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Research Article

Some Properties of Fractional Calculus and Linear Operators Associated with Certain Subclass of Multivalent Functions

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We investigate several distortion inequalities involving fractional calculus, Ruscheweyh derivatives, and some well-known integral operators. In special cases, the results presented in this paper provide new approaches to several previously known results.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{K}(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of all functions f of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\},\ a_{p+k} \geq 0), \tag{1.2}$$

For functions $f, g \in \mathcal{A}(p)$, given by

$$f(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \qquad g(z) := z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}, \tag{1.3}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.4}$$

The Ruscheweyh derivative of f of order $\delta + p - 1$ is defined by

$$D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\delta+p)_k}{(1)_k} a_{p+k} z^{p+k}, \tag{1.5}$$

where f is given by (1.1) and $\delta > -p$. The Ruscheweyh derivative $D^{\delta+p-1}$ has been studied by several authors; for example, see [1, 2].

For $\beta < 1$, $\gamma \geq 0$, $p \in \mathbb{N}$, and $\delta > -p$, let $\mathcal{T}_{\delta,p,\gamma}(\beta)$ consist of functions $f \in \mathcal{K}(p)$ so that

$$\Re\left\{ e^{i\eta} \left((1-\gamma) \frac{D^{\delta+p-1} f(z)}{z^p} + \gamma \frac{\left(D^{\delta+p-1} f(z)\right)'}{p z^{p-1}} - \beta \right) \right\} > 0, \quad z \in \mathcal{U}, \tag{1.6}$$

for some $\eta \in \mathbb{R}$. In [3], the authors obtained four containment results for the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. We denote $\mathcal{T}_{0,1,\gamma}(\beta) = \mathcal{T}_{\gamma}(\beta)$. The class $\mathcal{T}_{\gamma}(\beta)$ was studied by Swaminathan [4–6], Barnard et al. [7], Kim and Rønning [8], and others.

In the present paper, we investigate several distortion inequalities involving fractional calculus, Ruscheweyh derivative, and some well-known integral operators defined on the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. In special cases, the results presented here provide new approaches to some previously known results.

Remark 1.1. Throughout this section, we assume that $\delta + p \geq 1$.

2. Definitions and Lemmas

For the function f given by (1.2), we define

$$q_{\delta,p,\gamma}(z) = (1-\gamma) \frac{D^{\delta+p-1} f(z)}{z^p} + \gamma \frac{\left(D^{\delta+p-1} f(z)\right)'}{p z^{p-1}}. \tag{2.1}$$

It is easy to verify that

$$q_{\delta,p,\gamma}(z) = 1 - \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} z^k, \quad a_{p+k} \geq 0. \tag{2.2}$$

Lemma 2.1. *Let the function f be given by* (1.2). *Then $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} \leq \frac{1}{2}\left(\left| 1+e^{i\eta}(1-\beta) \right| - \left| 1-e^{i\eta}(1-\beta) \right| \right) \tag{2.3}$$

for some $\eta \in \mathbb{R}$.

Proof. Using the fact that $\operatorname{Re}(\omega) \geq 0$ if and only if $|1+\omega| \geq |1-\omega|$, it suffices to show that

$$\left| 1+e^{i\eta}\left(q_{\delta,p,\gamma}(z)-\beta\right) \right| - \left| 1-e^{i\eta}\left(q_{\delta,p,\gamma}(z)-\beta\right) \right| \geq 0, \tag{2.4}$$

where $q_{\delta,p,\gamma}(z)$ is defined by (2.1). Letting

$$\mathcal{A}_k = \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k}, \tag{2.5}$$

and assuming (2.3), we obtain

$$\begin{aligned} &\left| 1+e^{i\eta}\left(q_{\delta,p,\gamma}(z)-\beta\right) \right| - \left| 1-e^{i\eta}\left(q_{\delta,p,\gamma}(z)-\beta\right) \right| \\ &\quad = \left| 1+e^{i\eta}(1-\beta) - e^{i\eta}\sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| - \left| 1-e^{i\eta}(1-\beta) + e^{i\eta}\sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| \\ &\quad \geq \left| 1+e^{i\eta}(1-\beta) \right| - \left| e^{i\eta} \right| \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| - \left| 1+e^{i\eta}(1-\beta) \right| - \left| e^{i\eta} \right| \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| \\ &\quad \geq \left| 1+e^{i\eta}(1-\beta) \right| - \left| 1-e^{i\eta}(1-\beta) \right| - 2\sum_{k=1}^{\infty} A_k a_{p+k} \geq 0, \end{aligned} \tag{2.6}$$

where $z \in \partial\mathcal{U} = \{z;\quad z \in \mathbb{C} \text{ and } \ |z| = 1\}$. By (2.3), the desired inequality (2.4) follows at once. Conversely, if $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\operatorname{Re}\left\{ e^{i\eta}\left(q_{\delta,p,\gamma}(z)-\beta\right) \right\} > 0, \tag{2.7}$$

or, equivalently (2.4). This yields

$$\left| 1+e^{i\eta}\left(1-\sum_{k=1}^{\infty} A_k a_{p+k} z^k - \beta \right) \right| \geq \left| 1-e^{i\eta}\left(1-\sum_{k=1}^{\infty} A_k a_{p+k} z^k - \beta \right) \right|, \tag{2.8}$$

which implies that

$$\left| 1+e^{i\eta}(1-\beta) - e^{i\eta}\sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| \geq \left| 1-e^{i\eta}(1-\beta) + e^{i\eta}\sum_{k=1}^{\infty} A_k a_{p+k} z^k \right|. \tag{2.9}$$

Squaring the above inequality, choosing the value of z on the half line $z = re^{-i\theta}(0 \le r < 1)$, and letting $r \to 1^-$ through this line, we obtain

$$\begin{aligned}
&\left|1+e^{i\eta}(1-\beta)\right|^2+\left|e^{i\eta}\right|^2\left|\sum_{k=1}^{\infty}A_k a_{p+k}z^k\right|^2-2\left|1+e^{i\eta}(1-\beta)\right|\left|\sum_{k=1}^{\infty}A_k a_{p+k}z^k\right| \\
&\quad\ge\left|1-e^{i\eta}(1-\beta)\right|^2+\left|e^{i\eta}\right|^2\left|\sum_{k=1}^{\infty}A_k a_{p+k}z^k\right|^2+2\left|1-e^{i\eta}(1-\beta)\right|\left|\sum_{k=1}^{\infty}A_k a_{p+k}z^k\right|.
\end{aligned} \tag{2.10}$$

Hence we get

$$\begin{aligned}
2\left|\sum_{k=1}^{\infty}A_k a_{p+k}z^k\right|\left(\left|1-e^{i\eta}(1-\beta)\right|+\left|1+e^{i\eta}(1-\beta)\right|\right) &\le \left|1+e^{i\eta}(1-\beta)\right|^2-\left|1-e^{i\eta}(1-\beta)\right|^2 \\
&=\left(\left|1+e^{i\eta}(1-\beta)\right|-\left|1-e^{i\eta}(1-\beta)\right|\right)\left(\left|1+e^{i\eta}(1-\beta)\right|+\left|1-e^{i\eta}(1-\beta)\right|\right),
\end{aligned} \tag{2.11}$$

which reduces to

$$2\sum_{k=1}^{\infty}A_k a_{p+k} \le \left|1+e^{i\eta}(1-\beta)\right|-\left|1-e^{i\eta}(1-\beta)\right|. \tag{2.12}$$

So the desired inequality (2.3) follows upon using (2.5). □

Setting $\eta = \delta = 0$ and $p = 1$ in Lemma 2.1, we get the following result.

Corollary 2.2 ([5, Theorem 2.4]). *Let $f(z)$ be of the form* (1.2). *Then necessary and sufficient condition for f to be in $\mathfrak{T}_\gamma(\beta)$ is*

$$\sum_{k=2}^{\infty}\left[1+\gamma(k-1)\right]a_k \le 1-\beta, \quad a_k \ge 0. \tag{2.13}$$

Throughout this paper, we define

$$E_{\eta,\beta} := \left|1+e^{i\eta}(1-\beta)\right|-\left|1-e^{i\eta}(1-\beta)\right|. \tag{2.14}$$

As an immediate consequence of Lemma 2.1, we have the following corollary.

Corollary 2.3. *Let the function f be defined by* (1.2). *If $f \in \mathfrak{T}_{\delta,p,\gamma}(\beta)$, then*

$$a_{p+k} \le \frac{p(1)_k E_{\eta,\beta}}{2(\delta+p)_k(p+k\gamma)} \tag{2.15}$$

for some $\eta \in \mathbb{R}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_q$ $(p, q \in \mathbb{N} \cup \{0\},\ p \leq q+1)$ be complex numbers such that $\beta_k \neq 0, -1, -2, \ldots$ for $k \in \{1, 2, \ldots, q\}$. The generalized hypergeometric function ${}_pF_q$ is given by

$$ {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \quad z \in \mathcal{U}, \tag{2.16} $$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)(x+2)\cdots(x+n-1), \quad \text{for } n \in \mathbb{N},\ (x)_0 = 1. \tag{2.17} $$

The operator ${}_pF_q$ has recently been studied by several authors; for example, [3, 5]. For $p = q+1 = 2$, the above series give rise to the Gaussian hypergeometric series $F(a, b; c; z)$.

In [9], Hohlov introduced the convolution operator $H_{a,b;c}$ by

$$ H_{a,b;c}(f)(z) := zF(a, b; c; z) * f(z), \quad f \in \mathbb{A}(1). \tag{2.18} $$

Motivated by the operator $H_{a,b;c}$, the authors in [3] defined the convolution operators G_f^p and $H_{a,b;c}^{p,d}$ as follows:

$$ G_f^p(a, b; z) := G(z) := \left(\sum_{k=1}^{\infty} \frac{(1+a)(1+b)}{(k+a)(k+b)} z^{k+p-1} \right) * f(z), \quad (a > -1,\ b > -1), \tag{2.19} $$

$$ H_{a,b;c}^{p,d}(f)(z) = H(z) = z^p\ {}_3F_2\left(a, b, 1+\frac{p}{d}; \frac{p}{d}, c; z\right) * f(z), \tag{2.20} $$

where $f \in \mathcal{A}(p)$, $p \in \mathbb{N}$, and $d \geq 0$. For $p = 1$, the operator $G_f^1(a, b; z)$ was introduced in [7].

In Section 3, we will make use of the following well-known fractional calculus operators $D_z^{-\mu}$, D_z^{μ}, and $D_z^{n+\mu}$. For an analytic function f defined in a simply connected region of the complex z-plane containing the origin, these operators are defined as follows (See [1, 10]):

$$ D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt \quad (\mu > 0), \tag{2.21} $$

where multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$;

$$ D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(t)}{(z-t)^{\mu}} dt \quad (0 \leq \mu < 1), \tag{2.22} $$

where the multiplicity of $(z-t)^{-\mu}$ is removed, as in the definition of $D_z^{-\mu} f(z)$;

$$ D_z^{n+\mu}\ f(z) := \frac{d^n}{dz^n} D_z^{\mu}\ f(z) \quad (0 \leq \mu < 1;\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{2.23} $$

By virtue of (2.21), (2.22), (2.23) and in terms of Gamma function, it is wellknown (see for details [11]) that

$$D_z^{-\mu} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\mu+1)} z^{k+\mu} \quad (k \in \mathbb{N},\ \mu > 0),$$
$$D_z^{\mu} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)} z^{k-\mu} \quad (k \in \mathbb{N},\ 0 \leq \mu < 1), \tag{2.24}$$
$$D_z^{q+\mu} z^k = \frac{d^q}{dz^q} D_z^{\mu} z^k = \frac{\Gamma(k+1)}{\Gamma(k-q-\mu+1)} z^{k-(q+\mu)},$$

where $q \in \mathbb{N}_0$, $k \in \mathbb{N}$, $0 \leq \mu < 1$, and $q \leq k$ for $\mu = 0$.

In Section 4, we will investigate the integral operator $J_{\delta,p}$ defined by

$$(J_{\delta,p} f)(z) = \frac{\delta + p}{z^p} \int_0^z t^{\delta-1} f(t) dt, \tag{2.25}$$

where $f \in \mathcal{A}(p)$, $\delta > -p$, and $p \in \mathbb{N}$. For $p = 1$ and $\delta = 0$, the operator was first defined by Bernardi [12]. Later on several authors studied the operator $J_{\delta,p}$; for example, see [1, 5].

3. Distortion Inequalities of Convolution Operators

Theorem 3.1. *Let the function f defined by* (1.2) *be in the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. Then*

$$|q_{\delta,p,\gamma}(z)| \geq 1 - \frac{1}{2} E_{\eta,\beta} |z|,$$
$$|q_{\delta,p,\gamma}(z)| \leq 1 + \frac{1}{2} E_{\eta,\beta} |z| \tag{3.1}$$

for some $\eta \in \mathbb{R}$. Here, $q_{\delta,p,\gamma}(z)$ and $E_{\eta,\beta}$ are defined, respectively, by (2.1) *and* (2.14).

Proof. From (2.2), we have

$$|q_{\delta,p,\gamma}(z)| \geq 1 - \left| \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} z^k \right|$$
$$\geq 1 - \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} |z|^k \tag{3.2}$$
$$\geq 1 - |z| \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k}.$$

Making use of Lemma 2.1, we get

$$|q_{\delta,p,\gamma}(z)| \geq 1 - \frac{1}{2} E_{\eta,\beta} |z| \tag{3.3}$$

for some $\eta \in \mathbb{R}$. Similarly,

$$\left|q_{\delta,p,\gamma}(z)\right| \leq 1 + |z| \sum_{k=1}^{\infty} \frac{(\delta+p)_k (1+(k\gamma/p))}{(1)_k} a_{p+k} \leq 1 + \frac{1}{2} E_{\eta,\beta} |z| \tag{3.4}$$

for some $\eta \in \mathbb{R}$. This completes the proof. □

We next obtain distortion inequalities for the fractional operaters D_z^{μ} and $D_z^{-\mu}$.

Theorem 3.2. *Suppose $\mu \leq (1+k+p)/(k+b+2)$ and $p < b+1$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then for some $\eta \in \mathbb{R}$, one has*

$$\begin{aligned} \left|D_z^{\mu} G(z)\right| &\geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 - \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)} |z| E_{\eta,\beta}\right), \\ \left|D_z^{\mu} G(z)\right| &\leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 + \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)} |z| E_{\eta,\beta}\right), \end{aligned} \tag{3.5}$$

where $0 \leq \mu < 1$, $z \in \mathcal{U}$, $p \in \mathbb{N}$, and the operator $G(z) := G_f^p(a,b;z)$ was defined by (2.19).

Proof. By using (2.19), we deduce that

$$D_z^{\mu} G(z) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} \frac{(1+a)(1+b)\Gamma(k+p+1)}{(k+a+1)(k+b+1)\Gamma(k+p-\mu+1)} a_{p+k} z^{p+k-\mu}. \tag{3.6}$$

Then

$$\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu} G(z) = 1 - \sum_{k=1}^{\infty} \theta(k) a_{p+k} z^k, \tag{3.7}$$

where

$$\theta(k) = \frac{(1+a)(1+b)\Gamma(k+p+1)\Gamma(p-\mu+1)}{(k+a+1)(k+b+1)\Gamma(k+p-\mu+1)\Gamma(p+1)} \quad (k,p \in \mathbb{N},\ 0 \leq \mu < 1). \tag{3.8}$$

Since $\theta(k)$ is a decreasing function of k, when $\mu \leq (1+k+p)/(k+b+2)$, then

$$0 < \theta(k) \leq \theta(1) = \frac{(1+a)(1+b)(1+p)}{(2+a)(2+b)(1-\mu+p)}. \tag{3.9}$$

Also, according to Lemma 2.1 and $\delta + p \geq 1$, we have

$$\begin{aligned}\frac{1}{p}(\delta+p)(p+\gamma)\sum_{k=1}^{\infty}a_{p+k} &= \frac{\Gamma(\delta+p+1)(1+(\gamma/p))}{\Gamma(\delta+p)}\sum_{k=1}^{\infty}a_{p+k}\\ &\leq \sum_{k=1}^{\infty}\frac{(\delta+p)_k(1+(k\gamma/p))}{(1)_k}a_{p+k}\\ &\leq \frac{1}{2}E_{\eta,\beta}\end{aligned} \tag{3.10}$$

for some $\eta \in \mathbb{R}$. Then

$$\sum_{k=1}^{\infty}a_{p+k} \leq \frac{p}{2(\delta+p)(p+\gamma)}E_{\eta,\beta} \tag{3.11}$$

for some $\eta \in \mathbb{R}$. From (3.7) and (3.9), we obtain

$$\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}z^{\mu-p}D_z^{\mu}G(z) \geq 1-\theta(1)\sum_{k=1}^{\infty}a_{p+k}z^k. \tag{3.12}$$

In view of (3.11), we conclude that

$$\left|\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}z^{\mu-p}D_z^{\mu}G(z)\right| \geq 1-\frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)}|z|E_{\eta,\beta} \tag{3.13}$$

for some $\eta \in \mathbb{R}$, and

$$\left|\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}z^{\mu-p}D_z^{\mu}G(z)\right| \leq 1+\frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)}|z|E_{\eta,\beta} \tag{3.14}$$

for some $\eta \in \mathbb{R}$, which yield (3.5). □

By letting $p = 1$ and $b = a-1 > 0$ in Theorem 3.2, we deduce the following consequence.

Corollary 3.3. *If $f \in \mathcal{T}_{\delta,1,b}(\beta)$, then for $0 \leq \mu < 1$, $z \in \mathcal{U}$, $\delta \geq 0$ and some $\eta \in \mathbb{R}$*

$$\begin{aligned}\left|D_z^{\mu}G(z)\right| &\geq \frac{1}{\Gamma(2-\mu)}|z|^{1-\mu}\left(1-\frac{1}{(3+b)(2-\mu)(1+\delta)}|z|E_{\eta,\beta}\right),\\ \left|D_z^{\mu}G(z)\right| &\leq \frac{1}{\Gamma(2-\mu)}|z|^{1-\mu}\left(1+\frac{1}{(3+b)(2-\mu)(1+\delta)}|z|E_{\eta,\beta}\right).\end{aligned} \tag{3.15}$$

Theorem 3.4. *Let $\mu > 0$, $z \in \mathcal{U}$ and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then*

$$\left|D_z^{-\mu}G(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1 - \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)}|z|E_{\eta,\beta}\right), \tag{3.16}$$

$$\left|D_z^{-\mu}G(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1 + \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)}|z|E_{\eta,\beta}\right) \tag{3.17}$$

for some $\eta \in \mathbb{R}$. The operator $G(z) := G_f^p(a,b;z)$ was defined by (2.19).

Proof. In view of (2.19) and (2.22), we have

$$\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)}z^{-(\mu+p)}D_z^{-\mu}G(z) = 1 - \sum_{k=1}^{\infty}\tau(k)a_{p+k}z^k, \tag{3.18}$$

where

$$\tau(k) = \frac{(1+a)(1+b)\Gamma(k+p+1)\Gamma(p+\mu+1)}{(k+a+1)(k+b+1)\Gamma(k+p+\mu+1)\Gamma(p+1)}. \tag{3.19}$$

Since τ is a decreasing function of k, it follows that

$$0 < \tau(k) \leq \tau(1) = \frac{(1+a)(1+b)(1+p)}{(2+a)(2+b)(1+\mu+p)}. \tag{3.20}$$

By using (3.11), (3.18), and (3.20), we get

$$\begin{aligned}\left|\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)}z^{-(\mu+p)}D_z^{-\mu}G(z)\right| &\geq 1 - \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1+\mu+p)(\delta+p)(\gamma+p)}|z|E_{\eta,\beta},\\ \left|\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)}z^{-(\mu+p)}D_z^{-\mu}G(z)\right| &\leq 1 + \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1+\mu+p)(\delta+p)(\gamma+p)}|z|E_{\eta,\beta}\end{aligned} \tag{3.21}$$

for some $\eta \in \mathbb{R}$. The last two inequalities yield (3.16) and (3.17), respectively. □

Letting $\delta = 0$, $p = 1$, and $a = b + 1 = \mu + 2$ in Theorem 3.4, we get the following result.

Corollary 3.5. *Let $G_f^1(\mu + 2, \mu + 1; z)$ be defined by* (2.19). *If $f \in \mathcal{T}_\gamma(\beta)$, then*

$$\left|D_z^{-\mu} G_f^1(\mu+2,\mu+1;z)\right| \geq \frac{1}{\Gamma(\mu+2)}|z|^{\mu+1}\left(1 - \frac{1}{(\mu+4)(\gamma+1)}|z|E_{\eta,\beta}\right),$$
$$\left|D_z^{-\mu} G_f^1(\mu+2,\mu+1;z)\right| \leq \frac{1}{\Gamma(\mu+2)}|z|^{\mu+1}\left(1 + \frac{1}{(\mu+4)(\gamma+1)}|z|E_{\eta,\beta}\right) \tag{3.22}$$

for some $\eta \in \mathbb{R}$, $\mu > 0$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$.

We next prove the distortion theorems involving fractional calculus and generalized convolution operator defined by (2.20).

Theorem 3.6. *Suppose $0 \leq \mu < 1$, $z \in \mathcal{U}$, $p \in \mathbb{N}$, and $\eta \in \mathbb{R}$. Also, let $a \leq 1$, $b \leq p - \mu + 1$, and $c \geq p + 1$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then*

$$\left|D_z^{\mu} H(f)(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1 - \frac{ab(p+1)}{2c(1-\mu+p)(\delta+p)}|z|E_{\eta,\beta}\right),$$
$$\left|D_z^{\mu} H(f)(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1 + \frac{ab(p+1)}{2c(1-\mu+p)(\delta+p)}|z|E_{\eta,\beta}\right) \tag{3.23}$$

for some $\eta \in \mathbb{R}$. Here, the operator $H(f)(z)$ is defined by (2.20).

Proof. By making use of (2.20), we have

$$D_z^{\mu} H(f)(z) = D_z^{\mu}\left\{z^p - \sum_{k=1}^{\infty} \frac{(a)_k (b)_k (1 + (p/\gamma))_k}{(p/\gamma)_k (c)_k (1)_k} a_{p+k} z^{p+k}\right\}$$
$$= \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} \frac{(a)_k (b)_k (1 + (p/\gamma))_k \Gamma(p+k+1)}{(p/\gamma)_k (c)_k (1)_k \Gamma(p+k-\mu+1)} a_{p+k} z^{p+k-\mu}. \tag{3.24}$$

It is easy to verify that

$$1 + \frac{k\gamma}{p} = \frac{(1 + (p/\gamma))_k}{(p/\gamma)_k}. \tag{3.25}$$

This implies that

$$\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu} H(f)(z) = 1 - \sum_{k=1}^{\infty} \lambda(k)\left(1 + \frac{k\gamma}{p}\right) a_{p+k} z^k, \tag{3.26}$$

where

$$\lambda(k) = \frac{(a)_k (b)_k \Gamma(p+k+1)\Gamma(p-\mu+1)}{(c)_k (1)_k \Gamma(p+k-\mu+1)\Gamma(p+1)}. \tag{3.27}$$

Since λ is a decreasing function of k, when $a \le 1$, $c \ge p+1$ and $b \le p-\mu+1$, we get

$$0 < \lambda(k) \le \lambda(1) = \frac{ab(p+1)}{c(p-\mu+1)}. \tag{3.28}$$

From Lemma 2.1 and $\delta + p \ge 1$, we obtain

$$(\delta+p)\sum_{k=1}^{\infty}\left(1+\frac{k\gamma}{p}\right)a_{p+k} \le \sum_{k=1}^{\infty}\frac{(\delta+p)_k(1+(k\gamma/p))}{(1)_k}a_{p+k} \le \frac{1}{2}E_{\eta,\beta} \tag{3.29}$$

for some $\eta \in \mathbb{R}$. It follows from (3.26) and (3.28) that

$$\begin{aligned}\left|\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}z^{\mu-p}D_z^{\mu}H(f)(z)\right| &\ge 1-\frac{ab(p+1)}{2c(p-\mu+1)(\delta+p)}|z|E_{\eta,\beta},\\ \left|\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}z^{\mu-p}D_z^{\mu}H(f)(z)\right| &\le 1+\frac{ab(p+1)}{2c(p-\mu+1)(\delta+p)}|z|E_{\eta,\beta}\end{aligned} \tag{3.30}$$

for some $\eta \in \mathbb{R}$, which yield (3.23). □

We state an obvious variant of Theorem 3.6 as follows.

Corollary 3.7. *Let the function f defined by* (1.2) *be in the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. Also let $a \le 1$, $b \le p-\mu+1$, $c \ge p+1$, and $\delta + p > 1$. Then*

$$\begin{aligned}\left|D_z^{\mu}H(f)(z)\right| &\ge \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1-\frac{1}{2}|z|E_{\eta,\beta}\right),\\ \left|D_z^{\mu}H(f)(z)\right| &\le \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1+\frac{1}{2}|z|E_{\eta,\beta}\right)\end{aligned} \tag{3.31}$$

for some$\eta \in \mathbb{R}$, $0 \le \mu < 1$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$.

The proof of Theorem 3.8 is much akin to that of Theorem 3.6, and so it is omitted here.

Theorem 3.8. *Let* $a \leq 1$, $b \leq p + \mu + 1$ *and* $c \geq p + 1$. *Also, let* $\mu > 0$, $p \in \mathbb{N}$, *and* $z \in \mathcal{U}$. *If* $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, *then*

$$\left|D_z^{-\mu} H(f)(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1 - \frac{ab(p+1)}{2c(1+\mu+p)(\delta+p)}|z|E_{\eta,\beta}\right),$$
$$\left|D_z^{-\mu} H(f)(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1 + \frac{ab(p+1)}{2c(1+\mu+p)(\delta+p)}|z|E_{\eta,\beta}\right) \tag{3.32}$$

for some $\eta \in \mathbb{R}$.

Next we prove the following.

Theorem 3.9. *Let* $a \leq 1$, $b \leq p - \mu + 1$, *and* $c \geq p + 1$. *Also, let* $0 \leq \mu < 1$, $p \in \mathbb{N}$, *and* $z \in \mathcal{U}$. *If* $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, *then*

$$\left|D_z^{\mu}\left(D^{\delta+p-1} H(f)(z)\right)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1 - \frac{ab(p+1)}{2c(1-\mu+p)}|z|E_{\eta,\beta}\right),$$
$$\left|D_z^{\mu}\left(D^{\delta+p-1} H(f)(z)\right)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1 + \frac{ab(p+1)}{2c(1-\mu+p)}|z|E_{\eta,\beta}\right) \tag{3.33}$$

for some $\eta \in \mathbb{R}$.

Proof. We have

$$D^{\delta+p-1} H(f)(z) = z^p - \sum_{k=1}^{\infty} A_k z^{p+k}, \tag{3.34}$$

where

$$A_k = \frac{(\delta+p)_k (a)_k (b)_k (1+(p/\gamma))_k}{(1)_k (p/\gamma)_k (c)_k (1)_k} a_{p+k}. \tag{3.35}$$

Therefore

$$D_z^{\mu}\left(D^{\delta+p-1} H(f)(z)\right) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} A_k \frac{\Gamma(p+k+1)}{\Gamma(p+k-\mu+1)} z^{p+k-\mu}. \tag{3.36}$$

So, from (3.25), we have

$$\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu}\left(D^{\delta+p-1} H(f)(z)\right) = 1 - \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} \lambda(k) a_{p+k} z^k, \tag{3.37}$$

where $\lambda(k)$ is defined by (3.27). Since λ is a decreasing function of k, when $a \le 1$, $b \le p-\mu+1$ and $c \ge p+1$, then

$$0 < \lambda(k) \le \lambda(1) = \frac{ab(p+1)}{c(p-\mu+1)}. \tag{3.38}$$

From (3.37), (3.38), and Lemma 2.1, we find that

$$\begin{gathered} \left|\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu}\left(D^{\delta+p-1} H(f)(z)\right)\right| \ge 1 - \frac{ab(p+1)}{2c(1-\mu+p)} |z| E_{\eta,\beta}, \\ \left|\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu}\left(D^{\delta+p-1} H(f)(z)\right)\right| \le 1 + \frac{ab(p+1)}{2c(1-\mu+p)} |z| E_{\eta,\beta} \end{gathered} \tag{3.39}$$

for some $\eta \in \mathbb{R}$. The above inequalities lead us to the desired inequalities (3.33). □

The proof of the following theorem is similar to Theorem 3.9, and so it is omitted here.

Theorem 3.10. *Let $a \le 1$, $b \le p+\mu+1$, and $c \ge p+1$. Also, let $\mu > 0$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then*

$$\begin{gathered} \left|D_z^{-\mu}\left(D^{\delta+p-1} H(f)(z)\right)\right| \ge \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 - \frac{ab(p+1)}{2c(1+\mu+p)} |z| E_{\eta,\beta}\right), \\ \left|D_z^{-\mu}\left(D^{\delta+p-1} H(f)(z)\right)\right| \le \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 + \frac{ab(p+1)}{2c(1+\mu+p)} |z| E_{\eta,\beta}\right) \end{gathered} \tag{3.40}$$

for some $\eta \in \mathbb{R}$.

Upon setting $\delta = 0$ and $p = 1$ in Theorems 3.6, 3.8, 3.9, and 3.10, we arrive at the following result.

Corollary 3.11. *Let $a \le 1$, $c \ge 2$, $\mu > 0$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then*

$$\begin{gathered} \left|D_z^{\mu} H(f)(z)\right| \ge \frac{1}{\Gamma(2-\mu)} |z|^{1-\mu} \left(1 - \frac{ab}{c(2-\mu)} |z| E_{\eta,\beta}\right), \\ \left|D_z^{\mu} H(f)(z)\right| \le \frac{1}{\Gamma(2-\mu)} |z|^{1-\mu} \left(1 + \frac{ab}{c(2-\mu)} |z| E_{\eta,\beta}\right) \end{gathered} \tag{3.41}$$

for some $\eta \in \mathbb{R}$ *and* $b \leq 2-\mu$. *Furthermore*

$$\begin{aligned}\left|D_z^{-\mu}H(f)(z)\right| &\geq \frac{1}{\Gamma(2+\mu)}|z|^{1+\mu}\left(1-\frac{ab}{c(2+\mu)}|z|E_{\eta,\beta}\right),\\ \left|D_z^{-\mu}H(f)(z)\right| &\leq \frac{1}{\Gamma(2+\mu)}|z|^{1+\mu}\left(1+\frac{ab}{c(2+\mu)}|z|E_{\eta,\beta}\right)\end{aligned} \tag{3.42}$$

for some $\eta \in \mathbb{R}$ *and* $b \leq 2+\mu$.

Remark 3.12. Under the hypothesis of Corollary 3.11, $D_z^{\mu}H(f)(z)$ and $D_z^{-\mu}H(f)(z)$ are included in disks with its center at origin and radii r and R, respectively, given by

$$\begin{aligned} r &= \frac{1}{\Gamma(2-\mu)}\left(1+\frac{ab}{c(2-\mu)}E_{\eta,\beta}\right) \quad (0 \leq \mu < 1,\ b \leq 2-\mu,\ \text{for some } \eta \in \mathcal{R}),\\ R &= \frac{1}{\Gamma(2+\mu)}\left(1+\frac{ab}{c(2+\mu)}E_{\eta,\beta}\right) \quad (\mu > 0,\ b \leq 2+\mu,\ \text{for some } \eta \in \mathcal{R}).\end{aligned} \tag{3.43}$$

4. Distortion Inequalities of Integral Operator

In this section, we obtain the distortion theorems involving the integral operator $J_{\delta,p}$ of functions in the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$ and fractional calculus operator.

Theorem 4.1. *Let* $0 \leq \mu < 1/(\delta+p+1)$, $z \in \mathcal{U}$, *and* $p \in \mathbb{N}$. *If* $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, *then*

$$\begin{aligned}\left|D_z^{\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right)\right| &\geq \frac{1}{\Gamma(2-\mu)}|z|^{-\mu}\left(1-\frac{p}{2(\delta+p+1)(p+\gamma)}|z|E_{\eta,\beta}\right),\\ \left|D_z^{\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right)\right| &\leq \frac{1}{\Gamma(2-\mu)}|z|^{-\mu}\left(1+\frac{p}{2(\delta+p+1)(p+\gamma)}|z|E_{\eta,\beta}\right)\end{aligned} \tag{4.1}$$

for some $\eta \in \mathbb{R}$. *The operator* $(J_{\delta,p}f)(z)$ *is defined by* (2.21).

Proof. Using the definition (2.25), for function $f \in \mathcal{K}(p)$ of the form (1.2), we have

$$(J_{\delta,p}f)(z) = z^{\delta} - \sum_{k=1}^{\infty}\frac{\delta+p}{p+k+\delta}a_{p+k}. \tag{4.2}$$

So

$$D_z^{\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right) = \frac{1}{\Gamma(2-\mu)}z^{-\mu} - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+1)}{(\delta+p+k)\Gamma(k-\mu+1)}a_{p+k}z^{k-\mu}. \tag{4.3}$$

Therefore, we obtain

$$\Gamma(2-\mu)z^{\mu}D_z^{\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right) = 1 - \sum_{k=1}^{\infty}\phi(k)a_{p+k}z^{k}, \tag{4.4}$$

where

$$\phi(k) = \frac{(\delta+p)\Gamma(k+1)\Gamma(2-\mu)}{(\delta+p+k)\Gamma(k-\mu+1)}. \tag{4.5}$$

Since $\phi(k)$ is a decreasing function of k, when $\mu < 1/(\delta+p+1)$, then

$$0 < \phi(k) \leq \phi(1) = \frac{\delta+p}{\delta+p+1}. \tag{4.6}$$

By using (3.11), (4.4), and (4.6), we get

$$\begin{gathered}\left|\Gamma(2-\mu)z^{\mu}D_z^{\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right)\right| \geq 1 - \frac{p}{2(\delta+p+1)(p+\gamma)}|z|E_{\eta,\beta}, \\ \left|\Gamma(2-\mu)z^{\mu}D_z^{\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right)\right| \leq 1 + \frac{p}{2(\delta+p+1)(p+\gamma)}|z|E_{\eta,\beta}\end{gathered} \tag{4.7}$$

for some $\eta \in \mathbb{R}$, which prove the inequalities (4.1). □

The proof of the following theorem is similar to Theorem 4.1, and so it is omitted here.

Theorem 4.2. *Let $\mu > 0$, $p \in \mathbb{N}$, and $z \in \mathcal{U}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then*

$$\begin{gathered}\left|D_z^{-\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right)\right| \geq \frac{1}{\Gamma(\mu+1)}|z|^{\mu}\left(1 - \frac{p}{2(\delta+p+1)(\mu+1)(p+\gamma)}|z|E_{\eta,\beta}\right), \\ \left|D_z^{-\mu}\left(z^{-\delta}(J_{\delta,p}f)(z)\right)\right| \leq \frac{1}{\Gamma(\mu+1)}|z|^{\mu}\left(1 + \frac{p}{2(\delta+p+1)(\mu+1)(p+\gamma)}|z|E_{\eta,\beta}\right)\end{gathered} \tag{4.8}$$

for some $\eta \in \mathbb{R}$.

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