algebraic and topological properties. Possible limitations on uncountable products of such rings are discussed.

1. Terminology and Notation

Ring, module, and topological terminology and notation are standard. $\mathcal{R}$ denotes a unital ring with identity 1, and $\mathcal{R}^\omega$ denotes its countable product, that is, the set of all sequences of elements of $\mathcal{R}$. Of course $\omega = \{0, 1, \ldots\}$. $\mathcal{R}^\omega$ is both a left and right $\mathcal{R}$-module; it will be treated as a left $\mathcal{R}$-module unless otherwise stated. Results for left $\mathcal{R}$-modules carry over *mutatis mutandis* to right $\mathcal{R}$-modules.

The ith coordinate of an element $x \in \mathcal{R}^\omega$ will be denoted $x(i)$. Let $\mathcal{F}$ denote the submodule of $\mathcal{R}^\omega$, consisting of those elements of $\mathcal{R}^\omega$ which have only finitely many nonzero coordinates. Let $e^n \in \mathcal{F}$ have coordinates $e^n(i) = 1$ when $i = n$ and 0 otherwise; that is, $e^n(i) = \delta_{ni}$, the Kronecker delta. $\mathcal{F}$ is then the free submodule of $\mathcal{R}^\omega$, generated by the e^n, which form a basis of $\mathcal{F}$.

Let π_n denote the nth canonical projection of $\mathcal{R}^\omega$ onto $\mathcal{R}$; that is, for $x = (x(i) : i \in \omega) \in \mathcal{R}^\omega$, $\pi_n(x) = x(n)$. Note that $x = 0 \in \mathcal{R}^\omega$ if and only if $\pi_n(x) = 0$ for all n, and that $\pi_n(e^i) = \delta_{ni}$.

$\mathcal{R}$ is said to be slender if every $\mathcal{R}$-module homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$ is 0 on all but finitely many e^n. As will be demonstrated, this condition determines the structure of the endomorphism ring of $\mathcal{R}^\omega$, $\mathrm{End}_{\mathcal{R}}(\mathcal{R}^\omega)$.

As discussed in Section 2, the "original" slender ring was the integers $\mathbb{Z}$. Familiar nonslender rings are the p-adic integers for any prime $p \in \mathbb{Z}$. By using the method in [1, page 159, (d)] and known injective properties of the p-adics, it is easy to see that they are not slender.

2. The Origin of Slender Rings

Slender rings had their origin in the seminal paper by Specker [2], who proved the results in this Section for the ring of integers $\mathbb{Z}$ and its countable product $\mathbb{Z}^\omega$, which has become known as the Baer-Specker group.

The proof of the first theorem is an elaboration on the proof of a somewhat different theorem [3, page 172].

Theorem 2.1. *Every homomorphism h from the Baer-Specker group to $\mathbb{Z}$ satisfies $h(e^n) = 0$ for all but finitely many n.*

Proof. Suppose that $h(e^n) \neq 0$ for infinitely many e^n; it may be assumed that $h(e^n) \neq 0$ for all n. There exists a prime, say p_0, which does not divide $h(e^0)$. Define a sequence $b^n = a_n e^n$ of elements of $\mathbb{Z}^\omega$ as follows, where $a_n \in \mathbb{Z}$.

Set $b^0 = e^0$ and $a_0 = 1$. Suppose that $b^0, \ldots, b^n$ and the corresponding $a_0, \ldots, a_n$ have been defined and primes $p_0, \ldots, p_n$ have been chosen, with $p_i \mid h(b^0 + \cdots + b^i), i = 0, \ldots, n \neq 0$. Now choose a prime p_{n+1} different from $p_0, \ldots, p_n$ such that p_{n+1} does not divide $h(e^{n+1})$, $p_{n+1} \nmid h(e^{n+1})$. Then let b^{n+1} be a multiple k of $p_0 \cdots p_n e^{n+1}$ such that $h(b^0 + \cdots + b^n + b^{n+1})$ is divisible by p_{n+1}.

To see that the distinctiveness of the primes p_i makes such a choice of b^{n+1}possible, note that $h(b^0 + \cdots + b^n + b^{n+1}) = h(b^0 + \cdots + b^n) + kp_0 \cdots p_n h(e^{n+1})$, so that the problem reduces to finding integers k and l that satisfy an integral equation of the form $kp_{n+1} = j + lm$, in which p_{n+1} and $m = p_0 \cdots p_n h(e^{n+1})$ are relatively prime and $j = h(b^0 + \cdots + b^n)$. Being relatively prime, m and p_{n+1} generate all of $\mathbb{Z}$, including $j = h(b^0 + \cdots + b^n)$, so that suitable k and l indeed can be found. Set $a_{n+1} = kp_0 \cdots p_n$.

Let $a \in \mathbb{Z}^\omega$ with $a(i) = a_i$. Now $p_0 \nmid h(e^0)$, so $p_0 \mid a(i)$ for all $i > 0$ imply that $h(a) = h(b^0) + h(0, a(1), a(2), \ldots) = h(e^0) + p_0 h(0, a(1)/p_0, a(2)/p_0, \ldots)$ is not divisible by p_0. Thus $h(a) \neq 0$. But for each $i > 0$, $h(a) = h(b^0 + \cdots + b^i) + p_i h(0, \ldots, 0, a(i+1)/p_i, a(i+2)/p_i, \ldots)$ so that $h(a)$ is divisible by p_i because $p_i \mid h(b^0 + \cdots + b^i)$. The only integer divisible by an infinite number of primes is 0, so that $h(a) = 0$, a contradiction. □

The following proof has been utilized in a number of contexts; for a more general proof, see Theorem 3.1.

Theorem 2.2. *If a homomorphism h from the Baer-Specker group to $\mathbb{Z}$ satisfies $h(e^n) = 0$ for all n, then $h = 0$.*

Proof. Clearly $h(\mathcal{F}) = 0$. Let p and q be distinct primes. Let A be the subgroup of $\mathbb{Z}^\omega$, consisting of all sequences of the form $(a_n p^{n+1} : a_n \in \mathbb{Z}, n \in \omega)$. The elements of $A/\mathcal{F}$ are divisible by every power of p. Thus $h(\mathcal{F}) = 0$ necessitates $h(A) = 0$, as otherwise h would induce a nontrivial homomorphism from $A/\mathcal{F}$ to $\mathbb{Z}$, an impossibility in light of the divisibility of the elements of $A/\mathcal{F}$. Similarly, if C consists of all sequences of the form $(c_n q^{n+1})$, $h(C) = 0$. Now each element $b \in \mathbb{Z}^\omega$ can be written as $b = a + c$ with $a \in A$ and $c \in C$, because for each n, $1 = a_n p^{n+1} + c_n q^{n+1}$ for some $a_n, c_n \in \mathbb{Z}$, so that $b(n) = b(n)a_n p^{n+1} + b(n)c_n q^{n+1}$. Thus $\mathbb{Z}^\omega = A + C$ and so $h(\mathbb{Z}^\omega) = h(A) + h(C) = 0 + 0 = 0$. □

Remark 2.3. Theorem 2.2 frequently is stated in terms of the homomorphism being 0 on $\mathcal{F}$ instead of just the e^n. The important point is that the homomorphism is completely determined by its values on the e^n (or on $\mathcal{F}$).

The proofs of Theorems 2.1 and 2.2 use some number-theoretic properties of $\mathbb{Z}$ and so can be extended to only slender rings having the requisite primes, such as rings of polynomials over fields. Nevertheless, the results below hold for any ring with identity.

3. Homomorphisms of Countable Products of Slender Rings

The proof of the following theorem is adapted from a proof for $\mathbb{Z}$ [3, page 170].

Theorem 3.1. *If $\mathcal{R}$ is slender and h is a homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$, such that $h(e^n) = 0$ for all n, then $h = 0$.*

Proof. Suppose that $h \neq 0$ for some $p \in \mathcal{R}^\omega$. Define a homomorphism $\widetilde{h}$ from $\mathcal{F}$ to $\mathcal{R}^\omega$ by $\widetilde{h}(e^0) = p$ and $\widetilde{h}(e^n) = p - p(0)e^0 - \cdots - p(n-1)e^{n-1}$ for $n > 0$. Extend $\widetilde{h}$ to an endomorphism of $\mathcal{R}^\omega$ by defining $[\widetilde{h}(x)](n) = \sum_{i=0}^{n} x(i)[\widetilde{h}(e^i)](i)$. Now $h\widetilde{h}$ is a homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$ such that $h\widetilde{h}(e^n) = h[p - p(0)e^0 - \cdots - p(n-1)e^{n-1}] = h(p) - p(0)h(e^0) - \cdots - p(n-1)h(e^{n-1}) = h(p) \neq 0$ for all n, a contradiction to the slenderness of $\mathcal{R}$. □

Specker proved a less precise formulation of the next theorem for $\mathbb{Z}$ and $\mathbb{Z}^\omega$ [2, Satz III]. The proof given here seems shorter and simpler.

Theorem 3.2. *If $\mathcal{R}$ is slender, then every nonzero homomorphism h from $\mathcal{R}^\omega$ to $\mathcal{R}$ has a unique expression of the form $h = \pi_{n_0}h(e^{n_0}) + \cdots + \pi_{n_k}h(e^{n_k})$, with k and the n_i depending upon h and each $h(e^{n_i}) \neq 0$.*

Proof. Because $\mathcal{R}$ is slender, h is 0 except at some $e^{n_0}, \ldots, e^{n_k}$, in order $n_0 < \cdots < n_k$ if $k > 0$. Let $g = h - \pi_{n_0}h(e^{n_0}) - \cdots - \pi_{n_k}h(e^{n_k})$. Now $g(e^i) = 0$ for all i, so $g = 0$ by Theorem 3.1, and thus h has the form claimed. It is clear that h cannot be expressed as any other linear combination of canonical projections with nonzero coefficients. □

Note that if $\mathcal{R}$ is slender, then every homomorphism from its product $\mathcal{R}^\omega$ to $\mathcal{R}$ is effectively a homomorphism from a finite subproduct to $\mathcal{R}$.

Theorem 3.3. *If $\mathcal{R}$ is a slender ring and g is an endomorphism of $\mathcal{R}^\omega$, which satisfies $g(e^n) = 0$ for all n, then $g = 0$.*

Proof. If $g \neq 0$, then for some $p \in \mathcal{R}^\omega$, $g(p) \neq 0$ so that $\pi_k(g(p)) \neq 0$ for some k. But $\pi_k g$ would be a nonzero homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$, which is 0 at each e^n, thereby contradicting Theorem 3.1. □

Theorem 3.3 simply says that, for a slender ring, each endomorphism of the product $\mathcal{R}^\omega$ is determined by the endomorphism's values on the standard basis of $\mathcal{F}$, or on $\mathcal{F}$ itself.

The results of this Section can be used to determine the structure of $\mathrm{End}_{\mathcal{R}}(\mathcal{R}^\omega)$.

4. Infinite $\mathcal{R}$-Matrices and $\mathrm{End}_{\mathcal{R}}(\mathcal{R}^\omega)$

An infinite matrix over a ring $\mathcal{R}$ is a 2-dimensional array of elements from $\mathcal{R}$, of the form $M = (M(i,j) : M(i,j) \in \mathcal{R};\ i,j \in \omega)$. Such an infinite $\mathcal{R}$-matrix is said to be row-finite if, for each i, $M(i,j) = 0$ for almost all j; that is, only finitely many $M(i,j) \neq 0$ for each i.

If M and N are row-finite $\mathcal{R}$-matrices, their sum $M+N$ is defined to be the row-finite $\mathcal{R}$-matrix with entries $(M+N)(i,j) = M(i,j) + N(i,j)$. Their product MN is defined to be the row-finite $\mathcal{R}$-matrix with entries

$$MN(i,j) = \sum_{k\in\omega} N(k,j)M(i,k). \tag{4.1}$$

This order is required to encompass the case of noncommutative rings, when functions act from the left. The usual order of $M(i,k)N(k,j)$ works when functions act from the right. Every such sum is finite because, for each i, only finitely many $M(i,k)$ are nonzero. Multiplication of row-finite $\mathcal{R}$-matrices is associative, a property not always found in the multiplication of infinite matrices. With these definitions of addition and multiplication, the set of row-finite infinite $\mathcal{R}$-matrices forms a ring with identity I having entries $I(i,j) = \delta_{ij}$.

For purposes of matrix-vector multiplication, the elements of $\mathcal{R}^\omega$ are viewed as column vectors. If M is a row-finite $\mathcal{R}$-matrix and $x \in \mathcal{R}^\omega$, the product of M and x is denoted Mx and is defined to be the sequence in $\mathcal{R}^\omega$ with $Mx(i) = \sum_{k\in\omega} x(k)M(i,k)$. It is easy to see that M induces an endomorphism of $\mathcal{R}^\omega$ via such matrix-vector multiplication.

Theorem 4.1. *Every row-finite infinite $\mathcal{R}$-matrix induces an endomorphism of $\mathcal{R}^\omega$ via matrix-vector multiplication.*

What is most interesting is the converse for slender rings:

Theorem 4.2. *If $\mathcal{R}$ is slender, then every endomorphism of $\mathcal{R}^\omega$ is induced by multiplication of its vectors by a row-finite infinite $\mathcal{R}$-matrix.*

Proof. Let $g \in \mathrm{End}_{\mathcal{R}}(\mathcal{R}^\omega)$ and define an $\mathcal{R}$-matrix M by $M(i,j) = [g(e^j)](i)$ for $i,j \in \omega$. For each i, $\pi_i g$ is a homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$ and so by slenderness is 0 for almost all e^j, $j \in \omega$. But $\pi_i g(e^j) = [g(e^j)](i) = M(i,j)$ so that M is row-finite.

By Theorem 4.1, M induces an endomorphism of $\mathcal{R}^\omega$. According to Theorem 3.3, to prove that multiplication by M is the same as mapping by g, it suffices to demonstrate that $Me^j = g(e^j)$ for all j; that is, that $(Me^j)(i) = [g(e^j)](i)$. Now $(Me^j)(i) = \sum_{k\in\omega} e^j(k)M(i,k) = \sum_{k\in\omega} \delta_{jk}M(i,k) = \delta_{jj}M(i,j) = M(i,j) = [g(e^j)](i)$. □

Corollary 4.3. *If $\mathcal{R}$ is a slender ring, then the endomorphism ring $End_{\mathcal{R}}(\mathcal{R}^\omega)$ under function addition and composition is isomorphic to the ring of row-finite infinite $\mathcal{R}$-matrices.*

Proof. Note that the columns of the $\mathcal{R}$-matrix M in the proof of Theorem 4.2 are the values of the endomorphism g at the e^n. M is thus unique. There is then a 1-1 correspondence between the ring of row-finite infinite $\mathcal{R}$-matrices under addition and multiplication and $\mathrm{End}_{\mathcal{R}}(\mathcal{R}^\omega)$ under function addition and composition. It is easy to see that this correspondence preserves addition and multiplication. □

5. Topologies for a Ring $\mathcal{R}$ and Its Product $\mathcal{R}^\omega$

The foregoing algebraic properties of slender rings can be characterized topologically. Let a ring $\mathcal{R}$ be equipped with the discrete topology, that is, the topology in which every subset is open. The singletons, $\{r\}, r \in \mathcal{R}$, form a basis of this discrete topology, which is metrizable by defining $d(r,r) = 0$ for all $r \in \mathcal{R}$ and $d(r,s) = 1$ for distinct $r,s \in \mathcal{R}$. Under the discrete topology of $\mathcal{R}$, addition and scalar multiplication are continuous.

Let $\mathcal{R}^\omega$ be equipped with the product topology, that is, the topology in which the sets, $\pi_n^{-1}(U), n \in \omega, U \subseteq \mathcal{R}$, form a subbasis. In fact, the sets $\pi_n^{-1}(\{r\}), n \in \omega, r \in \mathcal{R}$, form a subbasis. Under the product topology of $\mathcal{R}^\omega$, the canonical projections are continuous open maps and addition and scalar multiplication are continuous.

This product topology is metrizable by defining $d(x,x) = 0$ for all $x \in \mathcal{R}^\omega$ and, for distinct $x,y \in \mathcal{R}^\omega$, $d(x,y) = 2^{-n}$ if $x(i)$ and $y(i)$ first differ at $n \in \omega$. Thus, if $d(x,y) < 2^{-n}$, then $x(i) = y(i)$ for at least $i = 0,\dots,n$. In general, $d(x,y) = d(x-y,0)$, an important fact used in the following discussions. Note also that $d(rx,0) \leq d(x,0)$ for all $r \in \mathcal{R}$ and $x \in \mathcal{R}^\omega$.

To see that d induces the product topology on $\mathcal{R}^\omega$, let $x \in \mathcal{R}^\omega$ and consider the neighborhood of x, $N = \{y \in \mathcal{R}^\omega : d(x,y) < 2^{-n}\}$. For all $y \in N$, $x(i) = y(i), i = 0,\dots,n$, so that $y \in O = \pi_0^{-1}(\{x(0)\}) \cap \cdots \cap \pi_n^{-1}(\{x(n)\})$ and so $N \subseteq O$. Conversely, any $y \in O$ satisfies $y(i) = x(i), i = 0,\dots,n$, which means that $d(x,y) < 2^{-n}$ so that $O \subseteq N$.

A sequence of elements of $\mathcal{R}^\omega$, $(x^n : n \in \omega)$, converges to $x \in \mathcal{R}^\omega$ if and only if the initial components of the x^n become the initial components of x as n becomes large. It readily follows that x^n converges to x if and only if $(x - x^n : n \in \omega)$ converges to $0 \in \mathcal{R}^\omega$. Clearly the sequence (e^n) converges to 0.

For $x = (x(i) : i \in \omega) \in \mathcal{R}^\omega$, if $\overrightarrow{x}^n \in \mathcal{F}, n \in \omega$, is defined as $\overrightarrow{x}^n(i) = x(i)$ if $i \leq n$ and $\overrightarrow{x}^n(i) = 0$ for $i > n$, then the sequence $(\overrightarrow{x}^n)$ converges to x. Thus $\mathcal{F}$ is dense in $\mathcal{R}^\omega$. If $\overleftarrow{x}^n \in \mathcal{R}^\omega, n \in \omega$, is defined as $\overleftarrow{x}^n(i) = 0$ if $i \leq n$ and $\overleftarrow{x}^n(i) = x(i)$ for $i > n$, then the sequence $(\overleftarrow{x}^n)$ converges to 0.

Observe that every Cauchy sequence $(x^n : n \in \omega)$ in $\mathcal{R}^\omega$ converges to an element of $\mathcal{R}^\omega$, so that $\mathcal{R}^\omega$ is a complete metric space. To see this, simply observe that, at each coordinate i, the $x^n(i)$ become constant as n becomes large, so that x^n converges to x with $x(i) = \lim_{n\to\infty} x^n(i)$.

For any homomorphism of topological modules, continuity on the entire domain is determined by continuity at 0. Thus, if h is a homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$, then h is continuous on all of $\mathcal{R}^\omega$ if and only if it is continuous at 0. This is easy to see because if the sequence x^n converges to $x \in \mathcal{R}^\omega$, then $\lim_{n\to\infty}[h(x^n) - h(x)] = \lim_{n\to\infty} h(x^n - x) = 0$ if and only if h is continuous at 0. A similar result, of course, holds for any endomorphism of $\mathcal{R}^\omega$.

Convention re Topologies

Unless stated otherwise, the topology on a ring is always the discrete topology, and the topology on its product is always the product topology.

Theorem 5.1. *A homomorphism to a ring $\mathcal{R}$ from its product $\mathcal{R}^\omega$ is uniformly continuous on $\mathcal{R}^\omega$ if and only if it is continuous at* 0. *Ditto for an endomorphism of $\mathcal{R}^\omega$.*

Proof. Suppose that h is a homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$, which is continuous at 0. Given $\epsilon > 0$, there exists $\delta > 0$ such that $d(h(x),0) < \epsilon$ provided $d(x,0) < \delta$. If $(y^n : n \in \omega)$ converges to $y \in \mathcal{R}^\omega$, so long as $d(y,y^n) < \delta$, then $d(h(y),h(y^n)) < \epsilon$. This holds because $d(h(y),h(y^n)) = d(h(y) - h(y^n),0) = d(h(y - y^n),0)$ and $d(y,y^n) = d(y - y^n,0)$. Thus h is uniformly continuous.

The proof for endomorphisms of $\mathcal{R}^\omega$ is essentially the same. □

Theorem 5.2. *If a ring $\mathcal{R}$ is slender, then every homomorphism from its product $\mathcal{R}^\omega$ to $\mathcal{R}$ is uniformly continuous, as is every endomorphism of $\mathcal{R}^\omega$.*

Proof. Since 0 is trivial, let h be a nonzero homomorphism from $\mathcal{R}^{\omega}$ to $\mathcal{R}$ and let (x^n) be a sequence in $\mathcal{R}^{\omega}$, which converges to 0. According to Theorem 3.2, h is a linear combination of canonical projections, the last one of which is some n_k. For n sufficiently large, the first n_k coordinates of the x^n are 0 and so $h(x^n) = 0$. Thus h is continuous at 0 and hence uniformly continuous everywhere by Theorem 5.1.

Let g be nonzero endomorphism of $\mathcal{R}^{\omega}$ and let M be the row-finite infinite $\mathcal{R}$-matrix which induces g. Again, it suffices to check convergence at 0, so let (x^n) be a sequence of elements of $\mathcal{R}^{\omega}$, which converges to 0. As n gets large, the products Mx^n have initial coordinates = 0 because M is row-finite and the x^n have initial coordinates = 0. Specifically, given any $k \in \omega$, let n_k be the largest j such that $M(i,j) \neq 0, i \leq k$, and let $m_k \in \omega$ be such that coordinates 0-n_k of x^n are equal to 0 for all $n > m_k$. Then coordinates 0-k of Mx^n are 0 for all $n > m_k$ so that $\lim_{n\to\infty} Mx^n = 0$. Again Theorem 5.1 completes the proof. □

For slender $\mathcal{R}$, the continuity of an endomorphism g of $\mathcal{R}^{\omega}$ at 0 is reflected by the row finiteness of the matrix M inducing it. In particular, $\lim_{n\to\infty} e^n = 0$ means that $\lim_{n\to\infty} Me^n = 0$. Since Me^n is column n of M, the initial entries of the columns of M ($M(0,n)$, $M(1,n)$, etc.) increasingly = 0.

The ring properties previously described algebraically can be cast in topological terms.

Theorem 5.3. *Let $\mathcal{R}$ be a ring with* 1. *Then a homomorphism h from $\mathcal{R}^{\omega}$ to $\mathcal{R}$ satisfies $h(e^n) = 0$ for all but finitely many n if and only if it satisfies* $\lim_{n\to\infty} h(e^n) = 0$.

Proof. If $h(e^n) = 0$ for all $n > k$, then $d(h(e^n), 0) = 0$ for all $n > k$, so that $\lim_{n\to\infty} h(e^n) = 0$.

Conversely, if $\lim_{n\to\infty} h(e^n) = 0$, there is a positive integer k such that $d(h(e^n), 0) < 1$ for all $n > k$. But $d(h(e^n), 0) < 1$ implies that $h(e^n) = 0$ because only $x = 0$ satisfies $d(x, 0) < 1$ in the metric on $\mathcal{R}$. □

Corollary 5.4. *A ring $\mathcal{R}$ with* 1 *is slender if and only if every homomorphism h from $\mathcal{R}^{\omega}$ to $\mathcal{R}$ satisfies* $\lim_{n\to\infty} h(e^n) = 0$.

The next theorem is a casting of slenderness in topological terms.

Theorem 5.5. *A ring $\mathcal{R}$ with* 1 *is slender if and only if every homomorphism from $\mathcal{R}^{\omega}$ to $\mathcal{R}$ is continuous at* 0.

Proof. If $\mathcal{R}$ is slender, then according to Theorem 5.2, every homomorphism from $\mathcal{R}^{\omega}$ to $\mathcal{R}$ is certainly continuous at 0.

Conversely, if every homomorphism h from $\mathcal{R}^{\omega}$ to $\mathcal{R}$ is continuous at 0, then $\lim_{n\to\infty} h(e^n) = 0$ because $\lim_{n\to\infty} e^n = 0$. The slenderness of $\mathcal{R}$ follows from Corollary 5.4. □

The next theorem is the topological analog of Theorems 4.1 and 4.2.

Theorem 5.6. *Let $\mathcal{R}$ be a ring with* 1. *Then every endomorphism of its product $\mathcal{R}^{\omega}$ is induced by a row-finite matrix if and only if every endomorphism of $\mathcal{R}^{\omega}$ is continuous at* 0.

Proof. If g is an endomorphism of $\mathcal{R}^{\omega}$, which is induced by a row-finite matrix M, and if (x^n) is a sequence in $\mathcal{R}^{\omega}$, that converges to 0, then as n get large, the initial coordinates of the x^n become 0. Since M is row-finite, the initial coordinates of the Mx^n also become 0 so that $(Mx^n) = (g(x^n))$ converges to 0, as in the proof of Theorem 5.2.

Conversely, suppose that every endomorphism of $\mathcal{R}^\omega$ is continuous at 0, and let h be a homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$. Define an endomorphism g of $\mathcal{R}^\omega$ by $g(x) = (h(x), h(x), \ldots)$. By hypothesis, g is continuous at 0. Since $\lim_{n\to\infty} e^n = 0$, $\lim_{n\to\infty} g(e^n) = 0$; that is, the initial coordinates of $g(e^n)$, which all $= h(e^n)$, must be 0 for n sufficiently large. Thus $\lim_{n\to\infty} h(e^n) = 0$, so by Corollary 5.4, $\mathcal{R}$ is slender. Theorem 4.2 completes the proof. □

6. Equivalent Conditions for Slenderness

It is convenient to draw together some of the more useful conditions for a ring to be slender. There are, of course, other equivalences.

Theorem 6.1. *Let $\mathcal{R}$ be a ring with* 1*, equipped with the discrete topology, and let its product $\mathcal{R}^\omega$ be equipped with the product topology. Then conditions (I)–(VI) are equivalent.*

(I) *Every homomorphism h from $\mathcal{R}^\omega$ to $\mathcal{R}$ satisfies $h(e^n) = 0$ for all but finitely many n.*

(II) *$\mathcal{R}$ is slender.*

(III) *Every nonzero homomorphism h from $\mathcal{R}^\omega$ to $\mathcal{R}$ has a unique expression of the form $h = \pi_{n_0} h(e^{n_0}) + \cdots + \pi_{n_k} h(e^{n_k})$, with k and the n_i depending upon h and each $h(e^{n_i}) \neq 0$.*

(IV) *Every endomorphism of $\mathcal{R}^\omega$ is induced by multiplication of its vectors by a row-finite infinite matrix with entries from $\mathcal{R}$.*

(V) *Every homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$ is continuous at* 0*.*

(VI) *Every endomorphism of $\mathcal{R}^\omega$ is continuous at* 0*.*

Proof. (I)⇔(II): This equivalence is simply definitional.

(II)⇒(III): This is Theorem 3.2.

(III)⇒(II): If every homomorphism h from $\mathcal{R}^\omega$ to $\mathcal{R}$ has the form stated, it is clear that $h(e^n) = 0$ for all but finitely many n. Thus (II)⇔(III).

(II)⇔(V): This is Theorem 5.5. Thus (I), (II), (III), and (V) are equivalent.

(IV)⇔(VI): This is Theorem 5.6. It now suffices to demonstrate the equivalence of (V) and (VI).

(V)⇒(VI): Let g be an endomorphism of $\mathcal{R}^\omega$; by Theorem 5.1, it suffices to demonstrate continuity at 0. Suppose that (x^i) is a sequence in $\mathcal{R}^\omega$, which converges to 0. Each $\pi_n g$ is a continuous homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$, which satisfies $\pi_n g(x^i) = 0$ for all $i >$ some $k_n \in \omega$. Given any $l \in \omega$, there exist $k_n \in \omega$ such that $\pi_n g(x^i) = 0$ for all $i > k_n$, $n = 0, \ldots, l$. If $m = \max\{k_0, \ldots, k_l\}$, then at least coordinates 0-l of $g(x^i)$ are 0 for all $i > m$; that is, $\lim_{i\to\infty} g(x^i) = 0$ so that g is continuous at 0.

(VI)⇒(V): This was shown in the proof of Theorem 5.6. □

Remark 6.2. Conditions (V) and (VI) of Theorem 6.1 could, of course, be stated by deleting "at 0" inasmuch as Theorem 5.1 covers uniform continuity everywhere. Similarly, "continuous" could be strengthened to "uniformly continuous" without weakening the theorem. The equivalence of (I) and (V) is shown in [4, Theorem 3(3)] using "continuous" instead of "at 0".

7. Extensions to Uncountable Products of Slender Rings

Although there is nothing sacrosanct about countable products of slender rings, it turns out that little may be gained from considering uncountable products. Notation used in the countable case will be carried over to the uncountable without further comment.

It is clear that if $\mathcal{R}$ is slender, then every homomorphism h from $\mathcal{R}^I$ to $\mathcal{R}$ satisfies $h(e^i) = 0$ for all but finitely many i, for every infinite index set I. Obviously, if every homomorphism h from $\mathcal{R}^I$ to $\mathcal{R}$ satisfies $h(e^i) = 0$ for all but finitely many i, for some infinite index set I, then $\mathcal{R}$ is slender. The problem for uncountable products, even when $\mathcal{R}$ is slender, is that the analog of Theorem 3.1 may not hold; that is, there may exist a nonzero homomorphism from $\mathcal{R}^I$ to $\mathcal{R}$, which annihilates all e^i.

Whether uncountable products yield anything new depends upon the existence *vel non* of measurable cardinals. An uncountable cardinal number κ is said to be κ-measurable if there exists a set I of cardinality κ and a measure μ on the subsets of I, which (i) assumes only the values $0, 1$; (ii) satisfies $\mu(\emptyset) = 0 = \mu(\{i\})$ for all $i \in I$, and $\mu(I) = 1$; and (iii) is κ-additive in the sense that if $\{J_k : k \in K\}$ is a collection of mutually disjoint subsets of I with $|K| < \kappa$, then $\mu(\bigcup_{k\in K} J_k) = \sum_{k\in K} \mu(J_k)$. With this definition of measurable cardinals, uncountable products of rings may be subject to the limitation discussed in the proposition below, the proof of which is modeled after one for $\mathbb{Z}$ [1, page 161, *Remark*].

Proposition 7.1. *Let $\mathcal{R}$ be a ring with 1. If there exists a κ-measurable cardinal satisfying $\kappa > |\mathcal{R}|$, and I is an index set of cardinality κ and μ is a κ-additive measure on I, then there exists a nonzero homomorphism h from $\mathcal{R}^I$ to $\mathcal{R}$, such that $h(e^i) = 0$ for all $i \in I$.*

Proof. For every $a = (a(i) : i \in I) \in \mathcal{R}^I$ and $r \in \mathcal{R}$, define $X_r(a) = \{i \in I : a(i) = r\}$. Then the $X_r(a)$ are pairwise disjoint subsets of I whose union is I. Since $|\mathcal{R}| < \kappa$, $1 = \mu(I) = \mu(\bigcup_{r\in\mathcal{R}} X_r(a)) = \sum_{r\in R} \mu(X_r(a))$, so that exactly one of the sets, say $X_s(a)$, has measure 1. Set $\eta(a) = s$. Using the properties of μ, it will be shown that η preserves addition and scalar multiplication and $\eta(e^i) = 0$ for all $i \in I$.

First, to see that $\eta(e^i) = 0$ for all $i \in I$, observe that $I = X_0(e^i) \cup X_1(e^i) = \{I \setminus \{i\}\} \cup \{i\}$, so that $\mu(X_0(e^i)) = 1$. Thus $\eta(e^i) = 0$.

To confirm additivity, suppose that $\eta(a) = s$ and $\eta(b) = t$; then $\mu(\{i \in I : a(i) \neq s\}) = 0$ and $\mu(\{i \in I : b(i) \neq t\}) = 0$. Now $I \subseteq (X_s(a) \cap X_t(b)) \cup \{i \in I : a(i) \neq s\} \cup \{i \in I : b(i) \neq t\}$ so that

$$\begin{aligned} 1 = \mu(I) &\leq \mu(X_s(a) \cap X_t(b)) + \mu\{i \in I : a(i) \neq s\} + \mu\{i \in I : b(i) \neq t\} \\ &= \mu(X_s(a) \cap X_t(b)) \leq 1, \end{aligned} \tag{7.1}$$

so $\mu(X_s(a) \cap X_t(b)) = 1$. Since $X_s(a) \cap X_t(b) \subseteq X_{s+t}(a+b)$, $1 = \mu(X_s(a) \cap X_t(b)) \leq \mu(X_{s+t}(a+b)) \leq 1$; that is, $\mu(X_{s+t}(a+b)) = 1$, so $\eta(a+b) = s+t$.

Finally, to check that $\eta(ra) = rs = r\eta(a)$ for $r \in \mathcal{R}$, note that $X_s(a) \subseteq X_{rs}(ra)$ so that $1 = \mu(X_s(a)) \leq \mu(X_{rs}(ra)) \leq 1$ or $\mu(X_{rs}(ra)) = 1$. Thus $\eta(ra) = rs = r\eta(a)$ and η is an $\mathcal{R}$-homomorphism. □

Remark 7.2. In the Proof of Proposition 7.1, not only is it true that $\eta(\mathcal{F}) = 0$, but $\eta(\mathcal{F}_\kappa) = 0$ where $\mathcal{F}_\kappa$ is the submodule of $\mathcal{R}^I$ consisting of all $x \in \mathcal{R}^I$ with $\mu(\{i \in I : x(i) \neq 0\}) = 0$. Thus η has a large kernel.

Ulam has shown that if there exists an $\aleph_1$-measurable cardinal (the meaning of which should be clear), then there is a least one, call it $\mathfrak{m}$, and it in fact is $\mathfrak{m}$-measurable [5]. Thus $\mathfrak{m}$ often is referred to as the least measurable cardinal. Lady has shown that if $\mathcal{R}$ is slender and h is a homomorphism from $\mathcal{R}^I$ to $\mathcal{R}$, satisfying $h(e^i) = 0$ for all $i \in I$, then $h = 0$, provided

$|I| < m$ [4, Theorem 3(4)]. If there are no measurable cardinals, it then follows that, for slender $\mathcal{R}$, such a homomorphism h is zero regardless of the cardinality of I.

The state of uncountable products of rings can be summarized as follows.

Theorem 7.3. *Let $\mathcal{R}$ be a ring with 1, equipped with the discrete topology, let σ be an uncountable ordinal, and let the product $\mathcal{R}^\sigma$ be equipped with the product topology. If there are no measurable cardinals, then conditions (I)–(V) are equivalent.*

(I) *$\mathcal{R}$ is slender.*

(II) *Every nonzero homomorphism h from $\mathcal{R}^\sigma$ to $\mathcal{R}$ has a unique expression of the form $h = \pi_{\iota_0}h(e^{\iota_0}) + \cdots + \pi_{\iota_k}h(e^{\iota_k})$, with k and the ι_j depending upon h and each $h(e^{\iota_j}) \neq 0$.*

(III) *Every endomorphism of $\mathcal{R}^\sigma$ is induced by multiplication of its vectors by a row-finite infinite matrix with entries from $\mathcal{R}$ and with rows and columns indexed by σ.*

(IV) *Every homomorphism from $\mathcal{R}^\sigma$ to $\mathcal{R}$ is continuous at 0.*

(V) *Every endomorphism of $\mathcal{R}^\sigma$ is continuous at 0.*

If there exists a least measurable cardinal $\mathfrak{m}$, then conditions (I)–(V) are equivalent for all ordinals σ of cardinality $< m$.

Proof. Assume that no measurable cardinal exists. Note that the proof of Theorem 3.2 requires only that a homomorphism which is 0 on all e^i itself be zero, a result which follows from [4, Theorem 3(4)] in the absence of measurable cardinals. Further note that Theorems 4.1 and 4.2 do not depend upon a countable product of rings. From earlier Sections, the equivalence of conditions (I)–(III) is apparent.

Although metrizability of the product topology on $\mathcal{R}^\sigma$ is lost, the product topology has the property that an endomorphism g of $\mathcal{R}^\sigma$ is continuous if and only if $\pi_i g$ is continuous for all i. Continuity at 0 continues to be the litmus test for overall continuity (as with any topological module). Thus conditions (IV) and (V) are readily seen to be equivalent.

All that remains is to connect (I)–(III) with (IV)-(V). Suppose that (IV) holds and let h be a continuous homomorphism from $\mathcal{R}^\sigma$ to $\mathcal{R}$. Suppose that $h(e^{\iota_n}) \neq 0, \iota_n < \sigma$, for all $n \in \omega$. As in previous Sections, let $\mathcal{R}^\omega$ have canonical projections p_n and let $e^n, n \in \omega$, be the basis of the free submodule $\mathcal{F} \subset \mathcal{R}^\omega$. Inject $\mathcal{R}^\omega$ into $\mathcal{R}^\sigma$ via g defined for $x \in \mathcal{R}^\omega$ as $[g(x)](\iota_n) = x(n), n \in \omega$, and $[g(x)](\iota) = 0$ otherwise. This injection is continuous because $p_n = \pi_{\iota_n} g$ is the composition of continuous functions for all $n \in \omega$. Now hg is a continuous homomorphism from $\mathcal{R}^\omega$ to $\mathcal{R}$ and so, by Theorem 6.1, must be 0 on all but finitely many e^n. Since $g(e^n) = e^{\iota_n}$, h must be 0 except for those finitely many e^{ι_n}. Thus $\mathcal{R}$ is slender and so (IV)$\Longrightarrow$(I).

Finally, suppose that (II) holds and let $h = \pi_{\iota_0}h(e^{\iota_0}) + \cdots + \pi_{\iota_k}h(e^{\iota_k})$ be a homomorphism from $\mathcal{R}^\sigma$ to $\mathcal{R}$ and let $r \in \mathcal{R}$; it suffices to address subbasic open sets. For any $x \in \mathcal{R}^\sigma$, $x \in h^{-1}(\{r\})$ if and only if $h(x) = r$; that is, if and only if $x(\iota_0)h(e^{\iota_0}) + \cdots + x(\iota_k)h(e^{\iota_k}) = r$. Now $x \in \pi_{\iota_0}^{-1}(\{x(\iota_0)\}) \cap \cdots \cap \pi_{\iota_k}^{-1}(\{x(\iota_k)\}) \subseteq h^{-1}(\{r\})$ so that $h^{-1}(\{r\})$ is open and h is continuous. Thus (II)$\Longrightarrow$(IV), making conditions (I)–(V) equivalent, under the assumption that there are no measurable cardinals.

If measurable cardinals exist, the key to the equivalence of (I)–(V) remains demonstrating that only the zero homomorphism from $\mathcal{R}^\sigma$ to $\mathcal{R}$ annihilates all e^ι, and again [4, Theorem 3(4)] supplies the proof for $|\sigma| < \mathfrak{m}$. □

Remark 7.4. The equivalence of conditions (I)–(V) for all uncountable ordinals is equivalent to the nonexistence of measurable cardinals. It is, of course, unnecessary in Theorem 7.3 to use an ordinal for indexing, as any uncountable index set would suffice.

Acknowlegment

The author thanks Professor Phill Schultz of the University of Western Australia for discussions via email.

References

[1] L. Fuchs, *Infinite Abelian Groups II*, Academic Press, New York, NY, USA, 1973.
[2] E. Specker, "Additive Gruppen von Folgen ganzer Zahlen," *Portugaliae Mathematica*, vol. 9, pp. 131–140, 1950.
[3] L. Fuchs, *Abelian Groups*, International Series of Monographs on Pure and Applied Mathematics, Pergamon Press, New York, NY, USA, 1960.
[4] E. L. Lady, "Slender rings and modules," *Pacific Journal of Mathematics*, vol. 49, pp. 397–406, 1973.
[5] S. Ulam, "Zur masstheorie in der allgemeinen mengenlehre," *Fundamenta Mathematicae*, vol. 16, pp. 140–150, 1930.