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Research Article

On Rational Approximations to Euler's Constant γ and to $\gamma + \log(a/b)$

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The author continues to study series transformations for the Euler-Mascheroni constant γ. Here, we discuss in detail recently published results of A. I. Aptekarev and T. Rivoal who found rational approximations to γ and $\gamma + \log q$ ($q \in \mathbb{Q}_{>0}$) defined by linear recurrence formulae. The main purpose of this paper is to adapt the concept of linear series transformations with integral coefficients such that rationals are given by explicit formulae which approximate γ and $\gamma + \log q$. It is shown that for every $q \in \mathbb{Q}_{>0}$ and every integer $d \geq 42$ there are infinitely many rationals a_m/b_m for $m = 1, 2, \ldots$ such that $|\gamma + \log q - a_m/b_m| \ll ((1-1/d)^d/(d-1)4^d)^m$ and $b_m \mid Z_m$ with $\log Z_m \sim 12d^2m^2$ for m tending to infinity.

1. Introduction

Let

$$s_n := \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2). \tag{1.1}$$

It is well known that the sequence $(s_n)_{n\geq 1}$ converges to Euler's constant $\gamma = 0,577\ldots$, where

$$s_n = \gamma + \mathcal{O}\left(\frac{1}{n}\right) \quad (n \geq 1). \tag{1.2}$$

Nothing is known on the algebraic background of such mathematical constants like Euler's constant γ. So we are interested in better diophantine approximations of these numbers, particularly in rational approximations.

In 1995 the author [1] introduced a linear transformation for the series $(s_n)_{n\geq 1}$ with integer coefficients which improves the rate of convergence. Let τ be an additional positive integer parameter.

Proposition 1.1 (see [1]). *For any integers $n \geq 1$ and $\tau \geq 2$ one has*

$$\left|\sum_{k=0}^{n}(-1)^{n+k}\binom{n+k+\tau-1}{n}\binom{n}{k}\cdot s_{k+\tau}-\gamma\right| \leq \frac{(\tau-1)!}{2n(n+1)(n+2)\cdots(n+\tau)}. \tag{1.3}$$

Particularly, by choosing $\tau = n \geq 2$, one gets the following result.

Corollary 1.2. *For any integer $n \geq 2$one has*

$$\left|\sum_{k=0}^{n}(-1)^{n+k}\binom{2n+k-1}{n}\binom{n}{k}\cdot s_{n+k}-\gamma\right| \leq \frac{1}{2n^2\binom{2n}{n}} \leq \frac{1}{n^{3/2}\cdot 4^n}. \tag{1.4}$$

Some authors have generalized the result of Proposition 1.1 under various aspects. At first one cites a result due to Rivoal [2].

Proposition 1.3 (see [2]). *For n tending to infinity, one has*

$$\left|\gamma-\frac{1}{(-2)^n}\sum_{k=0}^{n}(-1)^{k}\binom{2n+2k}{n}\binom{n}{k}s_{2k+n+1}\right| = \mathcal{O}\left(\frac{1}{n\ 27^{n/2}}\right). \tag{1.5}$$

Kh. Hessami Pilehrood and *T. Hessami Pilehrood* have found some approximation formulas for the logarithms of some infinite products including Euler's constant γ. These results are obtained by using Euler-type integrals, hypergeometric series, and the Laplace method [3].

Proposition 1.4 ([3]). *For n tending to infinity the following asymptotic formula holds:*

$$\left|\gamma-\sum_{k=0}^{n}(-1)^{n+k}\binom{n+k}{n}\binom{n}{k}s_{k+n+1}\right| = \frac{1}{4^{n+o(n)}}. \tag{1.6}$$

Recently the author has found series transformations involving three parameters n, τ_1 and τ_2, [4]. In Propositions 1.5 and 1.6 certain integral representations of the (discrete) series transformations are given, which exhibit important (analytical) tools to estimate the error terms of the transformations.

Proposition 1.5 (see [4]). *Let $n \geq 1$, $\tau_1 \geq 1$, and $\tau_2 \geq 1$ be integers. Additionally one assumes that*

$$1+\tau_1 \leq \tau_2. \tag{1.7}$$

Then one has

$$\sum_{k=0}^{n}(-1)^{n+k}\binom{n+\tau_1+k}{n}\binom{n}{k}\cdot s_{k+\tau_2}-\gamma = (-1)^{n+1}\int_0^1\left(\frac{1}{1-u}+\frac{1}{\log u}\right)\cdot u^{\tau_2-\tau_1-1}\cdot\frac{\partial^n}{\partial u^n}\left(\frac{u^{n+\tau_1}(1-u)^n}{n!}\right)du. \tag{1.8}$$

Proposition 1.6 (see [4]). *Let $n \geq 1$, $\tau_1 \geq 1$ and $\tau_2 \geq 1$ be integers. Additionally one assumes that*

$$1+\tau_1 \leq \tau_2 \leq 1+n+\tau_1. \tag{1.9}$$

Then one has

$$\sum_{k=0}^{n}(-1)^{n+k}\binom{n+\tau_1+k}{n}\binom{n}{k}\cdot s_{k+\tau_2}-\gamma = (-1)^{n+\tau_2-\tau_1}\int_0^1\int_0^1 w(t)\cdot\frac{(1-u)^{n+\tau_1}u^n(1-t)^{\tau_2-\tau_1-1}t^{n+\tau_1-\tau_2+1}}{(1-ut)^{n+1}}\,du\,dt, \tag{1.10}$$

with

$$w(t) := \frac{1}{t\cdot\left(\pi^2+\log^2\left(\frac{1}{t}-1\right)\right)}. \tag{1.11}$$

Setting

$$n=\tau_2=dm, \qquad \tau_1=(d-1)m-1, \quad (d\geq 2), \tag{1.12}$$

one gets an explicit upper bound from Proposition 1.6

Corollary 1.7. *For integers $m \geq 2$, $d \geq 3$, one has*

$$\left|\sum_{k=0}^{dm}(-1)^{dm+k}\binom{(2d-1)m+k-1}{dm}\binom{dm}{k}\cdot s_{k+dm}-\gamma\right| < C_d\cdot\left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^{m-2}, \tag{1.13}$$

where $0 < C_d \leq 1/16\pi^2$ is some constant depending only on d. For $d = 2$ one gets

$$\left|\sum_{k=0}^{2m}(-1)^{k}\binom{3m+k-1}{2m}\binom{2m}{k}\cdot s_{k+2m}-\gamma\right| < \left(\frac{16}{7\pi}\right)^2\cdot\frac{1}{64^m} \quad (m\geq 1). \tag{1.14}$$

For an application of Corollary 1.7 let the integers B_m and A_m be defined by

$$B_m := \text{l.c.m.}\ (1, 2, 3, \dots, 4m),$$
$$A_m := B_m \sum_{k=0}^{2m} (-1)^k \binom{3m+k-1}{2m} \binom{2m}{k} \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{k+2m-1}\right). \tag{1.15}$$

$\Lambda(k)$ denotes the von Mangoldt function. By [5, Theorem 434] one has

$$\psi(m) := \sum_{k \le m} \Lambda(k) \sim m. \tag{1.16}$$

Then, for $\varepsilon := (\log 55)/4 - 1 > 0.0018$, there is some integer m_0 such that

$$B_m = e^{\psi(4m)} < e^{4(1+\varepsilon)m} = 55^m \quad (m \ge m_0). \tag{1.17}$$

Multiplying (1.14) by B_m, we deduce the following corollary.

Corollary 1.8. *There is an integer m_0 such that one has for all integers $m \ge m_0$ that*

$$\left| B_m \sum_{k=0}^{2m} (-1)^k \binom{3m+k-1}{2m} \binom{2m}{k} \cdot \log(k+2m) + \gamma B_m - A_m \right| < \left(\frac{16}{7\pi}\right)^2 \cdot \left(\frac{55}{64}\right)^m. \tag{1.18}$$

2. Results on Rational Approximations to γ

In 2007, Aptekarev and his collaborators [6] found rational approximations to γ, which are based on a linear third-order recurrence. For the sake of brevity, let $D(n) = \text{l.c.m.}\ (1, 2, \dots, n)$.

Proposition 2.1 (see [6]). *Let $(p_n)_{n\ge0}$ and $(q_n)_{n\ge0}$ be two solutions of the linear recurrence*

$$\begin{aligned}(16n-15)(n+1)u_{n+1} &= \left(128n^3 + 40n^2 - 82n - 45\right)u_n \\ &\quad - n\left(256n^3 - 240n^2 + 64n - 7\right)u_{n-1} + n(n-1)(16n+1)u_{n-2}\end{aligned} \tag{2.1}$$

with $p_0 = 0$, $p_1 = 2$, $p_2 = 31/2$ and $q_0 = 1$, $q_1 = 3$, $q_2 = 25$. Then, one has $q_n \in \mathbb{Z}$, $D(n)p_n \in \mathbb{Z}$, and

$$\left|\gamma - \frac{p_n}{q_n}\right| \sim c_0 e^{-2\sqrt{2n}}, \quad |q_n| \sim \frac{c_1}{n^{1/4}} \frac{(2n)!}{n!} e^{\sqrt{2n}}, \tag{2.2}$$

with two positive constants c_0, c_1.

It seems interesting to replace the fraction p_n/q_n by

$$\frac{A_n}{B_n} := \frac{D(n)p_n}{D(n)q_n}, \tag{2.3}$$

and to estimate the remainder in terms of B_n.

Corollary 2.2. *Let $0 < \varepsilon < 1$. Then there are two positive constants c_2, c_3, such that for all sufficiently large integers n one has*

$$\begin{aligned} &c_2 \exp\left(-2(1+\varepsilon)\sqrt{2}\sqrt{\log B_n / \log\log B_n}\right) \\ &\quad < \left|\gamma - \frac{A_n}{B_n}\right| < c_3 \exp\left(-2(1-\varepsilon)\sqrt{2}\sqrt{\log B_n / \log\log B_n}\right). \end{aligned} \tag{2.4}$$

Recently, Rivoal [7] presented a related approach to the theory of rational approximations to Euler's constant γ, and, more generally, to rational approximations for values of derivatives of the Gamma function. He studied simultaneous Padé approximants to Euler's functions, from which he constructed a third-order recurrence formula that can be applied to construct a sequence in $\mathbb{Q}(z)$ that converges subexponentially to $\log(z) + \gamma$ for any complex number $z \in \mathbb{C} \setminus (-\infty, 0]$. Here, log is defined by its principal branch. We cite a corollary from [7].

Proposition 2.3 (see [7]). *(i) The recurrence*

$$\begin{aligned} &(n+3)^2(8n+11)(8n+19)U_{n+3} \\ &\quad = \left(24n^2 + 145n + 215\right)(8n+11)U_{n+2} \\ &\qquad - \left(24n^3 + 105n^2 + 124n + 25\right)(8n+27)U_{n+1} \\ &\qquad + (n+2)^2(8n+19)(8n+27)U_n\,, \end{aligned} \tag{2.5}$$

provides two sequences of rational numbers $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ with $p_0 = -1$, $p_1 = 4$, $p_2 = 77/4$ and $q_0 = 1$, $q_1 = 7$, $q_2 = 65/2$ such that $(p_n/q_n)_{n\geq 0}$ converges to γ.

(ii) The recurrence

$$\begin{aligned} &(n+1)(n+2)(n+3)U_{n+3} \\ &\quad = \left(3n^2 + 19n + 29\right)(n+1)U_{n+2} \\ &\qquad - \left(3n^3 + 6n^2 - 7n - 13\right)U_{n+1} + (n+2)^3 U_n, \end{aligned} \tag{2.6}$$

provides two sequences of rational numbers $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ with $p_0 = -1$, $p_1 = 11$, $p_2 = 71$ and $q_0 = 0$, $q_1 = 8$, $q_2 = 56$ such that $(p_n/q_n)_{n\geq 0}$ converges to $\log(2) + \gamma$.

The goal of this paper is to construct rational approximations to $\gamma + \log(a/b)$ without using recurrences by a new application of series transformations. The transformed sequences of rationals are constructed as simple as possible, only with few concessions to the rate of convergence (see Theorems 2.4 and 6.2 below).

In the following we denote by B_{2n} the Bernoulli numbers, that is, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, and so on (In Sections 3–6 the Bernoulli numbers cannot be confused with the integers B_n from Corollary 2.2.) In this paper we will prove the following result.

Theorem 2.4. *Let $a \geq 1$, $b \geq 1$, $d \geq 42$ and $m \geq 1$ be positive integers, and*

$$S_n := \sum_{j=1}^{an-1} \frac{1}{j} - \sum_{j=1}^{bn-1} \frac{1}{j} + 2\sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{n^2-1} \frac{1}{j} - \frac{1}{2n^2} + \sum_{j=1}^{dm} \frac{B_{2j}}{2j} \left(\frac{1}{n^{2j}} \left(\frac{1}{a^{2j}} - \frac{1}{b^{2j}} + 1 \right) - \frac{1}{n^{4j}} \right), \quad (n \geq 1). \tag{2.7}$$

Then,

$$\left| \sum_{k=0}^{dm} (-1)^{dm+k} \binom{(2d-1)m+k-1}{dm} \binom{dm}{k} S_{k+dm} - \gamma - \log \frac{a}{b} \right| < c_4 \cdot \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^m, \tag{2.8}$$

where c_4 is some positive constant depending only on d.

3. Proof of Theorem 2.4

Lemma 3.1. *One has for positive integers d and m*

$$g(k) := \binom{(2d-1)m+k-1}{dm} \binom{dm}{k} < 16^{dm} \quad (0 \leq k \leq dm). \tag{3.1}$$

Proof. Applying the well known inequality $\binom{g}{h} \leq 2^g$, we get

$$\binom{(2d-1)m+k-1}{dm} \binom{dm}{k} \leq 2^{(2d-1)m+dm-1} 2^{dm} = 2^{4dm-m-1} < 16^{dm}. \tag{3.2}$$

This proves the lemma. □

$g(k)$ takes its maximum value for $k = k_0$ with

$$k_0 = \frac{\sqrt{5d^2 - 4d + 1} - d + 1}{2} m + \mathcal{O}(1), \tag{3.3}$$

which leads to a better bound than 16^{dm} in Lemma 3.1. But we are satisfied with Lemma 3.1. A main tool in proving Theorem 2.4 is Euler's summation formula in the form

$$\sum_{i=1}^{n} f(i) = \int_1^n f(x)\, dx + \frac{f(1)+f(n)}{2} + \sum_{j=1}^{r} \frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(n) - f^{(2j-1)}(1)\right) + R_r, \tag{3.4}$$

where $r \in \mathbb{N}$ is a suitable chosen parameter, and the remainder R_r is defined by a periodic Bernoulli polynomial $P_{2r+1}(x)$, namely

$$R_r = \frac{1}{(2r+1)!}\int_1^n P_{2r+1}(x) f^{(2r+1)}(x)dx, \tag{3.5}$$

with

$$P_{2r+1}(x) = (-1)^{r-1}(2r+1)!\sum_{j=1}^{\infty}\frac{2\sin(2\pi jx)}{(2\pi j)^{2r+1}}. \tag{3.6}$$

Applying the summation formula to the function $f(x) = 1/x$, we get (see [8, equation (5)])

$$\sum_{i=1}^{n-1}\frac{1}{i} = \log n + \frac{1}{2} - \frac{1}{2n} + \sum_{j=1}^{r}\frac{B_{2j}}{2j}\left(1 - \frac{1}{n^{2j}}\right) - \int_1^n \frac{P_{2r+1}(x)}{x^{2r+2}}dx, \quad (n, r \in \mathbb{N}). \tag{3.7}$$

It follows that

$$\sum_{i=n}^{n^2-1}\frac{1}{i} - \log n = \frac{1}{2n} - \frac{1}{2n^2} + \sum_{j=1}^{r}\frac{B_{2j}}{2j}\left(\frac{1}{n^{2j}} - \frac{1}{n^{4j}}\right) - \int_n^{n^2}\frac{P_{2r+1}(x)}{x^{2r+2}}dx, \quad (n, r \in \mathbb{N}). \tag{3.8}$$

We prove Theorem 2.4 for $a \geq b$. The case $a < b$ is treated similarly. So we have again by the above summation formula that

$$\begin{aligned}\sum_{i=bn}^{an-1}\frac{1}{i} - \log\frac{a}{b} &= \left(\frac{1}{b} - \frac{1}{a}\right)\frac{1}{2n} + \sum_{j=1}^{r}\frac{B_{2j}}{2jn^{2j}}\left(\frac{1}{b^{2j}} - \frac{1}{a^{2j}}\right) \\ &\quad - \int_{bn}^{an}\frac{P_{2r+1}(x)}{x^{2r+2}}dx, \quad (n, r \in \mathbb{N}).\end{aligned} \tag{3.9}$$

First, we estimate the integral on the right-hand side of (3.8). We have

$$\begin{aligned}
\left|\int_n^{n^2} \frac{P_{2r+1}(x)}{x^{2r+2}} dx\right| &\leq \int_n^{n^2} \frac{|P_{2r+1}(x)|}{x^{2r+2}} dx \leq \int_n^{\infty} \frac{|P_{2r+1}(x)|}{x^{2r+2}} dx \\
&\leq 2(2r+1)! \int_n^{\infty} \frac{1}{x^{2r+2}} \sum_{j=1}^{\infty} \frac{1}{(2\pi j)^{2r+1}} dx \\
&= \frac{2(2r+1)!}{(2\pi)^{2r+1}} \left[-\frac{1}{(2r+1)x^{2r+1}}\right]_{x=n}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j^{2r+1}} \\
&= \frac{2(2r)!}{(2\pi)^{2r+1} n^{2r+1}} \zeta(2r+1) < \frac{3(2r)!}{(2\pi)^{2r+1} n^{2r+1}},
\end{aligned} \tag{3.10}$$

since $2\zeta(2r+1) \leq 2\zeta(3) < 3$. Next, we assume that $n \geq a$. Hence $[bn, an] \subseteq [n, n^2]$, and therefore we estimate the integral on the right-hand side in (3.9) by

$$\begin{aligned}
\left|\int_{bn}^{an} \frac{P_{2r+1}(x)}{x^{2r+2}} dx\right| &\leq \int_{bn}^{an} \frac{|P_{2r+1}(x)|}{x^{2r+2}} dx \\
&\leq \int_n^{n^2} \frac{|P_{2r+1}(x)|}{x^{2r+2}} dx \leq \frac{3(2r)!}{(2\pi)^{2r+1} n^{2r+1}}.
\end{aligned} \tag{3.11}$$

In the sequel we put $r = dm$. Moreover, in the above formula we now replace n by $dm + k$ with $0 \leq k \leq dm$. In order to estimate $(2r)!$ we use Stirling's formula

$$\sqrt{2\pi m} \left(\frac{m}{e}\right)^m < m! < \sqrt{2\pi(m+1)} \left(\frac{m}{e}\right)^m, \quad (m > 0). \tag{3.12}$$

Then, it follows that

$$\begin{aligned}
\left|\int_{dm+k}^{(dm+k)^2} \frac{P_{2r+1}(x)}{x^{2r+2}} dx\right| &\leq \frac{3(2r)!}{(2\pi)^{2r+1}(dm+k)^{2r+1}} \leq \frac{3(2r)!}{(2\pi)^{2r+1}(dm)^{2r+1}} \\
&= \frac{3(2dm)!}{(2\pi)^{2dm+1}(dm)^{2dm+1}} \\
&\leq \frac{3\sqrt{\pi(2dm+1)}}{(2\pi\, dm)^{2dm+1}} \cdot \left(\frac{2dm}{e}\right)^{2dm} \\
&\leq \frac{3\sqrt{3\pi dm}}{(2\pi dm)(\pi e)^{2dm}},
\end{aligned} \tag{3.13}$$

and similarly we have

$$\left|\int_{b(dm+k)}^{a(dm+k)} \frac{P_{2r+1}(x)}{x^{2r+2}}\, dx\right| \leq \frac{3\sqrt{3\pi dm}}{(2\pi dm)(\pi e)^{2dm}}, \quad (dm \geq a). \tag{3.14}$$

By using the definition of S_n in Theorem 2.4, the formula (1.1) for s_n, and the identities (3.8), (3.9), it follows that

$$\begin{aligned} S_n - \gamma - \log\frac{a}{b} &= (s_n - \gamma) + \left(s_n - s_n^2\right) + (s_{an} - s_{bn}) - \frac{1}{2n^2} + \sum_{j=1}^{r}\frac{B_{2j}}{2j}\left(\frac{1}{n^{2j}}\left(\frac{1}{a^{2j}} - \frac{1}{b^{2j}} + 1\right) - \frac{1}{n^{4j}}\right) \\ &= (s_n - \gamma) + \frac{1}{2n}\left(\frac{1}{b} - \frac{1}{a} - 1\right) + \int_n^{n^2}\frac{P_{2r+1}(x)}{x^{2r+2}}\, dx - \int_{bn}^{an}\frac{P_{2r+1}(x)}{x^{2r+2}}\, dx, \end{aligned} \tag{3.15}$$

where r is specified to $r = dm$ and n to $n = dm + k$. Moreover, we know from [4, Lemma 2] that

$$\sum_{k=0}^{dm}(-1)^{dm+k}g(k) = 1, \quad (m \geq 1). \tag{3.16}$$

By setting $n = dm + k$, the above formula for the series transformation of S_{dm+k} simplifies to

$$\begin{aligned} &\left|\sum_{k=0}^{dm}(-1)^{dm+k}g(k)S_{dm+k} - \gamma - \log\frac{a}{b}\right| \\ &\quad= \left|\sum_{k=0}^{dm}(-1)^{dm+k}g(k)(s_{dm+k} - \gamma) + \frac{1}{2}\left(\frac{1}{b} - \frac{1}{a} - 1\right)\sum_{k=0}^{dm}\frac{(-1)^{dm+k}g(k)}{dm+k}\right. \\ &\quad\quad + \sum_{k=0}^{dm}(-1)^{dm+k}g(k)\int_{dm+k}^{(dm+k)^2}\frac{P_{2r+1}(x)}{x^{2r+2}}\, dx \\ &\quad\quad \left. - \sum_{k=0}^{dm}(-1)^{dm+k}g(k)\int_{b(dm+k)}^{a(dm+k)}\frac{P_{2r+1}(x)}{x^{2r+2}}\, dx\right| \\ &\quad\leq \left|\sum_{k=0}^{dm}(-1)^{dm+k}g(k)(s_{dm+k} - \gamma)\right| + \sum_{k=0}^{dm}g(k)\left|\int_{dm+k}^{(dm+k)^2}\frac{P_{2r+1}(x)}{x^{2r+2}}\, dx\right| \\ &\quad\quad + \sum_{k=0}^{dm}g(k)\left|\int_{b(dm+k)}^{a(dm+k)}\frac{P_{2r+1}(x)}{x^{2r+2}}\, dx\right| \\ &\quad< C_d\cdot\left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^{m-2} + \sum_{k=0}^{dm}g(k)\frac{3\sqrt{3\pi dm}}{\pi dm(\pi e)^{2dm}}, \end{aligned} \tag{3.17}$$

where $dm \geq a$, $m \geq 2$, and $d \geq 3$. Here, we have used the results from Corollary 1.7, (3.13), and (3.14). The sum

$$\sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{dm+k} \tag{3.18}$$

vanishes, since for every real number $x > -dm$ we have

$$\sum_{k=0}^{dm} \frac{(-1)^{dm+k} \binom{(2d-1)m+k-1}{dm} \binom{dm}{k}}{dm+k+x} = \frac{(1-(d-1)m+x)\cdots(m+x)}{(dm+x)_{dm+1}}, \tag{3.19}$$

where on the right-hand side for an integer x with $-m \leq x \leq (d-1)m-1$ one term in the numerator equals to zero.

The inequality

$$\left(\frac{64}{(\pi e)^2} \right)^d < \frac{(1-1/d)^d}{2(d-1)} \tag{3.20}$$

holds for all integers $d \geq 42$. Now, using Lemma 3.1, we estimate the right-hand side in (3.17) for $dm \geq a$ and $d \geq 42$ as follows:

$$\begin{aligned}
&\left| \sum_{k=0}^{dm} (-1)^{dm+k} g(k) S_{dm+k} - \gamma - \log \frac{a}{b} \right| \\
&\quad < C_d \cdot \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m-2} + \sum_{k=0}^{dm} \frac{3\sqrt{3\pi dm}}{\pi dm} \frac{16^{dm}}{(\pi e)^{2dm}} \\
&\quad = C_d \cdot \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m-2} + \frac{3(dm+1)\sqrt{3dm}}{dm\sqrt{\pi}} \frac{1}{4^{dm}} \left(\frac{64}{(\pi e)^2} \right)^{dm} \\
&\quad \overset{(3.20)}{<} C_d \cdot \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m-2} + \frac{3(dm+1)\sqrt{3dm}}{dm 2^m \sqrt{\pi}} \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m} \\
&\quad \leq C_d \cdot \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m-2} + \frac{85}{28} \sqrt{\frac{3d}{\pi}} \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m} \leq c_4 \left(\frac{(1-1/d)^d}{(d-1)4^d} \right)^{m}.
\end{aligned} \tag{3.21}$$

The last but one estimate holds for all integers $m \geq 2$, $d \geq 42$, and c_4 is a suitable positive real constant depending on d. This completes the proof of Theorem 2.4.

4. On the Denominators of S_n

In this section we will investigate the size of the denominators b_m of our series transformations

$$\frac{a_m}{b_m} = \sum_{k=0}^{dm} (-1)^{dm+k} g(k) S_{k+dm}, \tag{4.1}$$

for m tending to infinity, where $a_m \in \mathbb{Z}$ and $b_m \in \mathbb{N}$ are coprime integers.

Theorem 4.1. *For every $m \geq 1$ there is an integer Z_m with $Z_m > 0$, $b_m | Z_m$, and*

$$\log Z_m \sim 12 d^2 m^2, \quad (m \longrightarrow \infty). \tag{4.2}$$

Proof. We will need some basic facts on the arithmetical functions $\vartheta(x)$ and $\psi(x)$. Let

$$\begin{aligned} \vartheta(x) &= \sum_{p \leq x} \log p, \quad (x > 1), \\ \psi(x) &= \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p, \quad (x > 1), \end{aligned} \tag{4.3}$$

where p is restricted on primes. Moreover, let $D_n :=$ l.c.m $(1, 2, \ldots, n)$ for positive integers n. Then,

$$\psi(n) = \log D_n, \quad (n \geq 1), \tag{4.4}$$

$$\psi(x) \sim \vartheta(x) \sim \pi(x) \log x \sim x, \quad (x \longrightarrow \infty), \tag{4.5}$$

where (4.5) follows from [5, Theorem 420] and the prime number theorem. By [5, Theorem 118] (von Staudt's theorem) we know how to obtain the prime divisors of the denominators of Bernoulli numbers B_{2k}: The denominators of B_{2k} are squarefree, and they are divisible exactly by those primes p with $(p-1) \mid 2k$. Hence,

$$B_{2k} \prod_{p \leq 2k+1} p \in \mathbb{Z}, \quad (k = 1, 2, \ldots). \tag{4.6}$$

Next, let $\max\{a, b\} \leq dm \leq n \leq 2dm$ ($n = k + dm$ are the subscripts of S_{k+dm} in Theorem 2.4). First, we consider the following terms from the series transformation in S_m:

$$\sum_{j=1}^{an-1} \frac{1}{j} - \sum_{j=1}^{bn-1} \frac{1}{j} + 2 \sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{n^2-1} \frac{1}{j} =: \sum_{j=1}^{n^2-1} \frac{e_j}{j}, \tag{4.7}$$

with

$$e_j := \begin{cases} 1, & \text{if } 1 \le j \le n-1 \\ -1, & \text{if } n \le j \le bn-1 \\ 0, & \text{if } bn \le j \le an-1 \\ -1, & \text{if } an \le j \le n^2-1 \end{cases} \quad (a \ge b),$$
$$e_j := \begin{cases} 1, & \text{if } 1 \le j \le n-1 \\ -1, & \text{if } n \le j \le an-1 \\ -2, & \text{if } an \le j \le bn-1 \\ -1, & \text{if } bn \le j \le n^2-1. \end{cases} \quad (a < b). \tag{4.8}$$

For every $m \ge 1$ there is a rational x_m/y_m defined by

$$\frac{x_m}{y_m} = \sum_{k=0}^{dm} (-1)^{dm+k} g(k) \sum_{j=1}^{(k+dm)^2-1} \frac{e_j}{j}, \tag{4.9}$$

where $x_m \in \mathbb{Z}$, $y_m \in \mathbb{N}$, $(x_m, y_m) = 1$, and

$$y_m \mid Y_m := D_{4d^2m^2}, \quad (dm \ge \max\{a,b\}). \tag{4.10}$$

Similarly, we define rationals u_m/v_m by

$$\frac{u_m}{v_m} = \sum_{k=0}^{dm} (-1)^{dm+k} g(k) \times \left(-\frac{1}{2(k+dm)^2} + \sum_{j=1}^{dm} \frac{B_{2j}}{2j} \left(\frac{1}{(k+dm)^{2j}} \left(\frac{1}{a^{2j}} - \frac{1}{b^{2j}} + 1 \right) - \frac{1}{(k+dm)^{4j}} \right) \right), \tag{4.11}$$

where $u_m \in \mathbb{Z}$, $v_m \in \mathbb{N}$ and $(u_m, v_m) = 1$. We have

$$(k+dm)^{2j} \mid (k+dm)^{4dm}, \quad (0 \le k \le dm, \ 1 \le j \le dm). \tag{4.12}$$

Therefore, using the conclusion (4.6) from von Staudt's theorem, we get

$$v_m \mid V_m := 2(ab)^{2dm} D_{dm} \left(\prod_{p \le 2dm+1} p \right) (D_{2dm})^{4dm}, \quad (dm \ge \max\{a,b\}). \tag{4.13}$$

Note that $D_{2dm} =$ l.c.m. $(dm, \ldots, 2dm)$, since every integer n_1 with $1 \le n_1 < dm$ divides at least one integer n_2 with $dm \le n_2 \le 2dm$.

From (4.10) and (4.13) we conclude on

$$b_m \mid Z_m := 2(ab)^{2dm} D_{dm} D_{4d^2m^2} (D_{2dm})^{4dm} \left(\prod_{p \le 2dm+1} p \right). \tag{4.14}$$

Hence we have from (4.4) and (4.5) that

$$\begin{aligned} \log Z_m &= \log 2 + 2dm \log(ab) + \psi(dm) + \psi\left(4d^2m^2\right) + 4dm\psi(2dm) + \vartheta(2dm+1) \\ &\sim \log 2 + 2dm \log(ab) + dm + 4d^2m^2 + 8d^2m^2 + (2dm+1) \\ &= 1 + \log 2 + (3 + 2\log(ab))dm + 12d^2m^2 \\ &\sim 12d^2m^2 \quad (m \longrightarrow \infty). \end{aligned} \tag{4.15}$$

The theorem is proved. □

Remark 4.2. On the one side we have shown that $\log Y_m \sim 4d^2m^2$ and $\log V_m \sim 8d^2m^2$. On the other side, every prime p dividing V_m satisfies $p \le \max\{a, b, dm, 2dm+1, 2dm\} = 2dm+1$ and therefore p divides $Y_m = D_{4d^2m^2}$. Conversely, all primes p with $2dm + 1 < p < 4d^2m^2$ divide Y_m, but not V_m. That means: V_m is much bigger than Y_m, but V_m is formed by powers of small primes, whereas Y_m is divisible by many big primes.

5. Simplification of the Transformed Series

Let

$$R_n := -\frac{1}{2n^2} + \sum_{j=1}^{dm} \frac{B_{2j}}{2j} \left(\frac{1}{n^{2j}} \left(\frac{1}{a^{2j}} - \frac{1}{b^{2j}} + 1 \right) - \frac{1}{n^{4j}} \right), \tag{5.1}$$

such that

$$S_n = \sum_{j=1}^{an-1} \frac{1}{j} - \sum_{j=1}^{bn-1} \frac{1}{j} + 2\sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{n^2-1} \frac{1}{j} + R_n. \tag{5.2}$$

In Theorem 2.4 the sequence S_n is transformed. In view of a simplified process we now investigate the transformation of the series $S_n - R_n$. Therefore we have to estimate the contribution of R_{k+dm} to the series transformation in Theorem 2.4. For this purpose, we define

$$\begin{aligned} E_m &:= \sum_{k=0}^{dm} (-1)^{dm+k} \binom{(2d-1)m+k-1}{dm} \binom{dm}{k} R_{k+dm} = -\frac{1}{2} \sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^2} \\ &+ \sum_{j=1}^{dm} \frac{B_{2j}}{2j} \left(\left(\frac{1}{a^{2j}} - \frac{1}{b^{2j}} + 1 \right) \sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{2j}} - \sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{4j}} \right). \end{aligned} \tag{5.3}$$

A major step in estimating E_m is to express the sums on the right-hand side by integrals.

Lemma 5.1. *For positive integers d, j and m one has*

$$\sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{2j}} = -\frac{(-1)^{dm}}{(dm)!(2j-1)!} \int_0^1 u^m (\log u)^{2j-1} \frac{\partial^{dm}}{\partial u^{dm}} \left(u^{(2d-1)m-1} (1-u)^{dm} \right) du. \tag{5.4}$$

Proof. For integers k, r and a real number ρ with $k + \rho > 0$ the identity

$$\frac{1}{(k+\rho)^r} = \frac{1}{(r-1)!} \int_0^\infty e^{-(k+\rho)t} t^{r-1} \, dt \tag{5.5}$$

holds, which we apply with $r = 2j$ and $\rho = dm$ to substitute the fraction $1/(dm+k)^{2j}$. Introducing the new variable $u := e^{-t}$, we then get

$$\begin{aligned} \sum_{k=0}^{2m} \frac{(-1)^{dm+k} g(k)}{(k+dm)^{2j}} &= -\frac{(-1)^{dm}}{(2j-1)!} \sum_{k=0}^{dm} (-1)^k g(k) \int_0^1 u^{k+dm-1} (\log u)^{2j-1} \, du \\ &= -\frac{(-1)^{dm}}{(2j-1)!} \int_0^1 \left(\sum_{k=0}^{dm} (-1)^k g(k) u^{k+(d-1)m-1} \right) u^m (\log u)^{2j-1} \, du. \end{aligned} \tag{5.6}$$

The sum inside the brackets of the integrand can be expressed by using the equation

$$\sum_{k=0}^{n} (-1)^k \binom{n+\tau+k}{n} \binom{n}{k} u^{\tau+k} = \frac{\partial^n}{\partial u^n} \left(\frac{u^{n+\tau} (1-u)^n}{n!} \right), \quad (n, \tau \in \mathbb{N} \cup \{0\}), \tag{5.7}$$

in which we put $n = dm$ and $\tau = (d-1)m - 1$. This gives the identity stated in the lemma. □

The following result deals with the case $j = 1$, in which we express the finite sum by a double integral on a rational function.

Corollary 5.2. *For every positive integer m one has*

$$\sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^2} = (-1)^{(d-1)m} \int_0^1 \int_0^1 \frac{(1-u)^{dm} (1-w)^m u^{(2d-1)m-1} w^{(d-1)m-1}}{(1-(1-u)w)^{dm+1}} \, du \, dw. \tag{5.8}$$

Proof. Set $j = 1$ in Lemma 5.1, and note that

$$\log u = -(1-u) \int_0^1 \frac{dw}{1-(1-u)w}. \tag{5.9}$$

Hence,

$$\sum_{k=0}^{dm}\frac{(-1)^{dm+k}g(k)}{(dm+k)^2} = -\frac{(-1)^{dm}}{(dm)!}\int_0^1 u^m \log u \,\frac{\partial^{dm}}{\partial u^{dm}}\left(u^{(2d-1)m-1}(1-u)^{dm}\right) du$$
$$= \frac{(-1)^{dm}}{(dm)!}\int_0^1\int_0^1 \frac{(1-u)u^m}{1-(1-u)w}\,\frac{\partial^{dm}}{\partial u^{dm}}\left(u^{(2d-1)m-1}(1-u)^{dm}\right) du\, dw. \tag{5.10}$$

Let s be any positive integer. Then we have the following decomposition of a rational function, in which u is considered as variable and w as parameter:

$$\frac{u^s}{1-(1-u)w} = \sum_{\nu=0}^{s-1}\frac{(w-1)^\nu}{w^{\nu+1}}\,u^{s-\nu-1} + \left(\frac{w-1}{w}\right)^s \frac{1}{1-(1-u)w}. \tag{5.11}$$

We additionally assume that $s-1 < dm$. Then, differentiating this identity dm-times with respect to u, the polynomial in u on the right-hand side vanishes identically:

$$\frac{\partial^{dm}}{\partial u^{dm}}\left(\frac{u^s}{1-(1-u)w}\right) = \left(\frac{w-1}{w}\right)^s \frac{(-1)^{dm}(dm)!\, w^{dm}}{(1-(1-u)w)^{dm+1}}. \tag{5.12}$$

Therefore, we get from (5.10) by iterated integrations by parts:

$$\sum_{k=0}^{dm}\frac{(-1)^{dm+k}g(k)}{(dm+k)^2} = \frac{1}{(dm)!}\int_0^1\int_0^1 u^{(2d-1)m-1}(1-u)^{dm}\frac{\partial^{dm}}{\partial u^{dm}}\left(\frac{u^m-u^{m+1}}{1-(1-u)w}\right) du\, dw$$
$$= \frac{1}{(dm)!}\int_0^1\int_0^1 u^{(2d-1)m-1}(1-u)^{dm}\left(\left(\frac{w-1}{w}\right)^m - \left(\frac{w-1}{w}\right)^{m+1}\right)$$
$$\times \frac{(-1)^{dm}(dm)!\, w^{dm}\, du\, dw}{(1-(1-u)w)^{dm+1}}. \tag{5.13}$$

The corollary is proved by noting that

$$\left(\frac{w-1}{w}\right)^m - \left(\frac{w-1}{w}\right)^{m+1} = (-1)^m \frac{(1-w)^m}{w^{m+1}}. \tag{5.14}$$

□

6. Estimating E_m

In this section we estimate E_m defined in (5.3). Substituting $1 - u$ for u into the integral in Lemma 5.1 and applying iterated integration by parts, we get

$$\sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{2j}} = -\frac{(-1)^{dm}}{(dm)!(2j-1)!} \int_0^1 \frac{\partial^{dm}}{\partial u^{dm}} \left((1-u)^m (\log(1-u))^{2j-1}\right)\left((1-u)^{(2d-1)m-1} u^{dm}\right) du. \tag{6.1}$$

Set

$$f(u) := (1-u)^m (\log(1-u))^{2j-1}, \tag{6.2}$$

where m and j are kept fixed. We have $f(0) = 0$. For an integer $k > 0$ we use Cauchy's formula

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{k+1}}\, dz \tag{6.3}$$

to estimate $|f^{(k)}(0)|$. Let C denote the circle in the complex plane centered around 0 with radius $R := 1 - 1/2k$. With $a = 0$ and $f(z)$ defined above, Cauchy's formula yields the identity

$$f^{(k)}(0) = \frac{k!}{2\pi R^k} \int_{-\pi}^{\pi} e^{-ik\phi} \left(1 - Re^{i\phi}\right)^m \log^{2j-1}\left(1 - Re^{i\phi}\right) d\phi. \tag{6.4}$$

For the complex logarithm function occurring in (6.4) we cut the complex plane along the negative real axis and exclude the origin by a small circle. All arguments ϕ of a complex number $z \notin (-\infty, 0]$ are taken from the interval $(-\pi, \pi)$. Therefore, using $1 - Re^{i\phi} = 1 - R\cos\phi - iR\sin\phi$, we get

$$\begin{aligned} \left|1 - Re^{i\phi}\right| &= \sqrt{1 + R^2 - 2R\cos\phi} =: \sqrt{A(R,\phi)}, \\ \arg\left(1 - Re^{i\phi}\right) &= -\arctan\left(\frac{R\sin\phi}{1 - R\cos\phi}\right). \end{aligned} \tag{6.5}$$

Hence,

$$\begin{aligned} \log\left(1 - Re^{i\phi}\right) &= \ln\sqrt{1 + R^2 - 2R\cos\phi} - i\arctan\left(\frac{R\sin\phi}{1 - R\cos\phi}\right) \\ &= \frac{1}{2}\ln(A(R,\phi)) - i\arctan\left(\frac{R\sin\phi}{1 - R\cos\phi}\right). \end{aligned} \tag{6.6}$$

Thus, it follows from (6.4) that

$$\begin{aligned}\left|f^{(k)}(0)\right| &\leq \frac{k!}{2\pi R^k}\int_{-\pi}^{\pi}\left|1-Re^{i\phi}\right|^m\cdot\left|\log\left(1-Re^{i\phi}\right)\right|^{2j-1}\,d\phi \\ &= \frac{k!}{2\pi R^k}\int_{-\pi}^{\pi}(A(R,\phi))^{m/2}\left(\frac{1}{4}\ln^2(A(R,\phi))+\arctan^2\left(\frac{R\sin\phi}{1-R\cos\phi}\right)\right)^{(2j-1)/2}d\phi \\ &= \frac{k!}{\pi R^k}\int_{0}^{\pi}(A(R,\phi))^{m/2}\left(\frac{1}{4}\ln^2(A(R,\phi))+\arctan^2\left(\frac{R\sin\phi}{1-R\cos\phi}\right)\right)^{(2j-1)/2}d\phi.\end{aligned} \tag{6.7}$$

From $0<R<1$ we conclude on

$$\begin{gathered}0<(1-R)^2=1+R^2-2R\leq 1+R^2-2R\cos\phi=A(R,\phi)<4,\quad (0\leq\phi\leq\pi), \\ 0\leq\frac{R\sin\phi}{1-R\cos\phi}\leq\frac{\sin\phi}{1-\cos\phi},\quad (0<\phi\leq\pi).\end{gathered} \tag{6.8}$$

Since arctan is a strictly increasing function, we get

$$\begin{aligned}\arctan\left(\frac{R\sin\phi}{1-R\cos\phi}\right)\leq\arctan\left(\frac{\sin\phi}{1-\cos\phi}\right) &= \arctan\cot\left(\frac{\phi}{2}\right) \\ &= \arctan\left(\tan\left(\frac{\pi-\phi}{2}\right)\right) \\ &= \frac{\pi-\phi}{2},\quad (0<\phi\leq\pi).\end{aligned} \tag{6.9}$$

For $0<R<1$, this upper bound also holds for $\phi=0$. Finally, we note that $R^k=(1-1/2k)^k\geq 1/2$. Altogether, we conclude from (6.7) on

$$\begin{aligned}\left|f^{(k)}(0)\right| &\leq \frac{k!}{\pi R^k}\int_0^{\pi}4^{m/2}\left(\frac{\ln^2 4}{4}+\arctan^2\left(\frac{\sin\phi}{1-\cos\phi}\right)\right)^{(2j-1)/2}d\phi \\ &\leq \frac{2^{m+1}k!}{\pi}\int_0^{\pi}\left(\ln^2 2+\left(\frac{\pi-\phi}{2}\right)^2\right)^{(2j-1)/2}d\phi \\ &\leq \frac{2^{m+1}k!}{\pi}\int_0^{\pi}\left(\ln^2 2+\frac{\pi^2}{4}\right)^{j-1/2}d\phi \\ &\leq \frac{2^{m+1}k!}{\pi}\int_0^{\pi}3^{j-1/2}\,d\phi\;\leq\;2^{m+1}3^j k!.\end{aligned} \tag{6.10}$$

It follows that the Taylor series expansion of $f(u)$,

$$f(u) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, u^k, \tag{6.11}$$

converges at least for $-1 < u < 1$. Then,

$$f^{(dm)}(u) = \sum_{k=dm}^{\infty} \frac{f^{(k)}(0)}{(k-dm)!}\, u^{k-dm} = \sum_{k=0}^{\infty} \frac{f^{(k+dm)}(0)}{k!}\, u^k, \tag{6.12}$$

and the estimate given by (6.10) implies for $0 < u < 1$ that

$$\begin{aligned} \left| f^{(dm)}(u) \right| &\le \sum_{k=0}^{\infty} \frac{\left| f^{(k+dm)}(0) \right|}{k!}\, u^k \ \le\ 2^{m+1} 3^j \sum_{k=0}^{\infty} \frac{(k+dm)!}{k!}\, u^k \\ &= 2^{m+1} 3^j (dm)! \sum_{k=0}^{\infty} \binom{k+dm}{k} u^k = \frac{2^{m+1} 3^j (dm)!}{(1-u)^{dm+1}}. \end{aligned} \tag{6.13}$$

Combining (6.13) with the result from (6.1), we get for $m > 1$

$$\begin{aligned} \left| \sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{2j}} \right| &\le \frac{2^{m+1} 3^j}{(2j-1)!} \int_0^1 (1-u)^{(d-1)m-2} u^{dm}\, du \\ &= \frac{2^{m+1} 3^j}{(2j-1)!} \, \frac{\Gamma(dm+1)\Gamma((d-1)m-1)}{\Gamma((2d-1)m)} \\ &= \frac{2d-1}{d-1} \cdot \frac{2^{m+1} 3^j}{(2j-1)!((d-1)m-1)} \cdot \frac{1}{\binom{(2d-1)m}{dm}}. \end{aligned} \tag{6.14}$$

We estimate the binomial coefficient by Stirling's formula (3.12). For this purpose we additionally assume that $m \ge 2d-1$:

$$\begin{aligned} \binom{(2d-1)m}{dm} &\ge \sqrt{\frac{(2d-1)m}{2\pi(dm+1)((d-1)m+1)}} \left(\frac{(2d-1)^{2d-1}}{d^d (d-1)^{d-1}} \right)^m \\ &\ge \sqrt{\frac{2d-1}{2\pi d^2 m}} \left(\frac{(2d-1)^{2d-1}}{d^d (d-1)^{d-1}} \right)^m. \end{aligned} \tag{6.15}$$

We now assume $m \ge 2d-1$ and substitute the above inequality into (6.14):

$$\left| \sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{2j}} \right| \le \frac{d(2d-1) 2^{m+1} 3^j \sqrt{2\pi m}}{(2j-1)!(d-1)((d-1)m-1)\sqrt{2d-1}} \left(\frac{d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m. \tag{6.16}$$

For all integers $m \geq 1$ and $d \geq 1$ we have

$$\frac{(2d-1)\sqrt{2\pi m}}{\sqrt{2d-1}} < 2\sqrt{\pi d m}, \qquad (d-1)((d-1)m-1) \geq (d-1)(d-2)m. \tag{6.17}$$

Thus we have proven the following result.

Lemma 6.1. *For all integers d, m with $d \geq 3$ and $m \geq 2d-1$ one has*

$$\left| \sum_{k=0}^{dm} \frac{(-1)^{dm+k} g(k)}{(dm+k)^{2j}} \right| < \frac{2^{m+2} 3^j \sqrt{\pi d^3}}{(2j-1)!(d-1)(d-2)\sqrt{m}} \left(\frac{d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m. \tag{6.18}$$

Next, we need an upper bound for the Bernoulli numbers B_{2j} (cf. [9, 23.1.15]):

$$\left|B_{2j}\right| \leq \frac{2(2j)!}{(2\pi)^{2j}\left(1-2^{1-2j}\right)} \leq \frac{4(2j)!}{(2\pi)^{2j}}, \quad (j \geq 1). \tag{6.19}$$

Let $d \geq 3$ and $m \geq \max\{2d-1, a/2\}$. Using this and Lemma 6.1, we estimate E_m in (5.3):

$$\begin{aligned}
|E_m| &< \frac{3}{2} \frac{\sqrt{\pi d^3}\, 2^{m+2}}{(d-1)(d-2)\sqrt{m}} \left(\frac{d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m + \frac{\sqrt{\pi d^3}\, 2^{m+2}}{(d-1)(d-2)\sqrt{m}} \left(\frac{d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m \\
&\quad \times \sum_{j=1}^{dm} \frac{B_{2j}}{2j} \left(\left| \frac{1}{a^{2j}} - \frac{1}{b^{2j}} + 1 \right| \frac{3^j}{(2j-1)!} + \frac{3^{2j}}{(4j-1)!} \right) \\
&\leq \frac{\sqrt{\pi d^3}\, 2^{m+2}}{(d-1)(d-2)\sqrt{m}} \left(\frac{d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m \\
&\quad \times \left(\frac{3}{2} + \sum_{j=1}^{dm} \frac{4(2j-1)!}{(2\pi)^{2j}} \left(\frac{2 \cdot 3^j}{(2j-1)!} + \frac{3^{2j}}{(4j-1)!} \right) \right) \\
&< \frac{4\sqrt{\pi d^3}}{(d-1)(d-2)\sqrt{m}} \left(\frac{2d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m \left(\frac{3}{2} + 8 \sum_{j=1}^{\infty} \left(\left(\frac{\sqrt{3}}{2\pi} \right)^{2j} + \left(\frac{3}{2\pi} \right)^{2j} \right) \right) \\
&< \frac{19\sqrt{\pi d^3}}{(d-1)(d-2)\sqrt{m}} \left(\frac{2d^d (d-1)^{d-1}}{(2d-1)^{2d-1}} \right)^m .
\end{aligned} \tag{6.20}$$

Now, let

$$T_n := \sum_{j=1}^{an-1}\frac{1}{j} - \sum_{j=1}^{bn-1}\frac{1}{j} + 2\sum_{j=1}^{n-1}\frac{1}{j} - \sum_{j=1}^{n^2-1}\frac{1}{j} = \sum_{j=1}^{n^2-1}\frac{e_j}{j}, \quad (n > 1), \tag{6.21}$$

with the numbers e_j introduced in the proof of Theorem 4.1. By definition of R_n and S_n we then have $T_n = S_n - R_n$, and therefore we can estimate the series transformation of T_n by applying the results from Theorem 2.4 and (6.20). Again, let $m \geq \max\{2d-1, a/2\}$ and $d \geq 42$.

$$\begin{aligned}
&\left|\sum_{k=0}^{dm}(-1)^{dm+k}\binom{(2d-1)m+k-1}{dm}\binom{dm}{k}T_{k+dm} - \gamma - \log\frac{a}{b}\right| \\
&\quad \leq \left|\sum_{k=0}^{dm}(-1)^{dm+k}g(k)S_{k+dm} - \gamma - \log\frac{a}{b}\right| + \left|\sum_{k=0}^{dm}(-1)^{dm+k}g(k)R_{k+dm}\right| \\
&\quad = \left|\sum_{k=0}^{dm}(-1)^{dm+k}g(k)S_{k+dm} - \gamma - \log\frac{a}{b}\right| + |E_m| \\
&\quad < c_4 \cdot \left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^m + \frac{19\sqrt{\pi d^3}}{(d-1)(d-2)\sqrt{m}}\left(\frac{2d^d(d-1)^{d-1}}{(2d-1)^{2d-1}}\right)^m.
\end{aligned} \tag{6.22}$$

By similar arguments we get the same bound when $b > a$. For $d \geq 3$ it can easily be seen that

$$\frac{2d^d(d-1)^{d-1}}{(2d-1)^{2d-1}} = \frac{2(2d-1)}{d-1}\cdot\frac{(1-1/d)^d}{(1-1/2d)^{2d}}\cdot\frac{1}{4^d} < \frac{18}{4^{d+1}}. \tag{6.23}$$

Thus, we finally have proven the following theorem.

Theorem 6.2. *Let*

$$T_n := \sum_{j=1}^{an-1}\frac{1}{j} - \sum_{j=1}^{bn-1}\frac{1}{j} + 2\sum_{j=1}^{n-1}\frac{1}{j} - \sum_{j=1}^{n^2-1}\frac{1}{j}, \quad (n > 1), \tag{6.24}$$

where a, b are positive integers. Let $d \geq 42$ be an integer. Then, there is a positive constant c_5 depending at most on a, b and d such that

$$\left|\sum_{k=0}^{dm}(-1)^{k}\binom{(2d-1)m+k-1}{dm}\binom{dm}{k}T_{k+dm} - \gamma - \log\frac{a}{b}\right| < \frac{c_5}{\sqrt{m}}\left(\frac{18}{4^{d+1}}\right)^m, \quad (m \geq 1). \tag{6.25}$$

7. Concluding Remarks

It seems that in Theorem 6.2 a smaller bound holds.

Conjecture 7.1. *Let* a, b *be positive integers. Let* $d \ge 2$ *be an integer. Then there is a positive constant* c_6 *depending at most on* a, b *and* d *such that for all integers* $m \ge 1$ *one has*

$$\left|\sum_{k=0}^{dm}(-1)^{dm+k}\binom{(2d-1)m+k-1}{dm}\binom{dm}{k}T_{k+dm}-\gamma-\log\frac{a}{b}\right| < c_6\cdot\left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^m. \tag{7.1}$$

A proof of this conjecture would be implied by suitable bounds for the integral stated in Lemma 5.1. For $j=1$ such a bound follows from the double integral given in Corollary 5.2:

$$\begin{aligned}\left|\sum_{k=0}^{dm}\frac{(-1)^{dm+k}g(k)}{(dm+k)^2}\right| &= \int_0^1\int_0^1\frac{(1-u)^{dm}(1-w)^m u^{(2d-1)m-1}w^{(d-1)m-1}}{(1-(1-u)w)^{dm+1}}du\,dw\\ &= \int_0^1\int_0^1\frac{(1-u)^2(1-w)u^2}{1-(1-u)w^3}\left(\frac{(1-u)u^2w}{1-(1-u)w}\right)^{d-2}\\ &\quad\times\left(\frac{(1-u)^d(1-w)u^{2d-1}w^{d-1}}{(1-(1-u)w)^d}\right)^{m-1}du\,dw\\ &\le \frac{1}{4^{d-2}}\left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^{m-1}\int_0^1\int_0^1\frac{(1-u)^2(1-w)u^2}{1-(1-u)w^3}du\,dw\\ &= \frac{2(d-1)}{3(1-1/d)^d}\cdot\left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^m,\quad (m\ge 1),\end{aligned} \tag{7.2}$$

where the double integral in the last but one line equals to $1/24$.

Note that the rational functions

$$\frac{(1-u)u^2w}{1-(1-u)w},\qquad \frac{(1-u)^d(1-w)u^{2d-1}w^{d-1}}{(1-(1-u)w)^d}, \tag{7.3}$$

take their maximum values 4^{2-d} and $(1-1/d)^d/((d-1)4^d)$ inside the unit square $[0,1]\times[0,1]$ at $(u,w)=(1/2,1)$ and $(u,w)=(1/2,(2d-2)/(2d-1))$, respectively. Finally, we compare the bound for the series transformation given by Theorem 2.4 with the bound proven for Theorem 6.2. In Theorem 2.4 the bound is

$$T_1(d,m) := c_4\cdot\left(\frac{(1-1/d)^d}{(d-1)4^d}\right)^m,\quad (d\ge 42,\quad m\ge 1), \tag{7.4}$$

whereas we have in Theorem 6.2 that

$$T_2(d,m) := \frac{c_5}{\sqrt{m}}\left(\frac{18}{4^{d+1}}\right)^m, \quad (d \geq 42,\ m \geq 1). \tag{7.5}$$

For fixed $d \geq 42$ and sufficiently large m it is clear on the one hand that $T_1(d,m) < T_2(d,m)$, but on the other hand we have

$$\lim_{m\to\infty}\lim_{d\to\infty}\frac{\log T_1(d,m)}{\log T_2(d,m)} = 1 = \lim_{d\to\infty}\lim_{m\to\infty}\frac{\log T_1(d,m)}{\log T_2(d,m)}. \tag{7.6}$$

Conversely, for d tending to infinity, one gets

$$-\log|T_1(d,m)| \gg dm\log 4, \qquad -\log|T_2(d,m)| \gg dm\log 4, \tag{7.7}$$

with implicit constants depending at most on m. For the denominators b_m of the transformed series S_{k+dm} in Theorem 2.4 we have the bound $\log b_m \ll d^2m^2$ from Theorem 4.1, and a similar inequality holds for the denominators of the transformed series T_{k+dm} in Theorem 6.2.

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