Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 830592, 9 pages
doi:10.1155/2009/830592

Research Article

Generalized Alpha-Close-to-Convex Functions

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Received 5 January 2009; Revised 26 June 2009; Accepted 13 September 2009

Recommended by Teodor Bulboacă

We define the classes $G_\beta(\alpha,k,\gamma)$ as follows: $f \in G_\beta(\alpha,k,\gamma)$ if and only if, for $z \in E = \{z \in \mathbb{C} : |z| < 1\}$, $|\arg\{(1-\alpha^2 z^2)f'(z)/e^{-i\beta}\phi'(z)\}| \le \gamma\pi/2$, $0 < \gamma \le 1$; $\alpha \in [0,1]$; $\beta \in (-\pi/2, \pi/2)$, where ϕ is a function of bounded boundary rotation. Coefficient estimates, an inclusion result, arclength problem, and some other properties of these classes are studied.

1. Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. By S, K, S^*, and C denote the subclasses of A which are univalent, close-to-convex, starlike, and convex in E, respectively. Let V_k be the class of functions of bounded boundary rotation. Paate [1] showed that a function f, defined by (1.1) and $f'(z) \neq 0$, is in V_k if and only if, for $z = re^{i\theta}$,

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \le k\pi. \tag{1.2}$$

It is geometrically obvious that $k \ge 2$ and $V_2 \equiv C$.

A class T_k of analytic functions related with the class V_k was introduced and studied in [2]. A function $f \in A$ is in T_k, $k \ge 2$, if and only if there exists a function $g \in V_k$ such that, for $z \in E$, $\operatorname{Re}\{f'(z)/g'(z)\} > 0$. It is clear that $T_2 \equiv K$.

Let P denote the class of analytic functions p defined by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.3}$$

with $\operatorname{Re} p(z) > 0$ for $z \in E$.

We denote $K(\gamma)$ as the class of strongly close-to-convex functions of order γ in the sense of Pommerenke [3]. A function $f \in A$ belongs to $K(\gamma)$ if and only if there exists $g \in S^*$ such that $|\operatorname{Arg}(zf'(z)/g(z))| \leq \pi\gamma/2$, for $z \in E$ and $\gamma \geq 0$.

Clearly $K(0) = C$, $K(1) = K$, and when $0 \leq \gamma < 1$, $K(\gamma)$ is a subset of K and hence contains only univalent functions. For $\gamma > 1$, $f \in K(\gamma)$ can be of infinite valence; see [4].

We now define the following.

Definition 1.1. A function $f \in A$ is said to belong to $G_\beta(\alpha, k, \gamma)$, where β is a real number, $\alpha \in \mathbb{C} : |\alpha| \leq 1$, $k \geq 2$, and $\gamma \in (0, 1]$ is called generalized alpha-close-to-convex with argument β if and only if there exists $\phi \in V_k$ such that

$$\left| \operatorname{Arg}\left\{ \frac{(1 - \alpha^2 z^2) f'(z)}{e^{-i\beta}\phi'(z)} \right\} \right| \leq \frac{\gamma\pi}{2}, \quad z \in E. \tag{1.4}$$

In (1.4), we choose this branch of argument which equals β, $|\beta| < \pi\gamma/2$, $\gamma \in (0, 1]$, when $z = 0$. We note that the condition $|\alpha| \leq 1$ implies that $G_\beta(\alpha, k, \gamma)$ is nonempty. From the normalization conditions $f'(0) = \phi'(0) = 1$, it follows from Definition 1.1 that $\operatorname{Re} e^{-i\beta} > 0$ and therefore $|\beta| < \gamma\pi/2$. Also, it follows from (1.4) that if $f \in G_\beta(\alpha, k, \gamma)$, then $f'(z) \neq 0$ for $z \in E$. Condition (1.4) is equivalent to the following $f \in G_\beta(\alpha, k, \gamma)$ if and only if there exists $p \in P$ such that

$$\frac{(1 - \alpha^2 z^2) f'(z)}{e^{-i\beta}\phi'(z)} = \left(p(z) \cos\frac{\beta}{\gamma} - i \sin\frac{\beta}{\gamma} \right)^{\gamma}, \quad \phi \in V_k. \tag{1.5}$$

We define $G(\alpha, k, \gamma)$ the class of generalized α-close-to-convex functions as

$$G(\alpha, k, \gamma) = \bigcup_{|\beta| < \pi/2} G_\beta(\alpha, k, \gamma). \tag{1.6}$$

If $\alpha = 0$ in (1.6), then the class $G(0, k, 1)$ is identical with the class T_k and $G(\alpha, 2, 1)$ is the class K of close-to-convex functions. Also $G_\beta(\alpha, 2, 1)$ in the class of close-to-convex function with argument β was defined by Goodman and Saff [5]. For details of special cases of $G_\beta(\alpha, 2, 1)$ with $\phi(z) = z$ in (1.4), we refer to [6]. The special case with $\gamma = 1 = \alpha$, $k = 2$, and $\phi(z) = z$ in (1.4) leads to the class of functions convex in the direction of the imaginary axis having special normalization; see [7].

2. Main Results

We now prove the main results as follows.

Theorem 2.1. *Let* $\alpha \in [0,1]$. *Then* $G(\alpha,k,\gamma) \subset G(0,k,\gamma_1)$, *where*

$$\gamma_1(\gamma,\alpha) = \gamma + \frac{2}{\pi}\arcsin\left(\alpha^2\right). \tag{2.1}$$

The constant $\gamma_1(\gamma,\alpha)$ *cannot be smaller.*

Proof. We will use an extended version of the method given in [8] to prove this result.

For $\alpha = 0$, the result is obvious. Let $f \in G(\alpha,k,\gamma)$. By (1.4), (1.5), and (1.6), then there exists a function $\phi \in V_k$ and a function $p \in P$, $|\beta| < \pi/2$ such that

$$\frac{f'(z)}{e^{-i\beta}\phi'(z)} = \frac{\left(p(z)\cos(\beta/\gamma) - i\sin(\beta/\gamma)\right)^{\gamma}}{1-\alpha^2 z^2}, \quad z \in E. \tag{2.2}$$

Let $q(z) = (p(z)\cos(\beta/\gamma) - i\sin(\beta/\gamma))^{\gamma}$, $z \in E$. Then we have

$$\left|\operatorname{Arg}\frac{f'(z)}{e^{-i\beta}\phi'(z)}\right| = \left|\operatorname{Arg} q(z) - \operatorname{Arg}\left(1-\alpha^2 z^2\right)\right| < \frac{\pi}{2}\left[\gamma + \frac{2}{\pi}\left|\operatorname{Arg}\left(1-\alpha^2 z^2\right)\right|\right]. \tag{2.3}$$

We choose in (2.3) this branch of argument which is equal $-\beta$ when $z = 0$.

Since $|\operatorname{Arg}(1-\alpha^2 z^2)| < \arcsin(\alpha^2)$, $z \in E$, we have from (2.3) $f \in G(0,k,\gamma_1)$, where γ_1 is given by (2.1). The constant $\gamma_1(\gamma,\alpha)$ cannot be smaller. Let $\alpha \in (0,1)$ be fixed. Let us consider the point $z_0 \in \mathbb{C}$ with $|z_0| = 1$ and $\operatorname{Arg}(1-\alpha^2 z^2) = -\arcsin(\alpha^2)$. Let $\phi_0 \in V_k$ be such that $\phi_0(z_0)$ is finite. Then, let

$$f_0'(z) = \frac{e^{-i\beta}\phi_0'(z)}{1-\alpha^2 z^2}\left[\left(p(z)\cos\frac{\beta}{\gamma} - i\sin\frac{\beta}{\gamma}\right)^{\gamma}\right], \quad z \in E, \ |\beta| < \frac{\pi}{2}, \tag{2.4}$$

where

$$\begin{aligned} P_0(z) &= \frac{1+\varepsilon z}{1-\varepsilon z}, \quad \varepsilon \in \mathbb{C}, \ |\varepsilon| < 1, \\ \phi_0'(z) &= \frac{(1+\delta_1 z)^{k/2-1}}{(1+\delta_2 z)^{k/2+1}}, \quad |\delta_1| = |\delta_2| = 1. \end{aligned} \tag{2.5}$$

Now, for $z \in E$,

$$\left|\operatorname{Arg}\frac{f_0'(z)}{e^{-i\beta}\phi_0'(z)}\right| = \left|\operatorname{Arg}\left(p_0(z)\cos\frac{\beta}{\gamma} i\sin\frac{\beta}{\gamma}\right)^{\gamma} - \operatorname{Arg}\left(1-\alpha^2 z^2\right)\right|, \tag{2.6}$$

and $\operatorname{Arg} e^{-i\beta} = -\beta$. Since p_0 maps the unit circle $|z| = 1$ onto imaginary axis, we may choose ε_0, $|\varepsilon_0| = 1$ such that $\varepsilon_0 \neq 1/z_0$, $P_0(z_0) = (1+\varepsilon_0 z_0)/(1-\varepsilon_0 z_0) \neq i \tan\beta$, $p_0(z_0) = ai$, $a > 0$. This means that $p_0(z_0)$ is finite and $\operatorname{Arg} p_0(z_0) = \pi/2$. Hence

$$\operatorname{Arg}\left[\left(p_0(z)\cos\frac{\beta}{\gamma} i \sin\frac{\beta}{\gamma}\right)^{\gamma}\right] = \frac{\gamma\pi}{2}. \tag{2.7}$$

Thus, from (2.4) and (2.6), we have

$$\left|\operatorname{Arg}\frac{f_0'(z)}{e^{-i\beta}\phi_0'(z)}\right| = \frac{\pi}{2}\left[\gamma + \frac{2}{\pi}\arcsin\left(\alpha^2\right)\right] = \gamma_1\frac{\pi}{2}. \tag{2.8}$$

Therefore γ_1 cannot be smaller.

For $\alpha = 1$, consider the sequence $\{z_n\}$, $z_n = e^{i\theta_n}$, $\theta_n \in (0, \pi/4)$, $n \in \mathbb{N} = 1$ such that $\lim_{n\to\infty} z_n = 1$. So

$$\lim_{n\to\infty}\operatorname{Arg}\left(1 - z_n^2\right) = -\frac{\pi}{2}. \tag{2.9}$$

Let $\phi \in V_k$ with $\phi(e^{i\theta})$ finite and $\theta \in (0, \pi/2)$. The function f_1 defined as

$$f_1'(z) = \frac{e^{-i\beta}\phi'(z)}{1-z^2}\left[\left(\left(\frac{1+z}{1-z}\right)\cos\frac{\beta}{\gamma} - i\sin\frac{\beta}{\gamma}\right)^{\gamma}\right], \quad z \in E, \ |\beta| < \frac{\pi}{2} \tag{2.10}$$

belongs to $G(1, k, \gamma)$. Thus, from (2.9), it follows that

$$\lim_{n\to\infty}\left|\operatorname{Arg}\frac{f_1'(z)}{e^{-i\beta}\phi'(z)}\right| = \lim_{n\to\infty}\left|\operatorname{Arg}\left\{\left(\left(\frac{1+z_n}{1-z_n}\right)\cos\frac{\beta}{\gamma} - i\sin\frac{\beta}{\gamma}\right)^{\gamma}\right\} - \operatorname{Arg}\left(1-z_n^2\right)\right| = (1+\gamma)\frac{\pi}{2}. \tag{2.11}$$

This means that $\gamma_1(1,\gamma) = 1 + \gamma$ is best possible.

We note that, for $\gamma = 1$, $k \geq 2$, we obtain a result proved in [8]. □

Theorem 2.2. *Let $f \in G(\alpha, k, \gamma)$, $\alpha \in [0, 1]$. Then, for every $\gamma \in (0, 1)$ and θ_1, θ_1 with $0 \leq \theta_2 - \theta_1 \leq 2\pi$, one has*

$$\int_{\theta_1}^{\theta_2}\operatorname{Re}\left\{1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\}d\theta > -\left(\gamma + \frac{k}{2} - 1 - \mathfrak{R}\right)\pi, \tag{2.12}$$

where

$$\mathfrak{R} = \frac{1}{\pi}\{\psi(r,\theta_2) - \psi(r,\theta_1)\},$$
$$\psi(r,\theta) = -\operatorname{Arg}\left(1 - \alpha^2\gamma^2 e^{2i\theta}\right) = \arctan\frac{\alpha^2 r^2 \sin 2\theta}{1 - \alpha^2 r^2\cos 2\theta}. \tag{2.13}$$

Proof. To prove this result, we shall essentially use the similar method given by Kaplan [9].

Let $f \in G(\alpha, k, \gamma)$ for fixed $\alpha \in [0,1]$. Then f satisfies the inequality (1.4) for some β, $|\beta| < \pi/2$ and $\phi \in V_k$. Let $\phi_1(z) = \phi(z)e^{i\beta}$, $z \in E$. Since $f'(z) \neq 0$, $\phi_1'(z) \neq 0$ for $z \in E$, we can define, for $z = re^{i\theta}$, $r \in (0,1), \theta$ is a real number, the following:

$$\wp(r,\theta) = \operatorname{Arg}\left\{\left(1-\alpha^2 r^2 e^{2i\theta}\right) f'\left(re^{i\theta}\right)\right\}, \tag{2.14}$$

$$V(r,\theta) = \operatorname{Arg} \phi_1'\left(re^{i\theta}\right), \tag{2.15}$$

$$\psi(r,\theta) = \operatorname{Arg}\left\{\left(1-\alpha^2 r^2 e^{2i\theta}\right) re^{i\theta} f'\left(re^{i\theta}\right)\right\} = \wp(r,\theta) + \theta, \tag{2.16}$$

$$V(r,\theta) = \operatorname{Arg}\left\{re^{i\theta}\phi_1'\left(re^{i\theta}\right)\right\} = \tau(r,\theta) + \theta. \tag{2.17}$$

The functions $\wp$, τ, ψ, and V are continuous and periodic with period 2π. From (1.4), we can choose the branches of argument of $\wp(z)$ and $\tau(z)$ as

$$\left|\wp(r,\theta) - \tau(r,\theta)\right| < \frac{\gamma\pi}{2}, \quad \gamma \in [0,1]. \tag{2.18}$$

Now, for $\phi_1 \in V_k$, it is known [10] that, for $\theta_1 < \theta_2$, $z = re^{i\theta}$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{\left(z\phi_1'(z)\right)'}{\phi_1'(z)}\right\} d\theta > -\left(\frac{k}{2} - 1\right)\pi. \tag{2.19}$$

From (2.16), (2.17), and (2.19), we have

$$\begin{aligned}
\psi(r,\theta_2) - \psi(r,\theta_1) &= \wp(r,\theta_2) + \theta_2 - \wp(r,\theta_1) - \theta_1 \\
&= \left[\wp(r,\theta_2) - \tau(r,\theta_2)\right] + \left[\tau(r,\theta_2) + \theta_2 - \tau(r,\theta_1) - \theta_1\right] - \left[\wp(r,\theta_1) - \tau(r,\theta_1)\right] \\
&> \gamma\pi - \left(\frac{k}{2} - 1\right)\pi = -\left(r + \frac{k}{2} - 1\right)\pi.
\end{aligned} \tag{2.20}$$

Moreover, by (2.16), we have

$$\frac{d}{d\theta}\psi(r,\theta) = \frac{d}{d\theta}\operatorname{Arg}\left(1 - \alpha^2 r^2 e^{2i\theta}\right) + \operatorname{Re}\left\{1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\}, \tag{2.21}$$

and therefore, from (2.20)

$$\begin{aligned}\int_{\theta_1}^{\theta_2}\operatorname{Re}\left\{1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\}d\theta &= \int_{\theta_1}^{\theta_2}\frac{d}{d\theta}\psi(r,\theta)d\theta-\int_{\theta_1}^{\theta_2}\operatorname{Arg}\left(1+\alpha^2r^2e^{2i\theta}\right)d\theta\\ &> -\left(\gamma+\frac{k}{2}-1\right)\pi-\left[\psi(r,\theta_1)-\psi(r,\theta_2)\right]\\ &= -\left(\gamma+\frac{k}{2}-1-\mathfrak{R}\right)\pi,\end{aligned} \tag{2.22}$$

where $\psi(r,\theta)$ and $\mathfrak{R}$ are defined by (2.13). This completes the proof. □

We note that, for $\gamma = 1$, $k = 2$, $\alpha = 0$, we obtain the necessary condition for f to be close-to-convex in E, proved in [9].

Remark 2.3. From Theorem 2.2, we can interpret some geometrical meaning for the functions in $G(\alpha,k,\gamma)$. For simplicity, let us suppose that the image domain is bounded by an analytic curve Γ. At a point on Γ, the outward drawn normal turns back at most $= -(\gamma + k/2 - 1 - \mathfrak{R})\pi$, where A is given by (2.13). This is a necessary condition for a function f to belong to $G(\alpha,k,\gamma)$. Goodman [4] showed that if $f \in K(\sigma)$, $\sigma \ge 0$, then, for $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$, $\int_{\theta_1}^{\theta_2}\operatorname{Re}((zf'(z))'/f'(z))d\theta > -\sigma\pi$.

We note that $f \in G(\alpha,k,\gamma)$ is univalent for $k + 2(\gamma - \mathfrak{R}) \le 4$, since

$$G(\alpha,k,\gamma) \subset K\left(\gamma+\frac{k}{2}-1-\mathfrak{R}\right). \tag{2.23}$$

The functions in $K(\gamma + k/2 - 1 - \mathfrak{R})$ need not even be finitely valent in E for $k + 2(\gamma - \mathfrak{R}) > 4$.

Remark 2.4. From Theorem 2.2 and [11, Lemma 1.3] by Pommerenke, it follows that $G(\alpha,k,\gamma)$ is a linearly invariant family of order $(\gamma + k/2 - \mathfrak{R})$. Therefore, the image of E under functions in $G(\alpha,k,\gamma)$ contains the schlicht disk $|z| < 1/(k + 2(\gamma - \mathfrak{R}))$.

Theorem 2.5. *Let $f \in G_\beta(\alpha,k,\gamma)$, $\gamma \in (0,1)$, $|\beta| < \pi/2$, be of the form* (1.1). *Then $|a_2| \le k/2 + ((1+\gamma)/2)|\cos(\beta/\gamma)|$. This estimate is best possible, extremal function being $f_0(z)$ defined by* (2.4).

Proof. Let $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ in (1.5).

Now, it is known that, for functions p of positive real part with $\gamma \in (0,1)$, p^γ is subordinate to $((1+z)/(1-z))^\gamma$. Also $|b_2| \le k/2$, see [1, 12]. Therefore, from (1.5), we have $2a_2 = 2b_2 + (e^{-i\beta}\cos(\beta/\gamma))(1+\gamma)$, and this gives us the required result. □

Remark 2.6. Let $f \in G(\alpha,k,\gamma)$, for $2 \le k \le 4 - 2(\gamma - \mathfrak{R})$, and be given by (1.1). Then f is univalent in E by Remark 2.3 and $w_0 f(z)/(w_0 - f(z))$ is univalent in E for $w_0 \ne 0$, $w_0 \ne f(z)$. Now

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right)z^2 + \cdots, \tag{2.24}$$

and therefore $|a_2+1/w_0| \leq 2$ and so $|1/w_0| \geq 2/(4+k+(1+\gamma)\cos(\beta/\gamma))$, on using Theorem 2.5. Hence it follows that the image of E under $f \in G(\alpha,k,\gamma)$ with $2 \leq k+2(\gamma-\mathfrak{R}) \leq 4$ contains the schlicht disc $|z| < 2/(4+k+(1+\gamma)\cos(\beta/\gamma))$.

From Remark 2.3, and the results proved for the class $K(\sigma)$, $\sigma \geq 0$ in [4], we at once have the following.

Theorem 2.7. *Let $f \in G(\alpha,k,\gamma)$ and be given by* (1.1)*. Let F_σ be defined by*

$$F_\sigma(z) = \frac{1}{2(\sigma+1)}\left[\left(\frac{1+z}{1-z}\right)^{\sigma+1} - 1\right] = z + \sum_{n=2}^{\infty} A_n(\sigma) z^n, \tag{2.25}$$

where $\sigma = (\gamma + k/2 - 1 - \mathfrak{R})$, and $\mathfrak{R}$ is given by (2.13)*. Clearly $F_\sigma \in G(\alpha,k,r)$.*

(i) *Denote by $L(r,f)$ the length of the image of the circle $|z| = r$ under f and by $A(r,f)$ the area of $f(|z| \leq r)$. Then, for $0 \leq r < 1$*

(a) $L(r,f) \leq L(r,F_\sigma)$,
(b) $A(r,f) \leq A(r,F_\sigma)$.

(ii) *For $z = re^{i\theta}$, $0 \leq r < 1$, $|f(z)| = (1/2(\sigma+1))[((1+z)/(1-z))^{\sigma+1} - 1]$.*

The function F_σ, defined by (2.25)*, shows that this upper bound is sharp.*

Theorem 2.8. *Let $f \in G(\alpha,k,\gamma)$. Then, for $0 < r < 1$, $\alpha, r \in (0,1)$, $k \geq 2$,*

$$L(r,f) \leq c(\alpha,k,r)\left(\frac{1}{1-r}\right)^{k/2+\gamma}, \tag{2.26}$$

where $c(\alpha,k,r)$ is a constant depending upon α, k, and γ only.

Proof. With $z = re^{i\theta}$,

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)| d\theta \\ &= \int_0^{2\pi} r\left|\frac{e^{-i\beta}\phi'(z)(p(z)\cos(\beta/\gamma) - i\sin(\beta/\gamma))^\gamma}{1-\alpha^2 z^2}\right| d\theta, \quad \phi \in V_k,\ p \in P,\ z \in E. \end{aligned} \tag{2.27}$$

For $\phi \in V_k$, it is known [10] that there exist $s_1, s_2 \in S^*$ such that

$$z\phi'(z) = \frac{(s_1(z))^{k/4+1/2}}{(s_2(z))^{k/4-1/2}}. \tag{2.28}$$

Also, for $p \in P$ we have for $z = re^{i\theta}$,

$$\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2} \tag{2.29}$$

(see [13]). Now, from (2.27), (2.28), and (2.29), we have

$$L(r,f) \le \frac{c_1(\alpha,k,\gamma)}{r^{(k/4-1/2)}}\left(\frac{1}{2\pi}\int_0^{2\pi}|s_1(z)|^{(k/4+1/2)(2/(2-\gamma))}d\theta\right)^{(2-\gamma)/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2\right)^{\gamma/2}$$
$$\le c(\alpha,k,\gamma)\left(\frac{1}{1-r}\right)^{k/2+\gamma}, \tag{2.30}$$

where we have used distortion theorems, subordination for the starlike functions, and Holder's inequality, and c and c_1 are constants. □

Theorem 2.9. *Let $f \in G(\alpha,k,\gamma)$ and be given by* (1.1)*. Then, for $\alpha,\gamma \in [0,1]$, $k \ge 2$, one has $a_n = o(1)n^{k/2+\gamma-1}$, $(n \to \infty)$ where $o(1)$ is a constant depending only on k, α, and γ.*

Proof. With $z = re^{i\theta}$, Cauchy's theorem gives

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)e^{-in\theta}d\theta. \tag{2.31}$$

Thus

$$n|a_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi}|zf'(z)|d\theta = (1/2\pi r^n)L(r,f). \tag{2.32}$$

Using Theorem 2.8 and putting $r = 1 - 1/n$, we prove this result. □

Acknowledgment

The authors would like to thank the referee for thoughtful comments and suggestions.

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