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Research Article

Vanishing Power Values of Commutators with Derivations on Prime Rings

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Let R be a prime ring of char $R \neq 2$, d a nonzero derivation of R and ρ a nonzero right ideal of R such that $[[d(x),x]_n,[y,d(y)]_m]^t = 0$ for all $x,y \in \rho$, where $n \geq 0$, $m \geq 0$, $t \geq 1$ are fixed integers. If $[\rho,\rho]\rho \neq 0$, then $d(\rho)\rho = 0$.

1. Introduction

Throughout this paper, unless specifically stated, R always denotes a prime ring with center $Z(R)$ and extended centroid C, Q the Martindale quotients ring. Let n be a positive integer. For given $a,b \in R$, let $[a,b]_0 = a$ and let $[a,b]_1$ be the usual commutator $ab - ba$, and inductively for $n > 1$, $[a,b]_n = [[a,b]_{n-1},b]$. By d we mean a nonzero derivation in R.

A well-known result proven by Posner [1] states that if $[[d(x),x],y] = 0$ for all $x,y \in R$, then R is commutative. In [2], Lanski generalized this result of Posner to the Lie ideal. Lanski proved that if U is a noncommutative Lie ideal of R such that $[[d(x),x],y] = 0$ for all $x \in U, y \in R$, then either R is commutative or char $R = 2$ and R satisfies S_4, the standard identity in four variables. Bell and Martindale III [3] studied this identity for a semiprime ring R. They proved that if R is a semiprime ring and $[[d(x),x],y] = 0$ for all x in a non-zero left ideal of R and $y \in R$, then R contains a non-zero central ideal. Clearly, this result says that if R is a prime ring, then R must be commutative.

Several authors have studied this kind of Engel type identities with derivation in different ways. In [4], Herstein proved that if char $R \neq 2$ and $[d(x),d(y)] = 0$ for all $x,y \in R$, then R is commutative. In [5], Filippis showed that if R is of characteristic different from 2 and ρ a non-zero right ideal of R such that $[\rho,\rho]\rho \neq 0$ and $[[d(x),x],[d(y),y]] = 0$ for all $x,y \in \rho$, then $d(\rho)\rho = 0$.

In continuation of these previous results, it is natural to consider the situation when $[[d(x),x]_n,[y,d(y)]_m]^t = 0$ for all $x, y \in \rho$, $n, m \geq 0$, $t \geq 1$ are fixed integers. We have studied this identity in the present paper.

It is well known that any derivation of a prime ring R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q. Moreover Q is a prime ring as well as R and the extended centroid C of R coincides with the center of Q. We refer to [6, 7] for more details.

Denote by $Q*_C C\{X,Y\}$ the free product of the C-algebra Q and $C\{X,Y\}$, the free C-algebra in noncommuting indeterminates X, Y.

2. The Case: R Prime Ring

We need the following lemma.

Lemma 2.1. *Let ρ be a non-zero right ideal of R and d a derivation of R. Then the following conditions are equivalent:* (i) *d is an inner derivation induced by some $b \in Q$ such that $b\rho = 0$;* (ii) *$d(\rho)\rho = 0$ (for its proof refer to [8, Lemma]).*

We mention an important result which will be used quite frequently as follows.

Theorem 2.2 (see Kharchenko [9]). *Let R be a prime ring, d a derivation on R and I a non-zero ideal of R. If I satisfies the differential identity $f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0$ for any $r_1, r_2, \ldots, r_n \in I$, then either* (i) *$I$ satisfies the generalized polynomial identity*

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0, \tag{2.1}$$

or (ii) *d is Q-inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0. \tag{2.2}$$

Theorem 2.3. *Let R be a prime ring of char $R \neq 2$ and d a derivation of R such that $[[d(x),x]_n,[[y,d(y)]_m]^t = 0$ for all $x, y \in R$, where $n \geq 0$, $m \geq 0$, $t \geq 1$ are fixed integers. Then R is commutative or $d = 0$.*

Proof. Let R be noncommutative. If d is not Q-inner, then by Kharchenko's Theorem [9]

$$g(x, y, u, v) = [[u,x]_n, [y,v]_m]^t = 0, \tag{2.3}$$

for all $x, y, u, v \in R$. This is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$, and R and $M_k(F)$ satisfy the same polynomial identity [10, Lemma 1]. But by choosing $u = e_{12}$, $x = e_{11}$, $v = e_{11}$ and $y = e_{21}$, we get

$$0 = [[u,x]_n, [y,v]_m]^t = (-1)^{tn}\left(e_{11} + (-)^t e_{22}\right), \tag{2.4}$$

which is a contradiction.

Now, let d be Q-inner derivation, say $d = ad(a)$ for some $a \in Q$, that is, $d(x) = [a, x]$ for all $x \in R$, then we have

$$[[a, x]_{n+1}, [y, [a, y]]_m]^t = 0, \tag{2.5}$$

for all $x, y \in R$. Since $d \neq 0$, $a \notin C$ and hence R satisfies a nontrivial generalized polynomial identity (GPI). By [11], it follows that RC is a primitive ring with $H = Soc(RC) \neq 0$, and eHe is finite dimensional over C for any minimal idempotent $e \in RC$. Moreover we may assume that H is noncommutative; otherwise, R must be commutative which is a contradiction.

Notice that H satisfies $[[a, x]_{n+1}, [y, [a, y]]_m]^t = 0$ (see [10, Proof of Theorem 1]). For any idempotent $e \in H$ and $x \in H$, we have

$$0 = [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^t. \tag{2.6}$$

Right multiplying by e, we get

$$\begin{aligned}
0 &= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^t e \\
&= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \\
&\quad \cdot \{[a, e]_{n+1}([ex(1-e), [a, ex(1-e)]]_m)e - ([ex(1-e), [a, ex(1-e)]]_m)[a, e]_{n+1}e\} \\
&= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \\
&\quad \cdot \left\{ [a, e]_{n+1} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, ex(1-e)]^j ex(1-e) [a, ex(1-e)]^{m-j} \right) e \right. \\
&\qquad \left. - \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, ex(1-e)]^j ex(1-e) [a, ex(1-e)]^{m-j} \right) [a, e]_{n+1} e \right\} \\
&= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \\
&\quad \cdot \left\{ 0 - \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (-ex(1-e)\ a)^j ex(1-e)(aex(1-e))^{m-j} \right) ae \right\} \\
&= -[[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \left(\sum_{j=0}^{m} \binom{m}{j} (ex(1-e)a)^{m+1} \right) e \\
&= -2^m [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} (ex(1-e)a)^{m+1} e \\
&= (-)^t 2^{mt} (ex(1-e)a)^{(m+1)t} e.
\end{aligned} \tag{2.7}$$

This implies that $0 = (-)^t 2^{mt}((1-e)aex)^{(m+1)t+1}$. Since char $R \neq 2$, $((1-e)aex)^{(m+1)t+1} = 0$. By Levitzki's lemma [12, Lemma 1.1], $(1-e)aex = 0$ for all $x \in H$. Since H is prime ring, $(1-e)ae = 0$, that is, $eae = ae$ for any idempotent $e \in H$. Now replacing e with $1-e$, we get that $ea(1-e) = 0$, that is, $eae = ea$. Therefore for any idempotent $e \in H$, we have $[a, e] = 0$.

So a commutes with all idempotents in H. Since H is a simple ring, either H is generated by its idempotents or H does not contain any nontrivial idempotents. The first case gives $a \in C$ contradicting $d \neq 0$. In the last case, H is a finite dimensional division algebra over C. This implies that $H = RC = Q$ and $a \in H$. By [10, Lemma 2], there exists a field F such that $H \subseteq M_k(F)$ and $M_k(F)$ satisfies $[[a,x]_{n+1},[y,[a,y]]_m]^t$. Then by the same argument as earlier, a commutes with all idempotents in $M_k(F)$, again giving the contradiction $a \in C$, that is, $d = 0$. This completes the proof of the theorem. □

Theorem 2.4. *Let R be a prime ring of char $R \neq 2$, d a non-zero derivation of R and ρ a non-zero right ideal of R such that $[[d(x),x]_n,[y,d(y)]_m]^t = 0$ for all $x,y \in \rho$, where $n \geq 0, m \geq 0, t \geq 1$ are fixed integers. If $[\rho,\rho]\rho \neq 0$, then $d(\rho)\rho = 0$.*

We begin the proof by proving the following lemma.

Lemma 2.5. *If $d(\rho)\rho \neq 0$ and $[[d(x),x]_n,[y,d(y)]_m]^t = 0$ for all $x,y \in \rho$, $m,n \geq 0$, $t \geq 1$ are fixed integers, then R satisfies nontrivial generalized polynomial identity (GPI).*

Proof. Suppose on the contrary that R does not satisfy any nontrivial GPI. We may assume that R is noncommutative; otherwise, R satisfies trivially a nontrivial GPI. We consider two cases.

Case 1. Suppose that d is Q-inner derivation induced by an element $a \in Q$. Then for any $x \in \rho$,

$$[[a,xX]_{n+1},[xY,[a,xY]]_m]^t \tag{2.8}$$

is a GPI for R, so it is the zero element in $Q *_C C\{X,Y\}$. Expanding this, we get

$$\begin{aligned}\Bigg(&[a,xX]_{n+1}\sum_{j=0}^{m}(-1)^j\binom{m}{j}[a,xY]^j xY[a,xY]^{m-j}\\ &-\sum_{j=0}^{m}(-1)^j\binom{m}{j}[a,xY]^j xY[a,xY]^{m-j}[a,xX]_{n+1}\Bigg)A(X,Y)=0,\end{aligned}\tag{2.9}$$

where $A(X,Y) = [[a,xX]_{n+1},[xY,[a,xY]]_m]^{t-1}$. If ax and x are linearly C-independent for some $x \in \rho$, then

$$\begin{aligned}\Bigg(&(axX)^{n+1}\sum_{j=0}^{m}(-1)^j\binom{m}{j}[a,xY]^j xY[a,xY]^{m-j}\\ &-\sum_{j=0}^{m}(-1)^j\binom{m}{j}(axY)^j xY[a,xY]^{m-j}[a,xX]_{n+1}\Bigg)A(X,Y)=0.\end{aligned}\tag{2.10}$$

Again, since ax and x are linearly C-independent, above relation implies that

$$(-xY[a,xY]^m[a,xX]_{n+1})A(X,Y)=0, \tag{2.11}$$

and so

$$\left(-xY(axY)^m(axX)^{n+1}\right)A(X,Y)=0. \tag{2.12}$$

Repeating the same process yields

$$\left(-xY(axY)^m(axX)^{n+1}\right)^t=0 \tag{2.13}$$

in $Q*_C C\{X,Y\}$. This implies that $ax=0$, a contradiction. Thus for any $x\in\rho$, ax and x are C-dependent. Then $(a-\alpha)\rho=0$ for some $\alpha\in C$. Replacing a with $a-\alpha$, we may assume that $a\rho=0$. Then by Lemma 2.1, $d(\rho)\rho=0$, contradiction.

Case 2. Suppose that d is not Q-inner derivation. If for all $x\in\rho$, $d(x)\in xC$, then $[d(x),x]=0$ which implies that R is commutative (see [13]). Therefore there exists $x\in\rho$ such that $d(x)\notin xC$, that is, x and $d(x)$ are linearly C-independent.

By our assumption, we have that R satisfies

$$[[d(xX),xX]_n,[xY,d(xY)]_m]^t=0. \tag{2.14}$$

By Kharchenko's Theorem [9],

$$[[d(x)X+xr_1,xX]_n,[xY,d(x)Y+xr_2]_m]^t=0, \tag{2.15}$$

for all $X,Y,r_1,r_2\in R$. In particular for $r_1=r_2=0$,

$$[[d(x)X,xX]_n,[xY,d(x)Y]_m]^t=0, \tag{2.16}$$

which is a nontrivial GPI for R, because x and $d(x)$ are linearly C-independent, a contradiction.

□

We are now ready to prove our main theorem.

Proof of Theorem 2.4. Suppose that $d(\rho)\rho\neq 0$, then we derive a contradiction. By Lemma 2.5, R is a prime GPI ring, so is also Q by [14]. Since Q is centrally closed over C, it follows from [11] that Q is a primitive ring with $H=Soc(Q)\neq 0$.

By our assumption and by [7], we may assume that

$$[[d(x),x]_n,[y,d(y)]_m]^t=0 \tag{2.17}$$

is satisfied by ρQ and hence by ρH. Let $e = e^2 \in \rho H$ and $y \in H$. Then replacing x with e and y with $ey(1-e)$ in (2.17), then right multiplying it by e, we obtain that

$$\begin{aligned} 0 &= \left[[d(e),e]_n, \left[ey(1-e), d\left(ey(1-e)\right)\right]_m\right]^t e \\ &= \left[[d(e),e]_n, \left[ey(1-e), d\left(ey(1-e)\right)\right]_m\right]^{t-1} \\ &\quad \cdot \left\{ [d(e),e]_n \sum_{j=0}^{m} (-1)^j \binom{m}{j} d\left(ey(1-e)\right)^j ey(1-e) d\left(ey(1-e)\right)^{m-j} e \right. \\ &\qquad \left. - \sum_{j=0}^{m} (-1)^j \binom{m}{j} d\left(ey(1-e)\right)^j ey(1-e) d\left(ey(1-e)\right)^{m-j} [d(e),e]_n e \right\}. \end{aligned} \tag{2.18}$$

Now we have the fact that for any idempotent e, $d(y(1-e))e = -y(1-e)d(e)$, $ed(e)e = 0$ and so

$$\begin{aligned} 0 &= \left[[d(e),e]_n, \left[ey(1-e), d\left(ey(1-e)\right)\right]_m\right]^{t-1} \\ &\quad \cdot \left\{ 0 - \sum_{j=0}^{m} (-1)^j \binom{m}{j} e\left(-y(1-e)d(e)\right)^j y(1-e) d\left(ey(1-e)\right)^{m-j} d(e)e \right\}. \end{aligned} \tag{2.19}$$

Now since for any idempotent e and for any $y \in R$, $(1-e)d(ey) = (1-e)d(e)y$, above relation gives

$$\begin{aligned} 0 &= \left[[d(e),e]_n, \left[ey(1-e), d\left(ey(1-e)\right)\right]_m\right]^{t-1} \\ &\quad \cdot \left\{ -e \sum_{j=0}^{m} \binom{m}{j} \left(y(1-e)d(e)\right)^j y(1-e) \left(d(e)y(1-e)\right)^{m-j} d(e)e \right\} \\ &= \left[[d(e),e]_n, \left[ey(1-e), d\left(ey(1-e)\right)\right]_m\right]^{t-1} \left\{ -e \sum_{j=0}^{m} \binom{m}{j} \left(y(1-e)d(e)\right)^{m+1} e \right\} \\ &= \left[[d(e),e]_n, \left[ey(1-e), d\left(ey(1-e)\right)\right]_m\right]^{t-1} \left\{ -2^m e\left(y(1-e)d(e)\right)^{m+1} e \right\} \\ &= \left\{ -2^m e\left(y(1-e)d(e)\right)^{m+1} \right\}^t e. \end{aligned} \tag{2.20}$$

This implies that $0 = (-1)^t 2^{mt} ((1-e)d(e)ey)^{(m+1)t+1}$ for all $y \in H$. Since char $R \neq 2$, we have by Levitzki's lemma [12, Lemma 1.1] that $(1-e)d(e)ey = 0$ for all $y \in H$. By primeness of H, $(1-e)d(e)e = 0$. By [15, Lemma 1], since H is a regular ring, for each $r \in \rho H$, there exists an idempotent $e \in \rho H$ such that $r = er$ and $e \in rH$. Hence $(1-e)d(e)e = 0$ gives $(1-e)d(e) = (1-e)d(e^2) = (1-e)d(e)e = 0$ and so $d(e) = ed(e) \in eH \subseteq \rho H$ and $d(r) = d(er) = d(e)er + ed(er) \in \rho H$. Hence for each $r \in \rho H$, $d(r) \in \rho H$. Thus $d(\rho H) \subseteq \rho H$. Set $J = \rho H$.

Then $\overline{J} = J/(J \cap l_H(J))$, a prime C-algebra with the derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in J$. By assumption, we have that

$$\left[\left[\overline{d}(\overline{x}), \overline{x}\right]_n, \left[\overline{y}, \overline{d}(\overline{y})\right]_m\right]^t = 0, \tag{2.21}$$

for all $\overline{x}, \overline{y} \in \overline{J}$. By Theorem 2.3, we have either $\overline{d} = 0$ or $\overline{\rho H}$ is commutative. Therefore we have that either $d(\rho H)\rho H = 0$ or $[\rho H, \rho H]\rho H = 0$. Now $d(\rho H)\rho H = 0$ implies that $0 = d(\rho\rho H)\rho H = d(\rho)\rho H \rho H$ and so $d(\rho)\rho = 0$. $[\rho H, \rho H]\rho H = 0$ implies that $0 = [\rho\rho H, \rho H]\rho H = [\rho, \rho H]\rho H \rho H$ and so $[\rho, \rho H]\rho = 0$, then $0 = [\rho, \rho\rho H]\rho = [\rho, \rho]\rho H \rho$ implying that $[\rho, \rho]\rho = 0$. Thus in all the cases we have contradiction. This completes the proof of the theorem. □

3. The Case: R Semiprime Ring

In this section we extend Theorem 2.3 to the semiprime case. Let R be a semiprime ring and U be its right Utumi quotient ring. It is well known that any derivation of a semiprime ring R can be uniquely extended to a derivation of its right Utumi quotient ring U and so any derivation of R can be defined on the whole of U [7, Lemma 2].

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 3.1 (see [16, Lemma 1 and Theorem 1] or [7, pages 31-32]). *Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P$ is a maximal ideal of C with U/PU 2-torsion free$\} = 0$.*

Theorem 3.2. *Let R be a 2-torsion free semiprime ring and d a non-zero derivation of R such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in R$, $n, m \geq 0, t \geq 1$ fixed are integers. Then d maps R into its center.*

Proof. Since any derivation d can be uniquely extended to a derivation in U, and R and U satisfy the same differential identities [7, Theorem 3], we have

$$\left[[d(x), x]_n, [y, d(y)]_m\right]^t = 0, \tag{3.1}$$

for all $x, y \in U$. Let P be any maximal ideal of C such that U/PU is 2-torsion free. Then by Lemma 3.1, PU is a prime ideal of U invariant under d. Set $\overline{U} = U/PU$. Then derivation d canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

$$\left[\left[\overline{d}(\overline{x}), \overline{x}\right]_n, \left[\overline{y}, \overline{d}(\overline{y})\right]_m\right]^t = 0, \tag{3.2}$$

for all $\overline{x}, \overline{y} \in \overline{U}$. By Theorem 2.3, either $\overline{d} = 0$ or $[\overline{U}, \overline{U}] = 0$, that is, $d(U) \subseteq PU$ or $[U, U] \subseteq PU$. In any case $d(U)[U, U] \subseteq PU$ for any maximal ideal P of C. By Lemma 3.1,

$\bigcap\{PU \mid P$ is a maximal ideal of C with U/PU 2-torsion free$\} = 0$. Thus $d(U)[U,U] = 0$. Without loss of generality, we have $d(R)[R,R] = 0$. This implies that

$$0 = d\left(R^2\right)[R,R] = d(R)R[R,R] + Rd(R)[R,R] = d(R)R[R,R]. \tag{3.3}$$

Therefore $[R,d(R)]R[R,d(R)] = 0$. By semiprimeness of R, we have $[R,d(R)] = 0$, that is, $d(R) \subseteq Z(R)$. This completes the proof of the theorem. □

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