Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 640948, 10 pages
doi:10.1155/2009/640948

Research Article

Unbounded Conditional Expectations for Partial O*-Algebras

Mayumi Takakura

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan

Correspondence should be addressed to Mayumi Takakura, mayumi@fukuoka-u.ac.jp

Received 18 September 2008; Accepted 26 January 2009

Recommended by Ingo Witt

The main purpose of this paper is to generalize studies of unbounded conditional expectations for O*-algebras to those for partial O*-algebras.

1. Introduction

In probability theory, conditional expectations play a fundamental role. Conditional expectations for von Neumann algebra have been studied in noncommutative probability theory. In particular, Takesaki [1] characterized the existence of conditional expectation using Tomita's modular theory. Thus a conditional expectation does not necessarily exist for a general von Neumann algebra. The study of conditional expectations for O*-algebras was begun by Gudder and Hudson [2]. After that, in [3, 4] we have investigated an unbounded conditional expectation which is a positive linear map $\mathcal{E}$ of an O*-algebra $\mathcal{M}$ onto a given O*-subalgebra $\mathcal{N}$ of $\mathcal{M}$. In this paper we will consider conditional expectations for partial O*-algebras. Suppose that $\mathcal{M}$ is a self-adjoint partial O*-algebra containing identity I on dense subspace $\mathfrak{D}$ of Hilbert space $\mathcal{H}$ with a strongly cyclic vector ξ_0, and $\mathcal{N}$ is a partial O*-subalgebra of $\mathcal{M}$ such that $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathcal{N}\xi_0}$, where $R^{\mathrm{w}}(\mathcal{M})$ is the set of all right multiplier of $\mathcal{M}$. The definitions of (self-adjoint) partial O*-algebra and a strongly cyclic vector are stated in Section 2. A map $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ is said to be a *weak conditional-expectation* of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$ if it satisfies $(AX\xi_0 \mid Y\xi_0) = (\mathcal{E}(A)X\xi_0 \mid Y\xi_0)$, for all $A \in \mathcal{M}$, for all $X, Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$; but, the range $\mathcal{E}(A)$ of the weak conditional-expectation $\mathcal{E}$ is not necessarily contained in $\mathcal{N}$, and so we have considered a map $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ satisfying the following:

(i) the domain $D(\mathcal{E})$ of $\mathcal{E}$ is a †-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$;

(ii) $\mathcal{E}$ is a projection; that is, it is hermitian ($\mathcal{E}(A)^\dagger = \mathcal{E}(A^\dagger)$, for all $A \in D(\mathcal{E})$) and $\mathcal{E}(X) = X$, for all $X \in \mathcal{N}$;

(iii) $\mathcal{E}(A\square X) = \mathcal{E}(A)\square X$, for all $A \in D(\mathcal{E})$, for all $X \in \mathcal{N} \cap R^{w}(\mathcal{M})$, $\mathcal{E}(X\square A) = X\square\mathcal{E}(A)$, for all $A \in D(\mathcal{E}) \cap R^{w}(\mathcal{N})$, for all $X \in \mathcal{N}$;

(iv) $\omega_{\xi_0}(\mathcal{E}(A)) = \omega_{\xi_0}(A)$, for all $A \in D(\mathcal{E})$, where ω_{ξ_0} is a state on $\mathcal{M}$ defined by $\omega_{\xi_0}(A) = (A\xi_0 \mid \xi_0)$, $A \in \mathcal{M}$;

and call it an *unbounded conditional expectation* of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. In particular, if $D(\mathcal{E}) = \mathcal{M}$, then $\mathcal{E}$ is said to be a *conditional expectation* of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$.

Finally, we will investigate the scale of the domain of unbounded conditional expectations of partial GW*-algebra which is unbounded generalizations of von Neumann algebras.

2. Preliminaries

In this section we review the definitions and the basic theory of partial O*-algebras, partial GW*-algebras and partial EW*-algebras. For more details, refer to [5].

A *partial* $*$*-algebra* is a complex vector space $\mathfrak{A}$ with an involution $x \rightarrow x^*$ and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ such that

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$, for all $\lambda, \mu \in \mathbb{C}$;

(iii) whenever $(x, y) \in \Gamma$, there exists a product $x \cdot y \in \mathfrak{A}$ with the usual properties of the multiplication: $x \cdot (y + \lambda z) = x \cdot y + \lambda(x \cdot z)$ and $(x \cdot y)^* = y^* \cdot x^*$ for $(x, y), (x, z) \in \Gamma$ and $\lambda \in \mathbb{C}$.

The element e of the $\mathfrak{A}$ is called a *unit* if $e^* = e$, $(e, x) \in \Gamma$ for all $x \in \mathfrak{A}$, and $e \cdot x = x \cdot e = x$, for all $x \in \mathfrak{A}$. Notice that the partial multiplication is not required to be associative. Whenever $(x, y) \in \Gamma$, x is called a *left multiplier* of y and y is called a *right multiplier* of x, and we write $x \in L(y)$ and $y \in R(x)$. For a subset $\mathcal{B} \subset \mathfrak{A}$, we write

$$L(\mathcal{B}) = \bigcap_{x \in \mathcal{B}} L(x), \qquad R(\mathcal{B}) = \bigcap_{x \in \mathcal{B}} R(x). \tag{2.1}$$

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot \mid \cdot)$ and $\mathfrak{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^{\dagger}(\mathfrak{D}, \mathcal{H})$ the set of all closable linear operators X such that $\mathfrak{D}(X) = \mathfrak{D}$, $\mathfrak{D}(X^*) \supseteq \mathfrak{D}$. The set $\mathcal{L}^{\dagger}(\mathfrak{D}, \mathcal{H})$ is a partial $*$-algebra with respect to the following operations: the usual sum $X + Y$, the scalar multiplication λX, the involution $X \rightarrow X^{\dagger}(= X^* \lceil \mathfrak{D})$, and the weak partial multiplication $X\square Y \equiv X^{\dagger *}Y$, defined whenever Y is a weak right multiplier of X ($X \in L^{w}(Y)$ or $Y \in R^{w}(X)$), that is, if and only if $Y\mathfrak{D} \subset \mathfrak{D}(X^{\dagger *})$ and $X^*\mathfrak{D} \subset \mathfrak{D}(Y^*)$. A partial $*$-subalgebra of $\mathcal{L}^{\dagger}(\mathfrak{D}, \mathcal{H})$ is called a *partial* O*-*algebra* on $\mathfrak{D}$.

Let $\mathcal{M}$ be a partial O*-algebra on $\mathfrak{D}$. The locally convex topology on $\mathfrak{D}$ defined by the family $\{\|\cdot\|_X;\ X \in \mathcal{M}\}$ of seminorms $\|\xi\|_X = \|\xi\| + \|X\xi\|$, $\xi \in \mathfrak{D}$ is called the *graph topology* on $\mathfrak{D}$ and denoted by $t_{\mathcal{M}}$. The completion of $\mathfrak{D}[t_{\mathcal{M}}]$ is denoted by $\widetilde{\mathfrak{D}}[t_{\mathcal{M}}]$. If the locally convex space $\mathfrak{D}[t_{\mathcal{M}}]$ is complete, then $\mathcal{M}$ is called *closed*. We also define the following domains:

$$\begin{gathered} \widehat{\mathfrak{D}}(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} \mathfrak{D}(\overline{X}), \qquad \mathfrak{D}^*(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} \mathfrak{D}(X^*), \\ \mathfrak{D}^{**}(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} \mathfrak{D}\big((X^* \lceil \mathfrak{D}^*(\mathcal{M}))^*\big), \end{gathered} \tag{2.2}$$

and then

$$\mathfrak{D} \subset \widetilde{\mathfrak{D}}(\mathcal{M}) \subset \widehat{\mathfrak{D}}(\mathcal{M}) \subset \mathfrak{D}^{**}(\mathcal{M}) \subset \mathfrak{D}^{*}(\mathcal{M}). \tag{2.3}$$

The partial O*-algebra $\mathcal{M}$ is called *fully closed* if $\mathfrak{D} = \widehat{\mathfrak{D}}(\mathcal{M})$, *self-adjoint* if $\mathfrak{D} = \mathfrak{D}^*(\mathcal{M})$, *essentially self-adjoint* if $\mathfrak{D}^*(\mathcal{M}) = \widehat{\mathfrak{D}}(\mathcal{M})$, and *algebraically self-adjoint* if $\mathfrak{D}^*(\mathcal{M}) = \mathfrak{D}^{**}(\mathcal{M})$.

We defined two weak commutants of $\mathcal{M}$. The *weak bounded commutant* $\mathcal{M}'_{\mathrm{w}}$ of $\mathcal{M}$ is the set

$$\mathcal{M}'_{\mathrm{w}} = \left\{C \in \mathcal{B}(\mathcal{H}); (CX\xi \mid \eta) = (C\xi \mid X^{\dagger}\eta) \text{ for every } X \in \mathcal{M},\ \xi,\eta \in \mathfrak{D}\right\}; \tag{2.4}$$

but the partial multiplication is not required to be associative, so we define the *quasi-weak bounded commutant* $\mathcal{M}'_{\mathrm{qw}}$ of $\mathcal{M}$ as the set

$$\mathcal{M}'_{\mathrm{qw}} = \left\{C \in \mathcal{M}'_{\mathrm{w}}; \left(CX_1^{\dagger}\xi \mid X_2\eta\right) = \left(C\xi \mid (X_1 \square X_2)\eta\right)\ \forall X_1 \in L(X_2),\ \xi,\eta \in \mathfrak{D}\right\}. \tag{2.5}$$

In general, $\mathcal{M}'_{\mathrm{qw}} \subsetneq \mathcal{M}'_{\mathrm{w}}$.

A $*$-*representation* of a partial $*$-algebra $\mathfrak{A}$ is a $*$-homomorphism of $\mathfrak{A}$ into $\mathcal{L}^{\dagger}(\mathfrak{D},\mathcal{H})$, satisfying $\pi(e) = I$ whenever $e \in \mathfrak{A}$, that is,

(i) π is linear;

(ii) $x \in L^{\mathrm{w}}(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^{\mathrm{w}}(\pi(y))$ and $\pi(x)\square\pi(y) = \pi(xy)$;

(iii) $\pi(x^*) = \pi(x)^{\dagger}$ for every $x \in \mathfrak{A}$.

Let π be a $*$-representation of a partial $*$-algebra $\mathfrak{A}$ into $\mathcal{L}^{\dagger}(\mathfrak{D},\mathcal{H})$. Then we define

$\widetilde{\mathfrak{D}}(\pi)$: the completion of $\mathfrak{D}$ with respect to the graph topology $t_{\pi(\mathfrak{A})}$,

$$\begin{aligned}
\widetilde{\pi}(x) &= \overline{\pi(x)}\lceil\widetilde{\mathfrak{D}}(\pi), \quad x \in \mathfrak{A};\\
\widehat{\mathfrak{D}}(\pi) &= \bigcap_{x\in\mathfrak{A}} \mathfrak{D}(\overline{\pi(x)}),\\
\widehat{\pi}(x) &= \overline{\pi(x)}\lceil\widehat{\mathfrak{D}}(\pi), \quad x \in \mathfrak{A};\\
\mathfrak{D}^*(\pi) &= \bigcap_{x\in\mathfrak{A}} \mathfrak{D}(\pi(x)^*),\\
\pi^*(x) &= \pi(x^*)^*\lceil\mathfrak{D}^*(\pi), \quad x \in \mathfrak{A}.
\end{aligned} \tag{2.6}$$

We say that π is *closed* if $\mathfrak{D} = \widetilde{\mathfrak{D}}(\pi)$; *fully closed* if $\mathfrak{D} = \widehat{\mathfrak{D}}(\pi)$; *essentially self-adjoint* if $\widehat{\mathfrak{D}}(\pi) = \mathfrak{D}^*(\pi)$; and *self-adjoint* if $\mathfrak{D} = \mathfrak{D}^*(\pi)$.

We introduce the weak and the quasi-weak commutants of a $*$-representaion π of a partial $*$-algebra $\mathfrak{A}$ as follows:

$$\pi(\mathfrak{A})'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}); (C\xi \mid \pi(x)\eta) = (C\pi(x^*)\xi \mid \eta),\ \forall x \in \mathfrak{A},\ \xi,\eta \in \mathfrak{D}(\pi)\},$$
$$\mathcal{C}_{\mathrm{qw}}(\pi) = \{C \in \pi(\mathfrak{A})'_{\mathrm{w}}; (C\pi(x_1^*)\xi \mid \pi(x_2)\eta) = (C\xi \mid \pi(x_1x_2)\eta),$$
$$\forall x_1, x_2 \in \mathfrak{A} \text{ such that } x_1 \in L(x_2), \text{ and all } \xi,\eta \in \mathfrak{D}(\pi)\}, \tag{2.7}$$

respectively.

We define the notion of strongly cyclic vector for a partial O*-algebra $\mathcal{M}$ on $\mathfrak{D}$ in $\mathcal{H}$. A vector ξ_0 in $\mathfrak{D}$ is said to be *strongly cyclic* if $R^{\mathrm{w}}(\mathcal{M})\xi_0$ is dense in $\mathfrak{D}[t_{\mathcal{M}}]$, and ξ_0 is said to be *separating* if $\overline{\mathcal{M}'_{\mathrm{w}}\xi_0} = \mathcal{H}$, where $R^{\mathrm{w}}(\mathcal{M}) = \{Y \in \mathcal{M}; X \Box Y \text{ is well-defined, for all } X \in \mathcal{M}\}$.

We introduce the notion of partial GW*-algebras and partial EW*-algebras which are unbounded generalizations of von Neumann algebras. A fully closed partial O*-algebra $\mathcal{M}$ on $\mathfrak{D}$ is called a *partial GW*-algebra* if there exists a von Neumann algebra $\mathcal{M}_0$ on $\mathcal{H}$ such that $\mathcal{M}'_0\mathfrak{D} \subset \mathfrak{D}$ and $\mathcal{M} = [\mathcal{M}_0\lceil\mathfrak{D}]^{s^*}$. A partial O*-algebra $\mathcal{M}$ on $\mathfrak{D}$ is said to be a partial EW*-algebra if $\overline{\mathcal{M}_b} \equiv \{A \in \mathcal{B}(\mathcal{H}); A\lceil\mathfrak{D} \in \mathcal{M}\}$ is a von Neumann algebra, $\mathcal{M}_b\mathfrak{D} \subset \mathfrak{D}$ and $\overline{\mathcal{M}_b}'\mathfrak{D} \subset \mathfrak{D}$.

3. Weak Conditional Expectations

In this section, let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing the identity I on $\mathfrak{D}$ in $\mathcal{H}$ with a strongly cyclic vector ξ_0 and let $\mathcal{N}$ be a partial O*-subalgebra of $\mathcal{M}$ such that

(N) $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathcal{N}\xi_0}$.

The following is easily shown.

Lemma 3.1. *Put*

$$\mathfrak{D}(\pi_{\mathcal{N}}) = (\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0,$$
$$\pi_{\mathcal{N}}(X)Y\xi_0 = (X \Box Y)\xi_0, \quad \forall X \in \mathcal{N},\ \forall Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}). \tag{3.1}$$

Then $\pi_{\mathcal{N}}$ is a $*$-representations of $\mathcal{N}$ in the Hilbert space $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathfrak{D}(\pi_{\mathcal{N}})}$.

We denote by $P_{\mathcal{N}}$ the projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathfrak{D}(\pi_{\mathcal{N}})}$. This projection $P_{\mathcal{N}}$ plays an important role in this reserch. First we have the following.

Lemma 3.2. *It holds that $P_{\mathcal{N}}\mathfrak{D} \subset \mathfrak{D}^*(\pi_{\mathcal{N}})$ and $\pi^*_{\mathcal{N}}(X)P_{\mathcal{N}}\xi = P_{\mathcal{N}}X\xi$, for all $X \in \mathcal{N}$ and for all $\xi \in \mathfrak{D}$.*

Proof. Take arbitrary $X \in \mathcal{N}$ and $\xi \in \mathfrak{D}$. For any $Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, we have

$$(\pi_{\mathcal{N}}(X^\dagger)Y\xi_0 \mid P_{\mathcal{N}}\xi) = ((X^\dagger \Box\, Y)\xi_0 \mid P_{\mathcal{N}}\xi) = (X^\dagger Y\xi_0 \mid \xi) = (Y\xi_0 \mid X\xi) = (Y\xi_0 \mid P_{\mathcal{N}}X\xi), \tag{3.2}$$

and so $P_{\mathcal{N}}\mathfrak{D} \subset \mathfrak{D}^*(\pi_{\mathcal{N}})$ and $\pi^*_{\mathcal{N}}(X)P_{\mathcal{N}}\xi = P_{\mathcal{N}}X\xi$. □

Definition 3.3. A map $\mathcal{E}$ of $\mathcal{M}$ into $\mathcal{L}^\dagger(\mathfrak{D}(\pi_\mathcal{N}), \mathcal{H}_\mathcal{N})$ is said to be a weak conditional-expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$ if it satisfies

$$(AX\xi_0 \mid Y\xi_0) = (\mathcal{E}(A)X\xi_0 \mid Y\xi_0), \quad \forall A \in \mathcal{M},\ \forall X, Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}). \tag{3.3}$$

For weak conditional-expectation we have the following.

Theorem 3.4. *There exists a unique weak conditional-expectation $\mathcal{E}(\cdot \mid \mathcal{N})$ of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$, and*

$$\mathcal{E}(A \mid \mathcal{N}) = P_\mathcal{N} A \lceil \mathfrak{D}(\pi_\mathcal{N}), \quad \forall A \in \mathcal{M}. \tag{3.4}$$

The weak conditional-expectation $\mathcal{E}(\cdot \mid \mathcal{N})$ of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$ satisfies the following:

(i) *$\mathcal{E}(\cdot \mid \mathcal{N})$ is linear,*

(ii) *$\mathcal{E}(\cdot \mid \mathcal{N})$ is hermitian, that is, $\mathcal{E}(A \mid \mathcal{N})^\dagger = \mathcal{E}(A^\dagger \mid \mathcal{N})$, for all $A \in \mathcal{M}$,*

(iii) *$\mathcal{E}(X \mid \mathcal{N}) = X \lceil \mathfrak{D}(\pi_\mathcal{N})$, for all $X \in \mathcal{N}$,*

(iv) *$\mathcal{E}(A^\dagger \square A \mid \mathcal{N}) \geq 0$, for all $A \in \mathcal{M}$ s.t. $A^\dagger \square A$ is well-defined,*

(v) *$\mathcal{E}(A \mid \mathcal{N})^\dagger \square \mathcal{E}(A \mid \mathcal{N}) \leq \mathcal{E}(A^\dagger \square A \mid \mathcal{N})$, for all $A \in \mathcal{M}$ s.t. $A^\dagger \square A$ and $\mathcal{E}(A \mid \mathcal{N})^\dagger \square \mathcal{E}(A \mid \mathcal{N})$ are well-defined,*

(vi) *$\mathcal{E}(A \mid \mathcal{N}) \square \pi_\mathcal{N}(X)$ is well-defined for any $A \in \mathcal{M}$ and $X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, and $\mathcal{E}(A \mid \mathcal{N}) \square \pi_\mathcal{N}(X) = \mathcal{E}(A \square X \mid \mathcal{N})$,*

(vii) *$\pi_\mathcal{N}(X) \square \mathcal{E}(A \mid \mathcal{N})$ is well-defined for any $A \in \mathcal{M} \cap R^{\mathrm{w}}(\mathcal{N})$ and for all $X \in \mathcal{N}$, and $\pi_\mathcal{N}(X) \square \mathcal{E}(A \mid \mathcal{N}) = \mathcal{E}(X \square A \mid \mathcal{N})$,*

(viii) *$\omega_{\xi_0}(\mathcal{E}(A \mid \mathcal{N})) = \omega_{\xi_0}(A)$, for all $A \in \mathcal{M}$.*

Proof. We put

$$\mathcal{E}(A \mid \mathcal{N}) = P_\mathcal{N} A \lceil \mathfrak{D}(\pi_\mathcal{N}), \quad \forall A \in \mathcal{M}. \tag{3.5}$$

By Lemma 3.2, $\mathcal{E}(A \mid \mathcal{N})$ is a linear map of $\mathfrak{D}(\pi_\mathcal{N})$ into $\mathfrak{D}^*(\pi_\mathcal{N})$ for any $A \in \mathcal{M}$, and furthermore we have $\mathcal{E}(A \mid \mathcal{N})^\dagger = \mathcal{E}(A^\dagger \mid \mathcal{N})$, for all $A \in \mathcal{M}$, so $\mathcal{E}(\cdot \mid \mathcal{N})$ is a map of $\mathcal{M}$ into $\mathcal{L}^\dagger(\mathfrak{D}(\pi_\mathcal{N}), \mathcal{H}_\mathcal{N})$.

Since

$$(\mathcal{E}(A \mid \mathcal{N})X\xi_0 \mid Y\xi_0) = (P_\mathcal{N} AX\xi_0 \mid Y\xi_0) = (AX\xi_0 \mid Y\xi_0) \tag{3.6}$$

for each $A \in \mathcal{M}$, $X, Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, $\mathcal{E}(\cdot \mid \mathcal{N})$ is a weak conditional-expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. It is easily shown that if $\mathcal{E}$ is a weak conditional-expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$, $\mathcal{E}(A) = \mathcal{E}(A \mid \mathcal{N})$ for each $A \in \mathcal{M}$. Thus the existence and uniqueness of weak conditional-expectations is shown. The statements (iii)–(viii) follow since $\mathcal{E}(A \mid \mathcal{N}) = P_\mathcal{N} A \lceil \mathfrak{D}(\pi_\mathcal{N})$, for all $A \in \mathcal{M}$. This completes the proof. □

4. Unbounded Conditional Expectations for Partial O*-Algebras

Let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing I on $\mathfrak{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathfrak{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N} \ni I$ is a partial O*-subalgebra of $\mathcal{M}$ satisfying (N): $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}}$. We introduce unbounded conditional expectations of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$.

Definition 4.1. A map $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ is said to be an unbounded conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$ if

(i) the domain $D(\mathcal{E})$ of $\mathcal{E}$ is a †-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$;

(ii) $\mathcal{E}$ is a projection; that is, it is hermitian ($\mathcal{E}(A)^\dagger = \mathcal{E}(A^\dagger)$, for all $A \in D(\mathcal{E})$) and $\mathcal{E}(X) = X$, for all $X \in \mathcal{N}$;

(iii) $\mathcal{E}(A \square X) = \mathcal{E}(A) \square X$, for all $A \in D(\mathcal{E})$, for all $X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, $\mathcal{E}(X \square A) = X \square \mathcal{E}(A)$, for all $A \in D(\mathcal{E}) \cap R^{\mathrm{w}}(\mathcal{N})$, for all $X \in \mathcal{N}$;

(iv) $\omega_{\xi_0}(\mathcal{E}(A)) = \omega_{\xi_0}(A)$, for all $A \in D(\mathcal{E})$.

In particular, if $D(\mathcal{E}) = \mathcal{M}$, then $\mathcal{E}$ is said to be a *conditional expectation* of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$.

For unbounded conditional expectations we have the following.

Lemma 4.2. *Let $\mathcal{E}$ be an unbounded conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. Then,*

$$\mathcal{E}(A)X\xi_0 = P_{\mathcal{N}}AX\xi_0 = \mathcal{E}(A \mid \mathcal{N})X\xi_0, \quad \forall A \in D(\mathcal{E}),\ \forall X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}). \tag{4.1}$$

Proof. For all $A \in D(\mathcal{E})$ and $X, Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, we have

$$\begin{aligned}(\mathcal{E}(A)X\xi_0 \mid Y\xi_0) &= (\mathcal{E}(A \square X)\xi_0 \mid Y\xi_0) = (\mathcal{E}(Y^\dagger \square A \square X)\xi_0 \mid \xi_0) = ((Y^\dagger \square A \square X)\xi_0 \mid \xi_0) \\ &= (AX\xi_0 \mid Y\xi_0) = (AX\xi_0 \mid P_{\mathcal{N}}Y\xi_0) = (P_{\mathcal{N}}AX\xi_0 \mid Y\xi_0).\end{aligned} \tag{4.2}$$

Hence, $\mathcal{E}(A)X\xi_0 = P_{\mathcal{N}}AX\xi_0 = \mathcal{E}(A \mid \mathcal{N})X\xi_0$, for all $A \in D(\mathcal{E})$, for all $X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$. □

Let $\mathfrak{E}$ be the set of all unbounded conditional expectations of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. Then $\mathfrak{E}$ is an ordered set with the following order $\subset$:

$$\mathcal{E}_1 \subset \mathcal{E}_2 \quad \text{iff } D(\mathcal{E}_1) \subset D(\mathcal{E}_2), \qquad \mathcal{E}_1(A) = \mathcal{E}_2(A), \quad \forall A \in D(\mathcal{E}_1). \tag{4.3}$$

Theorem 4.3. *There exists a maximal unbounded conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$, and it is denoted by $\mathcal{E}_{\mathcal{N}}$.*

Proof. We put

$$D(\mathcal{E}_0) \equiv \left\{A \in \mathcal{M}; P_{\mathcal{N}}A\lceil_{(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0} \in \mathcal{N}\lceil_{(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0}\right\}. \tag{4.4}$$

Then, for any $A \in D(\mathcal{E}_0)$, there exists a unique map $\mathcal{E}_0$ such that

$$\mathcal{E}_0(A)X\xi_0 = P_{\mathcal{N}}AX\xi_0 = \mathcal{E}(A \mid \mathcal{N})X\xi_0, \quad \forall X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}). \tag{4.5}$$

It is easily shown that $\mathcal{E}_0$ is an unbounded conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. Furthermore, $\mathcal{E}_0$ is maximal in $\mathfrak{E}$. Indeed, let $\mathcal{E} \in \mathfrak{E}$. Take an arbitrary $A \in D(\mathcal{E})$. Then by Lemma 4.2 we have

$$\mathcal{E}(A)X\xi_0 = P_{\mathcal{N}}AX\xi_0 = \mathcal{E}(A \mid \mathcal{N})X\xi_0, \quad X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}), \tag{4.6}$$

which implies $\mathcal{E}(A)X\xi_0 \in \mathcal{N}\lceil_{(\mathcal{N}\cap R^{\mathrm{w}}(\mathcal{M}))\xi_0}$. Hence $\mathcal{E} \subset \mathcal{E}_0$ and $\mathcal{E}_0$ is maximal in $\mathfrak{E}$. This completes the proof. □

5. Existence of Conditional Expectations for Partial O*-Algebras

Let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing I on $\mathfrak{D}$ in $\mathcal{H}$, $\xi_0 \in \mathfrak{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and $\mathcal{N} \ni I$ a partial O*-subalgebra of $\mathcal{M}$ such that

- (N) $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}}$,
- (N_1) $\mathcal{N}'_{\mathrm{w}}\widehat{\mathfrak{D}}(\mathcal{N}) \subset \widehat{\mathfrak{D}}(\mathcal{N})$,
- (N_2) $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is essentially self-adjoint for $\mathcal{N}$,
- (N_3) $\Delta''^{\,it}_{\xi_0}(\mathcal{N}'_{\mathrm{w}})'\Delta''^{\,-it}_{\xi_0} = (\mathcal{N}'_{\mathrm{w}})'$, for all $t \in \mathbb{R}$, where Δ''_{ξ_0} is the modular operator for the full Hilbert algebra $(\mathcal{M}'_{\mathrm{w}})'\xi_0$.

Lemma 5.1. *It holds that $D(\mathcal{E}_{\mathcal{N}}) = \{A \in \mathcal{M}; P_{\mathcal{N}}A\xi_0 \in \mathcal{N}\xi_0\}$.*

Proof. We put

$$D(\mathcal{E}) = \left\{A \in \mathcal{M}; P_{\mathcal{N}}A\xi_0 \in \mathcal{N}\xi_0\right\}. \tag{5.1}$$

By Lemma 4.2, we have

$$P_{\mathcal{N}}A\xi_0 = \mathcal{E}_{\mathcal{N}}(A)\xi_0 \in \mathcal{N}\xi_0 \tag{5.2}$$

for each $A \in D(\mathcal{E}_{\mathcal{N}})$. Hence, $D(\mathcal{E}_{\mathcal{N}}) \subset D(\mathcal{E})$. We show the converse inclusion. Since ξ_0 is separating vector for $\mathcal{M}$, it follows that for any $A \in D(\mathcal{E})$, there exists a unique element $\mathcal{E}(A)$ of $\mathcal{N}$ such that $P_{\mathcal{N}}A\xi_0 = \mathcal{E}(A)\xi_0$. Indeed, since $\mathcal{E}_{\mathcal{N}}$ is maximal in $\mathfrak{E}$, it is sufficient to show that $\mathcal{E}$ is an unbounded conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. By assumption (N_1) and [5, Proposition 2.3.5], we have

$$\overline{X} \text{ is affiliated with von Neumann algebra } (\mathcal{N}'_{\mathrm{w}})' \text{ for each } X \in \mathcal{N}, \tag{5.3}$$

$$\mathcal{N}'_{\mathrm{w}} = \mathcal{N}'_{\mathrm{qw}}. \tag{5.4}$$

Since $\mathcal{M}$ is self-adjoint and $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}}$, it follows that $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is a reducing subspace for $\mathcal{N}$, that is,

$$\mathcal{N}(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0 \subset \overline{(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0} = \overline{\mathcal{N}\xi_0}, \tag{5.5}$$

which implies by assumption (N_2) and [5, Theorem 7.4.4] that

$$P_{\mathcal{N}} \in N'_w, \qquad P_{\mathcal{N}}\widehat{\mathfrak{D}}(\mathcal{N}) \subset \widehat{\mathfrak{D}}(\mathcal{N}). \tag{5.6}$$

Furthermore, by (5.3) and (5.6), we have

$$\overline{\mathcal{N}\xi_0} = \overline{(\mathcal{N}'_{\mathrm{w}})'\xi_0}, \quad \text{that is,} \;\; P_{\mathcal{N}} = P_{(\mathcal{N}'_{\mathrm{w}})'}. \tag{5.7}$$

Let $S_{\xi 0}$ and $S''_{\xi 0}$ be the closures of the maps:

$$\begin{aligned} S_{\xi_0}A\xi_0 &= A^{\dagger}\xi_0, \quad A \in \mathcal{M}, \\ S''_{\xi_0}B\xi_0 &= B^{*}\xi_0, \quad B \in (\mathcal{M}'_{\mathrm{w}})'. \end{aligned} \tag{5.8}$$

By (5.3) we have

$$S_{\xi_0} \subset S''_{\xi_0}. \tag{5.9}$$

Takesaki proved in [1] that assumtion (N_3) implies

$$P_{(\mathcal{N}'_{\mathrm{w}})'}S''_{\xi_0} \subset S''_{\xi_0}P_{(\mathcal{N}'_{\mathrm{w}})'} \tag{5.10}$$

and there exists a conditional expectation $\mathcal{E}''$ of the von Neumann algebra $((\mathcal{M}'_{\mathrm{w}})', \xi_0)$ with respect to, $(\mathcal{N}'_{\mathrm{w}})'$.

By (5.6), (5.9), and (5.10), we have

$$\begin{aligned} \mathcal{E}(A^{\dagger})\xi_0 &= P_{\mathcal{N}}A^{\dagger}\xi_0 = P_{\mathcal{N}}S_{\xi_0}A\xi_0 = P_{\mathcal{N}}S''_{\xi_0}A\xi_0 \\ &= S''_{\xi_0}P_{\mathcal{N}}A\xi_0 = S''_{\xi_0}\mathcal{E}(A)\xi_0 = S_{\xi_0}\mathcal{E}(A)\xi_0 = \mathcal{E}(A)^{\dagger}\xi_0 \end{aligned} \tag{5.11}$$

for each $A \in D(\mathcal{E})$, which implies by the separateness of ξ_0 that $\mathcal{E}$ is hermitian.

It is clear that $\mathcal{E}(X) = X$, for all $X \in \mathcal{N}$. Take arbitrary $A \in D(\mathcal{E})$ and $X \in \mathcal{N} \cap L^{\mathrm{w}}(\mathcal{M})$. Since

$$\left(P_{\mathcal{N}}(X\square A)\xi_0 \mid Y\xi_0\right) = \left(P_{\mathcal{N}}A\xi_0 \mid X^{\dagger}Y\xi_0\right) = \left(\mathcal{E}(A)\xi_0 \mid X^{\dagger}Y\xi_0\right) = \left((X\square\mathcal{E}(A))\xi_0 \mid Y\xi_0\right) \tag{5.12}$$

for each $Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, it follows that $X\square A \in D(\mathcal{E})$ and $\mathcal{E}(X\square A) = X\square\mathcal{E}(A)$. Furthermore, since $\mathcal{E}$ is hermitian, it follows that $A\square X \in D(\mathcal{E})$ and $\mathcal{E}(A\square X) = \mathcal{E}(A)\square X$ for each $A \in D(\mathcal{E})$ and $X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$. It is clear that $\omega_{\xi_0}(\mathcal{E}(A)) = \omega_{\xi_0}(A)$ for each $A \in D(\mathcal{E})$. Thus $\mathcal{E}$ is an unbounded conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. This completes that proof. □

By Lemma 5.1, we have the following.

Theorem 5.2. *Let $\mathcal{M}$ be a self-adjoint partial O^*-algebra containing I on $\mathfrak{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathfrak{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N} \ni I$ is a partial O^*-subalgebra of $\mathcal{M}$ satisfying (N), (N_1), (N_2), and (N_3). Then there exists a conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$ if and only if $P_{\mathcal{N}}\mathcal{M}\xi_0 = \mathcal{N}\xi_0$.*

It is important to investigate the scale of the domain of an unbounded conditional expectation. We consider the case of partial GW*-algebras.

Theorem 5.3. *Let $\mathcal{M}$ be a partial GW^*-algebra on $\mathfrak{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathfrak{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N}$ be a partial GW^*-subalgebra of $\mathcal{M}$ satisfying (N), (N_1), (N_2), and (N_3).*

Then, $D(\mathcal{E}_{\mathcal{N}}) \supset$ linear span of $\{X\square A;\ X \in \mathcal{N},\ A \in (\mathcal{M}'_{\mathrm{w}})'$ s.t. $X\square A$ and $X\square\mathcal{E}''(A)$ are well defined$\} \supset$ linear span of $(\mathcal{M}'_{\mathrm{w}})'$ and $\mathcal{N}$.

In particular, if $\mathcal{N}_{P_{\mathcal{N}}}$ is a partial GW^*-algebra on $P_{\mathcal{N}}\mathfrak{D}$, then $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$.

Proof. Let $X \in \mathcal{N}$, and $A \in (\mathcal{M}'_{\mathrm{w}})'$ s.t. $X\square A$ and $X\square\mathcal{E}''(A)$ are all defined. Then, it follows since $\mathcal{N}$ is a partial GW*-subalgebra of $\mathcal{M}$ that

$$P_{\mathcal{N}}(X\square A)\xi_0 = P_{\mathcal{N}}X^{\dagger *}A\xi_0 = X^{\dagger *}P_{\mathcal{N}}A\xi_0 = (X\square\mathcal{E}''(A))\xi_0 \in \mathcal{N}\xi_0, \tag{5.13}$$

which implies by Lemma 5.1 that $X\square A \in D(\mathcal{E}_{\mathcal{N}})$ and $P_{\mathcal{N}}(X\square A)\xi_0 = (X\square\mathcal{E}''(A))\xi_0$. Suppose that $\mathcal{N}_{P_{\mathcal{N}}}$ is a partial GW*-algebra on $P_{\mathcal{N}}\mathfrak{D}$.

By the result of Takesaki [1] there exists a unique conditional expectation $\mathcal{E}''$ of the von Neumann algebra $(\mathcal{N}'_{\mathrm{w}})'$ such that $\mathcal{E}''(A_\alpha)_{P_{\mathcal{N}}} = P_{\mathcal{N}}AP_{\mathcal{N}}$ for each $A \in (\mathcal{M}'_{\mathrm{w}})'$. Since $\mathcal{M}$ is a partial GW*-algebra, for any $X \in \mathcal{M}$ there is a net $\{A_\alpha\} \in (\mathcal{M}'_{\mathrm{w}})'$ which converges strongly* to X. Then

$$\mathcal{E}''(A_\alpha)_{P_{\mathcal{N}}} \in \left((\mathcal{N}'_{\mathrm{w}})'\right)_{P_{\mathcal{N}}} = \left((\mathcal{N}_{P_{\mathcal{N}}})'_{\mathrm{w}}\right)', \tag{5.14}$$

and $\mathcal{E}''(A_\alpha)_{P_{\mathcal{N}}}$ converges strongly* to $P_{\mathcal{N}}X\lceil P_{\mathcal{N}}\mathfrak{D}$. Therefore, we have $P_{\mathcal{N}}X\lceil P_{\mathcal{N}}\mathfrak{D} \in \mathcal{N}$. Hence, $X \in D(\mathcal{E}_{\mathcal{N}})$ and $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation of $(\mathcal{M}, \xi_0)$ with respect to, $\mathcal{N}$. This completes the proof. □

Corollary 5.4. *Let $\mathcal{M}$ be a partial EW^*-algebra on $\mathfrak{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathfrak{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N}$ be a partial EW^*-subalgebra of $\mathcal{M}$ satisfying (N_2) and (N_3). Then,*

$$D(\mathcal{E}_{\mathcal{N}}) \supset \text{ linear span of } \mathcal{M}_b\mathcal{N} \text{ and } \mathcal{N}\mathcal{M}_b. \tag{5.15}$$

Proof. Since $\mathcal{M}_b \subset R^{\mathrm{w}}(\mathcal{M})$, it follows that $\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}) \supset \mathcal{N}_b$, and so clearly (N) holds. Furthermore, (N_1) holds since $\mathcal{N}'_{\mathrm{w}}\widehat{\mathfrak{D}}(\mathcal{N}) = \mathcal{N}_b\widehat{\mathfrak{D}}(\mathcal{N}) \subset \widehat{\mathfrak{D}}(\mathcal{N})$.This completes the proof. □

We consider the case of the well-known Segal L^p-space defined by τ.

Example 5.5. Let $\mathcal{M}_0$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful finite trace τ. We denote by $L^p(\tau)$ the Banach space completion of $\mathcal{M}_0$ with respect to, the norm

$$\|A\|_p \equiv \tau(|A|^p)^{1/p}, \quad A \in \mathcal{M}_0. \tag{5.16}$$

Then

$$\mathcal{M}_0 \equiv L^\infty(\tau) \subset L^p(\tau) \subset L^2(\tau) \subset L^q(\tau) \subset L^1(\tau), \quad 1 \leq q \leq 2 \leq p < \infty. \tag{5.17}$$

Let $2 \leq p < \infty$. Here we define a $*$-representation π of $L^p(\tau)$ by

$$\pi(X)A = XA, \quad X \in L^p(\tau),\ A \in L^\infty(\tau). \tag{5.18}$$

Then $\mathcal{M} \equiv \pi(L^p(\tau))$ is a partial EW*-algebra on $L^\infty(\tau)$ in $L^2(\tau)$ with $\mathcal{M}_b = \pi(L^\infty(\tau))$ which is integrable, that is, $\overline{\pi(X^\dagger)} = \pi(X)^*$ for each $X \in L^p(\tau)$. Furthremore, $\pi(L^p(\tau))$ has a strongly cyclic and separating vector $\xi_0 \equiv \lambda_\tau(I)$, where I is an identity operator on $\mathcal{H}$. Let $\mathcal{N}_0$ be a von Neumann subalgebra of $\mathcal{M}_0$. We put

$$\mathcal{N} = \left\{\pi(X);\ X \in L^p(\tau),\ \pi(X)\lambda_\tau(I) \in L^p(\tau\lceil\mathcal{N}_0)\right\}, \quad 2 \leq p \leq \infty. \tag{5.19}$$

Then $\mathcal{N}$ is an integrable partial EW*-subalgebra of $\mathcal{M}$ satisfying (N_2) and (N_3) and $P_{\mathcal{N}}\mathcal{M}\xi_0 = \mathcal{N}\xi_0$. By Theorem 5.2, there exists a conditional expectation of $(\mathcal{M}, \xi_0)$.

References

[1] M. Takesaki, "Conditional expectations in von Neumann algebras," *Journal of Functional Analysis*, vol. 9, pp. 306–321, 1972.

[2] S. P. Gudder and R. L. Hudson, "A noncommutative probability theory," *Transactions of the American Mathematical Society*, vol. 245, pp. 1–41, 1978.

[3] A. Inoue, M. Takakura, and H. Ogi, "Unbounded conditional expectations for O*-algebras," *Contemporary Mathematics*, vol. 427, pp. 225–234, 2007.

[4] A. Inoue, H. Ogi, and M. Takakura, "Conditional expectations for unbounded operator algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 80152, 22 pages, 2007.

[5] J.-P. Antoine, A. Inoue, and C. Trapani, *Partial *-Algebras and Their Operator Realizations*, vol. 553 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.