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Research Article

Probabilities as Values of Modular Forms and Continued Fractions

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We consider certain probability problems which are naturally related to integer partitions. We show that the corresponding probabilities are values of classical modular forms. Thanks to this connection, we then show that certain ratios of probabilities are specializations of the Rogers-Ramanujan and Ramanujan- Selberg- Gordon-Göllnitz continued fractions. One particular evaluation depends on a result from Ramanujan's famous first letter to Hardy.

1. Introduction and Statement of Results

In a recent paper [1], Holroyd et al. defined probability models whose properties are linked to the number theoretic functions $p_k(n)$, which count the number of integer partitions of n which do not contain k consecutive integers among its summands. Their asymptotic results for each k lead to a nice application: the calculation of thresholds of certain two dimensional cellular automata (see [1, Theorem 4]).

In a subsequent paper [2], Andrews explained the deep relationship between the generating functions for $p_k(n)$ and mock theta functions and nonholomorphic modular forms. Already in the special case of $p_2(n)$, one finds an exotic generating function whose unusual description requires both a modular form and a Maass form. This description plays a key role in the recent work of Bringmann and Mahlburg [3], who make great use of the rich properties of modular forms and harmonic Maass forms to obtain asymptotic results which improve on [1, Theorem 3] in the special case of $p_2(n)$.

It does not come as a surprise that the seminal paper of Holroyd et al. [1] has inspired further work at the interface of number theory and probability theory. (In addition to [2, 3], we point out the recent paper by Andrews et al. [4]) Here we investigate another family of probability problems which also turn out to be related to the analytical properties of classical partition generating functions.

These problems are relatives of the following standard undergraduate problem. Toss a fair coin repeatedly until it lands heads up. If one flips n tails before the first head, what is the probability that n is even? Since the probability of flipping n tails before the first head is $1/2^{n+1}$, the solution is

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\cdots=\frac{1}{1-1/2}-\frac{1}{1-1/4}=\frac{2}{3}. \tag{1.1}$$

Instead of continuing until the first head, consider the situation where a coin is repeatedly flipped: once, then twice, then three times, and so on. What is the probability of the outcome that each nth turn, where n is odd, has at least one head?

More generally, let $0 < p < 1$, and let $\{C_1, C_2, \ldots\}$ be a sequence of independent events where the probability of C_n is given by

$$\mathbb{P}_p(C_n) := 1 - p^n. \tag{1.2}$$

For each pair of integers $0 \le r < t$, we let

$$A(r,t) := \{\text{set of sequences where } C_n \text{ occurs if } n \not\equiv \pm r \bmod t\}. \tag{1.3}$$

In the case where $p = 1/2$, one can think of C_n as the event where at least one of n tosses of a coin is a head. Therefore, if $p = 1/2$, the problem above asks for the probability of the outcome $A(0,2)$.

Remark 1.1. Without loss of generality, we shall always assume that $0 \le r \le t/2$. In most cases $A(r,t)$ is defined by two arithmetic progressions modulo t. The only exceptions are for $r = 0$, and for even t when $r = t/2$.

It is not difficult to show that the problem of computing

$$\mathbb{P}_p(A(r,t)) := \text{“probability of A(r, t)”} \tag{1.4}$$

involves partitions. Indeed, if $p(r,t;n)$ denotes the number of partitions of n whose summands are congruent to $\pm r \pmod t$, then we shall easily see that the probabilities are computed using the generating functions

$$\mathcal{P}(r,t;q) := \sum_{n=0}^{\infty} p(r,t;n)q^n = \begin{cases} \displaystyle\prod_{n=0}^{\infty}\frac{1}{(1-q^{tn+r})(1-q^{tn+t-r})}, & \text{if } 0 < r < \dfrac{t}{2}, \\ \displaystyle\prod_{n=1}^{\infty}\frac{1}{(1-q^{tn})}, & \text{if } r = 0, \\ \displaystyle\prod_{n=0}^{\infty}\frac{1}{(1-q^{tn+t/2})}, & \text{if } r = \dfrac{t}{2}. \end{cases} \tag{1.5}$$

This critical observation is the bridge to a rich area of number theory, one involving class field theory, elliptic curves, and partitions. In these areas, modular forms play a central role, and so it is natural to investigate the number theoretic properties of the probabilities $\mathbb{P}_r(A(r,t))$ from this perspective.

Here we explain some elegant examples of results which can be obtained in this way. Throughout, we let $\tau \in \mathbb{H}$, the upper-half of the complex plane, and we let $q_\tau := e^{2\pi i\tau}$. The *Dedekind eta-function* is the weight $1/2$ modular form (see [5]) defined by the infinite product

$$\eta(\tau) := q_\tau^{1/24}\prod_{n=1}^{\infty}\left(1-q_\tau^n\right). \tag{1.6}$$

We also require some further modular forms, the so-called Siegel functions. For u and v real numbers, let $B_2(u) := u^2 - u + 1/6$ be the second Bernoulli polynomial, let $z := u - v\tau$, and let $q_z := e^{2\pi iz}$. Then the *Siegel function* $g_{u,v}(\tau)$ is defined by the infinite product

$$g_{u,v}(\tau) := -q_\tau^{\mathbf{B}_2(u)/2}e^{2\pi iv(u-1)/2}\prod_{n=1}^{\infty}\left(1-q_zq_\tau^n\right)\left(1-q_z^{-1}q_\tau^n\right). \tag{1.7}$$

These functions are weight 0 modular forms (e.g., see [5]).

For fixed integers $0 \le r \le t/2$, we first establish that $\mathbb{P}_p(A(r,t))$ is essentially a value of a single quotient of modular forms.

Theorem 1.2. *If $0 < p < 1$ and $\tau_p := -\log(p)\cdot i/2\pi$, then*

$$\mathbb{P}_p(A(r,t)) = \begin{cases} q_{\tau_p}^{(t-1)/24}\cdot\dfrac{\eta(\tau_p)}{\eta(t\tau_p)}, & \text{if } r = 0,\\[2ex] \dfrac{q_{\tau_p}^{-(t+2)/48}}{1-q_{\tau_p}^{t/2}}\cdot\dfrac{\eta(\tau_p)\eta(t\tau_p)}{\eta((t/2)\tau_p)}, & \text{if } r = \dfrac{t}{2},\\[2ex] -\dfrac{q_{\tau_p}^{-(2t+1)/24}e^{-\pi i(r/t)}}{1-q_{\tau_p}^{r}}\cdot\dfrac{\eta(\tau_p)}{g_{0,-r/t}(t\tau_p)}, & \text{otherwise.}\end{cases} \tag{1.8}$$

One of the main results in the work of Holroyd et al. (see [1, Theorem 2]) was the asymptotic behavior of their probabilities as $p \to 1$. In Section 4 we obtain the analogous results for $\log\mathbb{P}_p(A(r,t))$.

Theorem 1.3. *As $p \to 1$, one has*

$$-\log\mathbb{P}_p(A(r,t)) \sim \begin{cases} \dfrac{\pi^2}{6(1-p)}\left(1-\dfrac{1}{t}\right), & \text{if } r = 0,\ \dfrac{t}{2},\\[2ex] \dfrac{\pi^2}{6(1-p)}\left(1-\dfrac{2}{t}\right), & \text{otherwise.}\end{cases} \tag{1.9}$$

Table 1: Values of $L(p)$.

p	$\mathbb{P}_p(A(2,5))$	$\mathbb{P}_p(A(1,5))$	$L(p)$
0.3	$0.692\cdots$	$0.883\cdots$	$0.61607\cdots$
0.4	$0.576\cdots$	$0.776\cdots$	$0.61778\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.97	$6.43\cdots \times 10^{-14}$	$1.03\cdots \times 10^{-13}$	$0.61803\cdots$
0.98	$5.65\cdots \times 10^{-21}$	$9.11\cdots \times 10^{-21}$	$0.61803\cdots$
0.99	$3.11\cdots \times 10^{-42}$	$5.03\cdots \times 10^{-42}$	$0.61803\cdots$

To fully appreciate the utility of Theorem 1.2, it is important to note that the relevant values of Dedekind's eta-function and the Siegel functions can be reformulated in terms of the real-analytic Eisenstein series (see [5])

$$E_{u,v}(\tau,s) := \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} e^{2\pi i(mu+nv)}\frac{y^s}{|m\tau+n|^{2s}}, \quad \tau = x+iy\in\mathbb{H},\ \mathrm{Re}(s)>1. \tag{1.10}$$

One merely makes use of the *Kronecker Limit Formulas* (e.g., see [5, Part 4]). These limit formulas are prominent in algebraic number theory for they explicitly relate such values of modular forms to values of zeta-functions of number fields.

We give two situations where one plainly sees the utility of these observations. Firstly, one can ask for a more precise limiting behavior than is dictated by Theorem 1.3. For example, consider the limiting behavior of the quotient $L(p) := \mathbb{P}_p(A(2,5))/\mathbb{P}_p(A(1,5))$, as $p \to 1$. Table 1 is very suggestive.

Theorem 1.4. *The following limits are true:*

$$\begin{aligned}\lim_{p\to 1}\frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))} &= \frac{-1+\sqrt{5}}{2},\\ \lim_{p\to 1}\frac{\mathbb{P}_p(A(3,8))}{\mathbb{P}_p(A(1,8))} &= -1+\sqrt{2}.\end{aligned} \tag{1.11}$$

Remark 1.5. Notice that $(1/2)(-1+\sqrt{5}) = -1+\phi$, where ϕ is the *golden ratio*

$$\phi := \frac{1}{2}\left(1+\sqrt{5}\right) = 1.61803\ldots. \tag{1.12}$$

Theorem 1.4 is a consequence of our second application which concerns the problem of obtaining *algebraic* formulas for all of these ratios, not just the limiting values. As a function of p, one may compute the ratios of these probabilities in terms of continued fractions. To ease notation, we let

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+\cdots} \tag{1.13}$$

denote the continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}. \tag{1.14}$$

The following continued fractions are well known:

$$\begin{aligned} \frac{1}{1+}\frac{1}{1+}\frac{1}{1+\cdots} &= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = \frac{-1+\sqrt{5}}{2}, \\ \frac{1}{2+}\frac{1}{2+}\frac{1}{2+\cdots} &= \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}} = -1+\sqrt{2}. \end{aligned} \tag{1.15}$$

Theorem 1.4 is the limit of the following exact formulas.

Theorem 1.6. *If* $0 < p < 1$ *and* $\tau_p := -\log(p) \cdot (i/2\pi)$, *then one has that*

$$\begin{aligned} \frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))} &= \left(\frac{1}{1+}\frac{q_{\tau_p}}{1\ +}\frac{q_{\tau_p}^2}{1\ +}\frac{q_{\tau_p}^3}{1\ +\cdots} \right), \\ \frac{\mathbb{P}_p(A(3,8))}{\mathbb{P}_p(A(1,8))} &= \left(\frac{1}{1+q_{\tau_p}+}\frac{q_{\tau_p}^2}{1+q_{\tau_p}^3+}\frac{q_{\tau_p}^4}{1+q_{\tau_p}^5+}\frac{q_{\tau_p}^6}{1+q_{\tau_p}^7+\cdots} \right). \end{aligned} \tag{1.16}$$

Theorem 1.6 can also be used to obtain many further beautiful expressions, not just those pertaining to the limit as $p \to 1$. For example, we obtain the following simple corollary.

Corollary 1.7. *For* $p_1 = 1/e^{2\pi}$ *and* $p_2 = 1/e^{\pi}$, *one has that*

$$\begin{aligned} \frac{\mathbb{P}_{p_1}(A(2,5))}{\mathbb{P}_{p_1}(A(1,5))} &= e^{-2\pi/5}\left(-\phi + \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)} \right), \\ \frac{\mathbb{P}_{p_2}(A(3,8))}{\mathbb{P}_{p_2}(A(1,8))} &= e^{-\pi/2}\left(\sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}} \right). \end{aligned} \tag{1.17}$$

Remark 1.8. The evaluation of $\mathbb{P}_p(A(2,5))/\mathbb{P}_p(A,1,5))$ when $p = 1/e^{2\pi}$ follows from a formula in Ramanujan's famous first letter to Hardy dated January 16, 1913.

Theorem 1.6 and Corollary 1.7 follow from the fact that the Fourier expansions of the relevant modular forms in Theorem 1.2 are realized as the q-continued fractions of Rogers-Ramanujan and Ramanujan-Selberg-Gordon-Göllnitz. We shall explain these results in Section 5.

We chose to focus on two particularly simple examples of ratios, namely,

$$\frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))}, \qquad \frac{\mathbb{P}_p(A(3,8))}{\mathbb{P}_p(A(1,8))}. \tag{1.18}$$

One can more generally consider ratios of the form

$$\frac{\mathbb{P}_p(A(r_1,t))}{\mathbb{P}_p(A(r_2,t))}. \tag{1.19}$$

It is not the case that all such probabilities, for fixed r_1 and r_2, can be described by a single continued fraction. However, one may generalize these two cases, thanks to Theorem 1.2, by making use of the so-called *Selberg relations* [6] (see also [7]) which extend the notion of a continued fraction. Arguing in this way, one may obtain Theorem 1.6 in its greatest generality. Although we presented Theorem 1.4 as the limiting behavior of the continued fractions in Theorem 1.6, we stress that its conclusion also follows from the calculation of the explicit Fourier expansions at cusps of the modular forms in Theorem 1.2. In this way one may also obtain Theorem 1.4 in generality. Turning to the problem of obtaining explicit formulas such as those in Corollary 1.7, we have the theory of complex multiplication at our disposal. In general, one expects to obtain beautiful evaluations as algebraic numbers whenever τ_p is an algebraic integer in an imaginary quadratic extension of the field of rational numbers. We leave these generalizations to the reader.

2. Combinatorial Considerations

Here we give a lemma which expresses the probability $\mathbb{P}_p(A(r,t))$ in terms of the infinite products in (1.5).

Lemma 2.1. *If* $0 < p < 1$ *and* $\tau_p := -\log(p) \cdot (i/2\pi)$, *then*

$$\mathbb{P}_p(A(r,t)) = \wp\left(r,t;q_{\tau_p}\right) \cdot \prod_{n=1}^{\infty}\left(1-q_{\tau_p}^n\right). \tag{2.1}$$

Proof. By the Borel-Cantelli lemma, with probability 1 at most finitely many of the C_n's will fail to occur. Now, for each pair of integers $0 \le r \le t/2$, let $S(r,t)$ be the countable set of binary strings $a_1a_2a_3a_4\cdots \in \{0,1\}^{\mathbb{N}}$ in which $a_n = 1$ if $n \not\equiv \pm r \mod t$, and $a_n = 0$ for at most finitely many n satisfying $n \equiv \pm r \mod t$ (with the appropriate modifications on the arithmetic progressions modulo t for $r = 0$, and for t even with $r = t/2$). Then the event $A(r,t)$ can be written as the countable disjoint union

$$A(r,t) = \bigcup_{a_1a_2a_3\cdots \in S_{r,t}} \bigcap_{n:a_n=1} C_n \cap \bigcap_{n:a_n=0} C_n^c. \tag{2.2}$$

Using this, we find that

$$\begin{aligned}
\mathbb{P}_p(A(r,t)) &= \sum_{a_1a_2a_3\cdots\in S_{r,t}} \mathbb{P}_p\left(\bigcap_{n:a_n=1} C_n \cap \bigcap_{n:a_n=0} C_n^c\right) \\
&= \sum_{a_1a_2a_3\cdots\in S_{r,t}} \prod_{n:a_n=1}\left(1-q_{\tau_p}^n\right)\prod_{n:a_n=0} q_{\tau_p}^n \\
&= \prod_{i=1}^{\infty}\left(1-q_{\tau_p}^n\right) \sum_{a_1a_2a_3\cdots\in S_{r,t}} \prod_{n:a_n=0} \frac{q_{\tau_p}^n}{1-q_{\tau_p}^n} \\
&= \prod_{n=1}^{\infty}\left(1-q_{\tau_p}^n\right) \cdot \sum_{a_1a_2a_3\cdots\in S_{r,t}} \prod_{n:a_n=0}\left(q_{\tau_p}^n + q_{\tau_p}^{2n} + q_{\tau_p}^{3n} + \cdots\right) \\
&= \prod_{n=1}^{\infty}\left(1-q_{\tau_p}^n\right) \cdot \wp\left(r,t;q_{\tau_p}\right).
\end{aligned} \tag{2.3}$$

□

3. Proof of Theorem 1.2

We first note that $q_{\tau_p} = e^{2\pi i \tau_p} = p$, and by (1.6) we have

$$\prod_{n=1}^{\infty}\left(1-q_{\tau_p}^n\right) = q_{\tau_p}^{-1/24}\eta(\tau_p). \tag{3.1}$$

Now we prove each of the three relevant cases in turn. First, suppose that $r=0$. By Lemma 2.1 we need to show

$$q_{\tau_p}^{-1/24}\eta(\tau_p)\prod_{n=1}^{\infty}\left(1-q_{\tau_p}^{tn}\right)^{-1} = q_{\tau_p}^{(t-1)/24}\frac{\eta(\tau_p)}{\eta(t\tau_p)}. \tag{3.2}$$

This follows from the identity

$$\prod_{n=1}^{\infty}\left(1-q_{\tau_p}^{tn}\right)^{-1} = \prod_{n=1}^{\infty}\left(1-q_{t\tau_p}^{n}\right)^{-1} = q_{t\tau_p}^{1/24}\eta(t\tau_p)^{-1}. \tag{3.3}$$

Next, suppose that t is even and $r=t/2$. By Lemma 2.1 we need to show

$$q_{\tau_p}^{-1/24}\eta(\tau_p)\prod_{n=0}^{\infty}\left(1-q_{\tau_p}^{tn+(t/2)}\right)^{-1} = \frac{q_{\tau_p}^{-(t+2)/48}}{1-q_{\tau_p}^{t/2}}\frac{\eta(\tau_p)\eta(t\tau_p)}{\eta((t/2)\tau_p)}. \tag{3.4}$$

Observe that

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1-q_{\tau_p}^{tn+(t/2)}\right)^{-1}&=\prod_{n=1}^{\infty}\left(1-q_{(t/2)\tau_p}^{2n+1}\right)^{-1}\\&=\prod_{n=1}^{\infty}\left(1-q_{(t/2)\tau_p}^{2n}\right)\left(1-q_{(t/2)\tau_p}^{n}\right)^{-1}\\&=\prod_{n=1}^{\infty}\left(1-q_{t\tau_p}^{n}\right)\left(1-q_{(t/2)\tau_p}^{n}\right)^{-1}.\end{aligned}\tag{3.5}$$

Then (3.4) follows from the identities

$$\prod_{n=1}^{\infty}\left(1-q_{t\tau_p}^{n}\right)=q_{t\tau_p}^{-1/24}\eta(t\tau_p),\quad \prod_{n=1}^{\infty}\left(1-q_{(t/2)\tau_p}^{n}\right)^{-1}=q_{(t/2)\tau_p}^{1/24}\eta\left(\frac{t}{2}\tau_p\right)^{-1}.\tag{3.6}$$

Finally, suppose that $0<r<t/2$. By Lemma 2.1 we need to show

$$q_{\tau_p}^{-1/24}\eta(\tau_p)\prod_{n=0}^{\infty}\left(1-q_{\tau_p}^{tn+r}\right)^{-1}\left(1-q_{\tau_p}^{tn+t-r}\right)^{-1}=-\frac{q_{\tau_p}^{-(2t+1)/24}e^{-\pi i(r/t)}}{1-q_{\tau_p}^{r}}\frac{\eta(\tau_p)}{g_{0,-r/t}(t\tau_p)}.\tag{3.7}$$

Observe that

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1-q_{\tau_p}^{tn+r}\right)\left(1-q_{\tau_p}^{tn+t-r}\right)&=\left(1-q_{\tau_p}^{r}\right)\left(1-q_{\tau_p}^{t-r}\right)\prod_{n=1}^{\infty}\left(1-q_{\tau_p}^{tn+r}\right)\left(1-q_{\tau_p}^{tn+t-r}\right)\\&=\left(1-q_{\tau_p}^{r}\right)\left(1-q_{\tau_p}^{t-r}\right)\prod_{n=1}^{\infty}\left(1-q_{r\tau_p}q_{\tau_p}^{tn}\right)\left(1-q_{r\tau_p}^{-1}q_{\tau_p}^{t(n+1)}\right)\\&=\frac{\left(1-q_{\tau_p}^{r}\right)\left(1-q_{r\tau_p}^{-1}q_{t\tau_p}\right)}{\left(1-q_{r\tau_p}^{-1}q_{t\tau_p}\right)}\prod_{n=1}^{\infty}\left(1-q_{r\tau_p}q_{t\tau_p}^{n}\right)\left(1-q_{r\tau_p}^{-1}q_{t\tau_p}^{n}\right)\\&=\left(1-q_{\tau_p}^{r}\right)\prod_{n=1}^{\infty}\left(1-q_{r\tau_p}q_{t\tau_p}^{n}\right)\left(1-q_{r\tau_p}^{-1}q_{t\tau_p}^{n}\right)\end{aligned}\tag{3.8}$$

It follows from the definition of the Siegel function $g_{u,v}(\tau)$ that

$$\prod_{n=1}^{\infty}\left(1-q_{r\tau_p}q_{t\tau_p}^{n}\right)\left(1-q_{r\tau_p}^{-1}q_{t\tau_p}^{n}\right)=-q_{t\tau_p}^{1/12}e^{\pi i(r/t)}g_{0,-r/t}(t\tau_p),\tag{3.9}$$

from which we obtain

$$\prod_{n=0}^{\infty}\left(1-q_{\tau_p}^{tn+r}\right)^{-1}\left(1-q_{\tau_p}^{tn+t-r}\right)^{-1}=-q_{\tau_p}^{-t/12}e^{-\pi i(r/t)}\left(1-q_{\tau_p}^{r}\right)^{-1}g_{0,-r/t}(t\tau_p)^{-1}. \tag{3.10}$$

Substitute the preceding identity into the left-hand side of (3.7) to obtain the result.

4. Proof of Theorem 1.3

Recall that the partition generating function is defined by

$$G(x):=\prod_{n=1}^{\infty}(1-x^n)^{-1},\quad 0<x<1. \tag{4.1}$$

We now prove each of the relevant cases in turn. First, suppose that $r=0$. Then $\wp(0,t;q_{\tau_p})=G(q_{\tau_p}^t)$, and by Lemma 2.1,

$$-\log\mathbb{P}_p(A(0,t))=\log G\left(q_{\tau_p}\right)-\log G\left(q_{\tau_p}^t\right). \tag{4.2}$$

Make the change of variables $x=e^{-w}$. Then a straightforward modification of the analysis in [8, pages 19–21] shows that, for each integer $\alpha\geq 1$,

$$\log G(e^{-\alpha w})=\frac{\pi^2}{6\alpha w}+\frac{1}{2}\log(1-e^{-\alpha w})+C_\alpha+O(w)\quad\text{as } w\longrightarrow 0 \tag{4.3}$$

for some constant C_α. It follows from (4.3) that

$$\log G\left(q_{\tau_p}\right)-\log G\left(q_{\tau_p}^t\right)\sim\frac{\pi^2}{6(1-p)}\left(1-\frac{1}{t}\right)\quad\text{as } p\longrightarrow 1. \tag{4.4}$$

Next, suppose that t is even and $r=t/2$. Then

$$\wp\left(\frac{t}{2},t;q_{\tau_p}\right)=\frac{1}{1-q_{\tau_p}^{t/2}}\prod_{n=1}^{\infty}\left(1-q_{(t/2)\tau_p}^{n}\right)^{-1}\left(1-q_{t\tau_p}^{n}\right)=\frac{G\left(q_{\tau_p}^{t/2}\right)G\left(q_{\tau_p}^{t}\right)^{-1}}{1-q_{\tau_p}^{t/2}}, \tag{4.5}$$

and by Lemma 2.1 and (4.3),

$$\begin{aligned}-\log\mathbb{P}_p\left(A\left(\frac{t}{2},t\right)\right)&=\log\left(1-q_{\tau_p}^{t/2}\right)+\log G\left(q_{\tau_p}\right)-\log G\left(q_{\tau_p}^{t/2}\right)+\log G\left(q_{\tau_p}^{t}\right)\\&\sim\frac{\pi^2}{6(1-p)}\left(1-\frac{2}{t}+\frac{1}{t}\right)\quad\text{as } p\longrightarrow 1.\end{aligned} \tag{4.6}$$

Finally, suppose that $0 < r < t/2$. Then by Lemma 2.1,

$$\log \mathbb{P}_p(A(r,t)) = \log G\left(q_{\tau_p}\right) - \log\left(\prod_{n=0}^{\infty}\left(1 - q_{\tau_p}^{tn+r}\right)^{-1}\right) - \log\left(\prod_{n=0}^{\infty}\left(1 - q_{\tau_p}^{tn+t-r}\right)^{-1}\right). \tag{4.7}$$

One can show in a manner similar to (4.3) that for each integer $\beta \geq 1$,

$$\begin{aligned}\log\left(\prod_{n=0}^{\infty}\left(1 - e^{-(\alpha n+\beta)w}\right)^{-1}\right) &= -\log\left(1 - e^{-\beta w}\right) + \sum_{k=1}^{\infty}\frac{e^{-\beta w k}}{\alpha w k^2} \\ &\quad + \frac{1}{2}\log\left(1 - e^{-(\alpha+\beta)w}\right) + C_{\alpha,\beta} + O(w) \quad \text{as } w \longrightarrow 0\end{aligned} \tag{4.8}$$

for some constant $C_{\alpha,\beta}$. It follows from (4.3) and (4.8) that

$$\begin{aligned}&\log G\left(q_{\tau_p}\right) - \log\left(\prod_{n=0}^{\infty}\left(1 - q_{\tau_p}^{tn+r}\right)^{-1}\right) - \log\left(\prod_{n=0}^{\infty}\left(1 - q_{\tau_p}^{tn+t-r}\right)^{-1}\right) \\ &\quad\sim \frac{\pi^2}{6(1-p)}\left(1 - \frac{2}{t}\right) \quad \text{as } p \longrightarrow 1.\end{aligned} \tag{4.9}$$

5. Proof of Theorem 1.6 and Corollary 1.7

Here we prove Theorem 1.6 and Corollary 1.7. These results will follow from well-known identities of Rogers-Ramanujan, Selberg, and Gordon-Göllnitz (e.g., see [9]). We require the celebrated Rogers-Ramanujan continued fraction

$$R(q) = q^{1/5}\left(\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots}\right), \tag{5.1}$$

and the Ramanujan-Selberg-Gordon-Göllnitz continued fraction

$$H(q) = q^{1/2}\left(\frac{1}{1+q+}\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^6}{1+q^7+\cdots}\right). \tag{5.2}$$

It turns out that these q-continued fractions satisfy the following identities (e.g., see [10]):

$$\begin{aligned}R(q) &= q^{1/5}\prod_{n=1}^{\infty}\frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})} = q^{1/5}\frac{\wp(2,5;q)}{\wp(1,5;q)}, \\ H(q) &= q^{1/2}\prod_{n=1}^{\infty}\frac{(1-q^{8n-1})(1-q^{8n-7})}{(1-q^{8n-3})(1-q^{8n-5})} = q^{1/2}\frac{\wp(3,8;q)}{\wp(1,8;q)}.\end{aligned} \tag{5.3}$$

Theorem 1.6 now follows easily from Theorem 1.2.

To prove Corollary 1.7, notice that if $p = 1/e^{2\pi}$ (resp., $p = 1/e^{\pi}$), then $\tau_p = i$ (resp., $\tau_p = i/2$). In particular, we have that $q_{\tau_p} = e^{-2\pi}$ (resp., $q_{\tau_p} = e^{-\pi}$). Corollary 1.7 now follows from the famous evaluations (e.g., see [11] and [12, page xxvii] which is Ramanujan's first letter to Hardy)

$$R\left(e^{-2\pi}\right) = -\phi + \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)},$$
$$H\left(e^{-\pi}\right) = \sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}. \tag{5.4}$$

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