Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 603041, 6 pages
doi:10.1155/2009/603041

*Research Article*

# A Note on the Range of the Operator $X \mapsto TX - XT$ Defined on $\mathcal{C}_2(\mathcal{H})$

**Vasile Lauric**

*Department of Mathematics, Florida A&M University, Tallahassee, FL 32307, USA*

Correspondence should be addressed to Vasile Lauric, vasile.lauric@famu.edu

Received 10 November 2008; Accepted 11 May 2009

Recommended by Manfred H. Moller

We show how a proof of J. Stampfli can be extended to prove that the operator $X \mapsto TX - XT$ defined on the Hilbert-Schmidt class, when $T$ is an $M$-hyponormal, $p$-hyponormal, or log-hyponormal operator, has a closed range if and only if $\sigma(T)$ is finite.



## 1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$. The Hilbert-Schmidt class, denoted by $\mathcal{C}_2(\mathcal{H})$, is a Hilbert space with the $\|\cdot\|_2$-norm that arises from the inner product $\langle X, Y\rangle = \operatorname{tr}(XY^*)$, where tr is the scalar-valued trace. For $T \in \mathcal{L}(\mathcal{H})$, define $\Delta_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $\Delta_T(X) = TX - XT$, and let $\sigma(T)$ denote the spectrum of $T$. Let the range of a linear operator $S$ be denoted by $\mathcal{R}(S)$. For a normal operator $N \in \mathcal{L}(\mathcal{H})$, Anderson and Foiaş [1] proved that $\mathcal{R}(\Delta_N)$ is norm closed if and only if $\sigma(N)$ is a finite set. In [2], Stampfli extended this result to the class of hyponormal operators.

**Theorem A** ([2]). *Let $T \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator. Then $\mathcal{R}(\Delta_T)$ is norm closed if and only if $\sigma(T)$ is finite.*

In fact, Stampfli provided a proof of the "only if" implication which extends to a larger class of operators than the class of hyponormal operators (see Proposition 2.2). For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_{\text{nap}}(T)$ denote its *normal approximate point spectrum*, that is, the set of those $\lambda \in \mathbb{C}$ for which there exists an orthonormal sequence $\{\phi_n\}_n$ in $\mathcal{H}$ such that

$$\left\|(T-\lambda)\phi_n\right\| + \left\|(T-\lambda)^*\phi_n\right\| \longrightarrow 0. \tag{1.1}$$

Define the class $\mathcal{G}(\mathcal{H})$ as follows:

$$\mathcal{G}(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) \mid \sigma_{\text{nap}}(T) \text{ is an infinite set}\}. \tag{1.2}$$

Some classes of hyponormal related operators, such as $M$-hyponormal operators, that is,

$$m \cdot \|(T-\lambda)^*\phi\| \leq \|(T-\lambda)\phi\|, \quad (\forall)\phi \in \mathcal{H},\ (\forall)\lambda \in \mathbb{C},\ \text{for some} \ \ m > 0, \tag{1.3}$$

$p$-hyponormal operators, that is, $(T^*T)^p \geq (TT^*)^p$ for some $p > 0$, or log-hyponormal operators, that is, invertible operators such that $\log(T^*T) \geq \log(TT^*)$, have spectrum that is finite or they belong to $\mathcal{G}(\mathcal{H})$. Particularly, the hyponormal operators (i.e., 1-hyponormal) have this property.

In [3] Stampfli proved the following lemma which will be used in Section 2.

**Lemma B.** *Let $T \in \mathcal{G}(\mathcal{H})$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of distinct points of $\sigma_{nap}(T)$. Then for any sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers converging to zero, there exists an orthonormal sequence $\{\phi_n\}_{n=1}^{\infty}$ of vectors in $\mathcal{H}$ such that*

$$\|(T-\lambda_n)\phi_n\| + \|(T-\lambda_n)^*\phi_n\| < \varepsilon_n \quad \textit{for } n = 1, 2, \ldots, \tag{1.4}$$

$$\langle \phi_n, T\phi_k \rangle = 0 \quad \textit{for } k = 1, \ldots, n-1. \tag{1.5}$$

## 2. The Closedness of the Range of $\Delta_T^{(2)}$

The operator $\Delta_T$ defined on the Hilbert-Schmidt class will be denoted in the remainder of this note by $\Delta_T^{(2)}$, that is, $\Delta_T^{(2)} : \mathcal{C}_2(\mathcal{H}) \to \mathcal{C}_2(\mathcal{H})$, $\Delta_T^{(2)}(X) = TX - XT$. Let $H^M(\mathcal{H})$ denote the set of $M$-hyponormal operators.

**Proposition 2.1.** *Let $T \in H^M(\mathcal{H})$. If $\sigma(T)$ is finite, then $\mathcal{R}(\Delta_T^{(2)})$ is closed.*

*Proof.* It is well known that an operator $T \in H^M(\mathcal{H})$ with finite spectrum is normal. Indeed, for such an operator, the restriction to an invariant subspace $\mathcal{M}$ belongs to $H^M(\mathcal{M})$. On the other hand, if $T \in H^M(\mathcal{H})$ with $\sigma(T) = \{\lambda\}$, then $T = \lambda I$, (cf. [4]). Thus, we can write $T = \sum_{i=1}^{n_0} \lambda_i E_i$, where $E_i$'s are the spectral projections.

Let $X_n$ and $C$ be in $\mathcal{C}_2(\mathcal{H})$ such that $\|\Delta_T^{(2)}(X_n) - C\|_2 \to 0$. Therefore $\Delta_T(X_n) - C \to 0$ in the $\mathcal{L}(\mathcal{H})$ norm, and according to Theorem A, there exists $X^0 \in \mathcal{L}(\mathcal{H})$ such that $C = TX^0 - X^0T$. For an arbitrary $X \in \mathcal{L}(\mathcal{H})$, let $[X_{ij}]$ be the block-matrix representation of $X$ relative to the decomposition $\mathcal{H} = \sum_{i=1}^{n_0} \oplus E_i\mathcal{H}$. Thus

$$C_{ij} = (\lambda_i - \lambda_j) X_{ij}^0, \tag{2.1}$$

for all $i, j = 1, \ldots, n_0$. This implies that each $X_{ij}^0 = (1/(\lambda_i - \lambda_j)) C_{ij}$ is a Hilbert-Schmidt operator. Moreover $X_{ii}^0$ can be chosen 0, and thus $X^0 \in \mathcal{C}_2(\mathcal{H})$. □

**Proposition 2.2.** *Let $T \in \mathcal{G}(\mathcal{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is not closed.*

*Proof.* We will use same notation and circle of ideas as in [2]. Let $\{\lambda_n\}_{n\geq 1}$ be sequence of distinct points of $\sigma_{\text{nap}}(T)$ so that $\lambda_n \to \lambda_0$. Let

$$\eta_n = \max\left\{ \left|\lambda_{j+1} - \lambda_j\right|^{-1/2} \mid j = 1, \ldots, n \right\}, \tag{2.2}$$

and choose a nonincreasing sequence $\{\varepsilon_n\}_{n\geq 1}$ so that $0 < \varepsilon_n \leq |\lambda_{n+1} - \lambda_n|^2$, $n \geq 1$, and $\sum_{n\geq 1} \varepsilon_n^2 \eta_n^2 < \infty$. According to Lemma B, there exists an orthonormal sequence $\{\phi_n\}_{n\geq 1}$ that satisfies (1.4) and(1.5). Let $\mathcal{H}_1 = \vee\{\phi_n \mid n \geq 1\}$, $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$, and let $\delta_n$ such that

$$T\phi_n = \mu_n \phi_n + \delta_n, \quad \delta_n \perp \phi_n, \quad n \geq 1. \tag{2.3}$$

It results that

$$\left|\mu_n - \lambda_n\right| < \varepsilon_n, \quad \|\delta_n\| < 2\varepsilon_n, \quad n \geq 1. \tag{2.4}$$

Define $V : \mathcal{H} \to \mathcal{H}$ by $V\phi_n = |\lambda_{j+1} - \lambda_j|^{-1/2}\, \phi_{n+1}$, $n \geq 1$, and $Vg = 0$, $g \in \mathcal{H}_2$. Let $\mathcal{M}_n = \vee\{\phi_j \mid j = 1, \ldots, n\}$ and let $P_n$ be the orthogonal projection onto $\mathcal{M}_n$, and define $V_n = VP_n$. A tedious calculation shows that

$$\Delta_T(V_n)\phi_j = \begin{cases} v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} - V_n\delta_j, & j \leq n, \\ -V_n\delta_j, & j > n, \end{cases} \tag{2.5}$$

where $v_j = |\lambda_{j+1} - \lambda_j|^{-1/2}$. Denoting $\Delta_T(V_n) - \Delta_T(V_m)$ by $\Delta_T^{n,m}$, then for $n < m$,

$$\Delta_T^{n,m}\phi_j = \begin{cases} 0, & j \leq n, \\ -v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} + (V_m - V_n)\delta_j, & n < j \leq m, \\ (V_m - V_n)\delta_j, & j > m. \end{cases} \tag{2.6}$$

Furthermore, from (2.3) it results that

$$\delta_j \perp \phi_j, \phi_{j+1}, \phi_{j+2}, \ldots \tag{2.7}$$

and from (2.4)

$$\left\|V_n\delta_j\right\| \leq 2\eta_j\varepsilon_j, \quad \forall j, n \geq 1. \tag{2.8}$$

We will show next that $\|\Delta_T^{n,m}\|_2 \to 0$ when $m, n \to \infty$, thus there exists $C \in \mathcal{C}_2(\mathcal{H})$ such that $\|\Delta_T(V_n) - C\|_2 \to 0$, that is, $C \in \overline{\mathcal{R}(\Delta_T^{(2)})}$.

First, we will show that $\|\Delta_T^{n,m}|_{\mathscr{H}_1}\|_2^2 \to 0$, when $m,n \to \infty$. Indeed,

$$\begin{aligned}\|\Delta_T^{n,m}|_{\mathscr{H}_1}\|_2^2 &= \sum_{j=1}^{\infty} \|\Delta_T^{n,m}\phi_j\|^2 \overset{(2.6)}{=} \\ &= \sum_{j=n+1}^{m} \left\|-v_j(\mu_{j+1}-\mu_j)\phi_{j+1} + v_j\delta_{j+1} + (V_m - V_n)\delta_j\right\|^2 \\ &\quad + \sum_{j=m+1}^{\infty} \left\|(V_m - V_n)\delta_j\right\|^2.\end{aligned} \tag{2.9}$$

The first sum of the right-hand side of the above can be majorized by

$$2\cdot \sum_{j=n+1}^{m} \left\|-v_j(\mu_{j+1}-\mu_j)\phi_{j+1} + v_j\delta_{j+1}\right\|^2 + 2\cdot \sum_{j=n+1}^{m} \left\|(V_m - V_n)\delta_j\right\|^2. \tag{2.10}$$

Since $\phi_{j+1} \perp \delta_{j+1}$, we have

$$\|\Delta_T^{n,m}|_{\mathscr{H}_1}\|_2^2 \le 2\left[\sum_{j=n+1}^{m}\left(v_j^2\ \left|\mu_{j+1}-\mu_j\right|^2 + v_j^2\ \left\|\delta_{j+1}\right\|^2\right) + \sum_{j=n+1}^{\infty}\left\|(V_m - V_n)\delta_j\right\|^2\right]. \tag{2.11}$$

According to (2.8),

$$\left\|(V_m - V_n)\delta_j\right\|^2 \le 16\eta_j^2\varepsilon_j^2, \tag{2.12}$$

and according to (2.4),

$$\begin{gathered} v_j^2\left\|\delta_{j+1}\right\|^2 \le 4\eta_j^2\varepsilon_{j+1}^2 \le 4\eta_j^2\varepsilon_j^2, \\ \left|\mu_{j+1}-\mu_j\right|^2 \le \left(2\varepsilon_j + \left|\lambda_{j+1}-\lambda_j\right|\right)^2 \le 8\varepsilon_j^2 + 2\left|\lambda_{j+1}-\lambda_j\right|^2, \end{gathered} \tag{2.13}$$

which implies

$$v_j^2\left|\mu_{j+1}-\mu_j\right|^2 \le 8\eta_j^2\varepsilon_j^2 + 2\left|\lambda_{j+1}-\lambda_j\right|. \tag{2.14}$$

Therefore

$$\|\Delta_T^{n,m}|_{\mathscr{H}_1}\|_2^2 \le c_1 \cdot \sum_{j=n+1}^{\infty} \eta_j^2\varepsilon_j^2 + c_2 \cdot \sum_{j=n+1}^{m}\left|\lambda_{j+1}-\lambda_j\right|, \tag{2.15}$$

where $c_1$ and $c_2$ are some constants. After a careful review of the proof, one can see that the sequence $\{\lambda_n\}$ can be assumed to converge fast enough (otherwise choose a subsequence of it), more precisely $\sum_{j=n+1}^{m} |\lambda_{j+1}-\lambda_j| \to 0$ when $n,m \to \infty$.

We show next that $\|\Delta_T^{n,m}|_{\mathscr{H}_2}\|_2^2 \to 0$, when $m, n \to \infty$. Indeed, we can write

$$T^*\phi_n = \overline{\mu}_n \phi_n + \gamma_n \quad \text{with } \langle \gamma_n, \phi_n \rangle = 0, \quad \|\gamma_n\| \leq 2\varepsilon_n, \quad n \geq 1. \tag{2.16}$$

Obviously, we can write $T^*\phi_n = \theta_n \phi_n + \gamma_n$ with $\langle \gamma_n, \phi_n \rangle = 0$, which implies

$$\begin{gathered} \theta_n = \langle \theta_n \phi_n + \gamma_n, \phi_n \rangle = \langle T^*\phi_n, \phi_n \rangle = \langle \phi_n, T\phi_n \rangle = \langle \phi_n, \mu_n \phi_n + \delta_n \rangle = \overline{\mu}_n, \\ \|\gamma_n\| = \|(T^* - \overline{\mu}_n)\phi_n\| \leq \|(T - \lambda_n)^*\phi_n\| + \left|\overline{\lambda}_n - \overline{\mu}_n\right| \overset{(1.4),(2.4)}{\leq} 2\varepsilon_n. \end{gathered} \tag{2.17}$$

For an orthonormal basis $\{\psi_i\}_{i \geq 1}$ of $\mathscr{H}_2$, we will show that

$$\sum_{i=1}^{\infty} \left\| \Delta_T^{n,m} \psi_i \right\|^2 \longrightarrow 0 \quad \text{when } n, m \longrightarrow \infty. \tag{2.18}$$

For each $i$, write $T\psi_i = \sum_{k=1}^{\infty} a_k^{(i)} \phi_k + w_i$ with $w_i \in \mathscr{H}_2$. Thus

$$V_m T \psi_i = \sum_{k=1}^{m} a_k^{(i)} V_m \phi_k + V_m w_i = \sum_{k=1}^{m} a_k^{(i)} v_k \phi_{k+1}. \tag{2.19}$$

Since $V_m \psi_i = 0$, we have $\Delta_T(V_m)\psi_i = -V_m T \psi_i$, and consequently, for $n < m$,

$$\Delta_T^{n,m} \psi_i = \sum_{k=n+1}^{m} a_k^{(i)} v_k \phi_{k+1}. \tag{2.20}$$

Since the sequence $\{\phi_k\}$ is orthonormal, we have

$$\left\| \Delta_T^{n,m} \psi_i \right\|^2 = \sum_{k=n+1}^{m} \left| a_k^{(i)} \right|^2 \cdot v_k^2. \tag{2.21}$$

Therefore

$$\sum_{i=1}^{\infty} \left\| \Delta_T^{n,m} \psi_i \right\|^2 = \sum_{i=1}^{\infty} \sum_{k=n+1}^{m} \left| a_k^{(i)} \right|^2 \cdot v_k^2 = \sum_{k=n+1}^{m} v_k^2 \left( \sum_{i=1}^{\infty} \left| a_k^{(i)} \right|^2 \right). \tag{2.22}$$

For a fixed $k$,

$$\sum_{i=1}^{\infty}\left|a_k^{(i)}\right|^2 = \sum_{i=1}^{\infty}\left|\langle T\psi_i, \phi_k\rangle\right|^2 = \sum_{i=1}^{\infty}\left|\langle \psi_i, T^*\phi_k\rangle\right|^2$$
$$\overset{(2.16)}{=} \sum_{i=1}^{\infty}\left|\langle \psi_i, \overline{\mu}_k\phi_k + \gamma_k\rangle\right|^2 = \sum_{i=1}^{\infty}\left|\langle \psi_i, \gamma_k\rangle\right|^2 \le \|\gamma_k\|^2 \tag{2.23}$$
$$\overset{(2.16)}{\le} 4\varepsilon_k^2.$$

Consequently, $\sum_{i=1}^{\infty}\|\Delta_T^{n,m}\psi_i\|^2 \le 4\sum_{k=n+1}^{m} v_k^2 \cdot \varepsilon_k^2 \to 0$ for $n, m \to \infty$.

The operator $C$ is not in $\mathcal{R}(\Delta_T^{(2)})$ since, according to the proof of Theorem A in [2], $C \notin \mathcal{R}(\Delta_T)$. □

**Theorem 2.3.** *Let $T \in H^M(\mathscr{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is closed if and only if $\sigma(T)$ is finite.*

*Proof.* If $T \in H^M(\mathscr{H})$ and $\sigma(T)$ are finite, then according to Proposition 2.1, $\mathcal{R}(\Delta_T^{(2)})$ is closed. Conversely, if $T \in H^M(\mathscr{H})$ has an infinite spectrum, then there are infinitely many distinct points $\{\lambda_n\}_n$ that are either isolated points of the spectrum, in which case they are eigenvalues, or accumulation points of the spectrum, in which case they are in $\sigma_{\text{ap}}(T)$. Since $T \in H^M(\mathscr{H})$, we have $\sigma_p(T), \sigma_{\text{ap}}(T) \subseteq \sigma_{\text{nap}}(T)$. Thus $T \in \mathcal{G}(\mathscr{H})$ and according to Proposition 2.2, $\mathcal{R}(\Delta_T^{(2)})$ is not closed. □

## References


[1] J. Anderson and C. Foiaş, "Properties which normal operators share with normal derivations and related operators," *Pacific Journal of Mathematics*, vol. 61, no. 2, pp. 313–325, 1975.

[2] J. G. Stampfli, "On the range of a hyponormal derivation," *Proceedings of the American Mathematical Society*, vol. 52, pp. 117–120, 1975.

[3] J. G. Stampfli, "Compact perturbations, normal eigenvalues and a problem of Salinas," *Journal of the London Mathematical Society*, vol. 9, no. 1, pp. 165–175, 1974.

[4] A. Uchiyama and T. Yoshino, "Weyl's theorem for $p$-hyponormal or $M$-hyponormal operators," *Glasgow Mathematical Journal*, vol. 43, no. 3, pp. 375–381, 2001.