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Research Article

An Extension of Stolarsky Means to the Multivariable Case

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We give an extension of well-known Stolarsky means to the multivariable case in a simple and applicable way. Some basic inequalities concerning this matter are also established with applications in Analysis and Probability Theory.

1. Introduction

There is a huge amount of papers investigating properties of the so-called Stolarsky (or extended) two-parametric mean value, defined for positive values of x, y, as

$$E_{r,s}(x,y) := \left(\frac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)}, \qquad rs(r-s)(x-y) \neq 0. \tag{1.1}$$

E means can be continuously extended on the domain

$$\{(r,s;x,y) \mid r,\ s \in \mathbb{R};\ x,y \in \mathbb{R}_+\} \tag{1.2}$$

by the following:

$$E_{r,s}(x,y) = \begin{cases} \left(\dfrac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)} & rs(r-s) \neq 0; \\ \exp\left(-\dfrac{1}{s} + \dfrac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0; \\ \left(\dfrac{x^s - y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0,\ r = 0; \\ \sqrt{xy}, & r = s = 0; \\ x, & y = x > 0, \end{cases} \tag{1.3}$$

and in this form are introduced by Keneth Stolarsky in [1].

Most of the classical two-variable means are special cases of the class E. For example, $E_{1,2} = (x+y)/2$ is the arithmetic mean, $E_{0,0} = \sqrt{xy}$ is the geometric mean, $E_{0,1} = (x-y)/(\log x - \log y)$ is the logarithmic mean, $E_{1,1} = (x^x/y^y)^{1/(x-y)}/e$ is the identric mean, and so forth. More generally, the rth power mean $((x^r + y^r)/2)^{1/r}$ is equal to $E_{r,2r}$.

Recently, several papers are produced trying to define an extension of the class E to $n,\ n > 2$ variables. Unfortunately, this is done in a highly artificial mode (cf. [2–4]), without a practical background. Here is an illustration of this point; recently Merikowski [4] has proposed the following generalization of the Stolarsky mean $E_{r,s}$ to several variables:

$$E_{r,s}(\mathrm{X}) := \left[\frac{L(\mathrm{X}^s)}{L(\mathrm{X}^r)}\right]^{1/(s-r)}, \quad r \neq s, \tag{1.4}$$

where $X = (x_1, \ldots, x_n)$ is an n-tuple of positive numbers and

$$L(\mathrm{X}^s) := (n-1)! \int_{E_{n-1}} \prod_{i=1}^{n} x_i^{su_i}\, du_1 \cdots du_{n-1}. \tag{1.5}$$

The symbol E_{n-1} stands for the Euclidean simplex which is defined by

$$E_{n-1} := \{(u_1, \ldots, u_{n-1}) : u_i \geq 0,\ 1 \leq i \leq n-1;\ u_1 + \cdots + u_{n-1} \leq 1\}. \tag{1.6}$$

In this paper, we give another attempt to generalize Stolarsky means to the multivariable case in a simple and applicable way. The proposed task can be accomplished by founding a "weighted" variant of the class E, wherefrom the mentioned generalization follows naturally.

In the sequel, we will need notions of the weighted geometric mean $G = G(p,q;x,y)$ and weighted rth power mean $S_r = S_r(p,q;x,y)$, defined by

$$G := x^p y^q; \qquad S_r := \left(px^r + qy^r\right)^{1/r}, \tag{1.7}$$

where

$$p, q, x, y \in \mathbb{R}_+; \qquad p + q = 1; \qquad r \in \mathbb{R}/\{0\} \ . \tag{1.8}$$

Note that $(S_r)^r > (G)^r$ for $x \neq y,\ r \neq 0$, and $\lim_{r\to 0} S_r = G$.

1.1. Weighted Stolarsky Means

We introduce here a class W of weighted two-parameters means which includes the Stolarsky class E as a particular case. Namely, for $p, q, x, y \in R_+, p + q = 1, rs(r - s)(x - y) \neq 0$, we define

$$W = W_{r,s}(p, q; x, y) := \left(\frac{r^2}{s^2}\frac{(S_s)^s - (G)^s}{(S_r)^r - (G)^r}\right)^{1/(s-r)} = \left(\frac{r^2}{s^2}\frac{px^s + qy^s - x^{ps}y^{qs}}{px^r + qy^r - x^{pr}y^{qr}}\right)^{1/(s-r)}. \tag{1.9}$$

Various properties concerning the means W can be established; some of them are the following:

$$\begin{gathered} W_{r,s}(p, q; x, y) = W_{s,r}(p, q; x, y); \\ W_{r,s}(p, q; x, y) = W_{r,s}(q, p; y, x); \qquad W_{r,s}(p, q; y, x) = xyW_{r,s}\left(p, q; x^{-1}, y^{-1}\right); \\ W_{ar,as}(p, q; x, y) = \left(W_{r,s}(p, q; x^a, y^a)\right)^{1/a}, \quad a \neq 0. \end{gathered} \tag{1.10}$$

Note that

$$\begin{aligned} W_{2r,2s}\left(\frac{1}{2}, \frac{1}{2}; x, y\right) &= \left(\frac{r^2}{s^2}\frac{x^{2s} + y^{2s} - 2\left(\sqrt{xy}\right)^{2s}}{x^{2r} + y^{2r} - 2\left(\sqrt{xy}\right)^{2r}}\right)^{1/2(s-r)} \\ &= \left(\frac{r^2}{s^2}\frac{(x^s - y^s)^2}{(x^r - y^r)^2}\right)^{1/2(s-r)} = E(r, s; x, y). \end{aligned} \tag{1.11}$$

In the same manner, we get

$$\begin{aligned} W_{r,s}\left(\frac{2}{3}, \frac{1}{3}; x^3, y^3\right) &= \left(\frac{2x^s + y^s}{2x^r + y^r}\right)^{1/(s-r)} (E(r, s; x, y))^2; \\ W_{r,s}\left(\frac{3}{4}, \frac{1}{4}; x^4, y^4\right) &= \left(\frac{3x^{2s} - (xy)^s + y^{2s}}{3x^{2r} - (xy)^r + y^{2r}}\right)^{1/(s-r)} (E(r, s; x, y))^2. \end{aligned} \tag{1.12}$$

The weighted means from the class W can be extended continuously to the domain

$$D = \{(r, s; x, y) \mid \ r,\ s \in \mathbb{R};\ x, y \in \mathbb{R}_+\}. \tag{1.13}$$

This extension is given by

$$W_{r,s}(p,q;x,y) = \begin{cases} \left(\dfrac{r^2}{s^2}\dfrac{px^s+qy^s-x^{ps}y^{qs}}{px^r+qy^r-x^{pr}y^{qr}}\right)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0; \\ \left(2\dfrac{px^s+qy^s-x^{ps}y^{qs}}{pqs^2\log^2(x/y)}\right)^{1/s}, & s(x-y)\neq 0,\ r=0; \\ \exp\left(\dfrac{-2}{s}+\dfrac{px^s\log x+qy^s\log y-(p\log x+q\log y)x^{ps}y^{qs}}{px^s+qy^s-x^{ps}y^{qs}}\right), & s(x-y)\neq 0,\ r=s; \\ x^{(p+1)/3}y^{(q+1)/3}, & x\neq y,\ r=s=0; \\ x, & x=y. \end{cases} \tag{1.14}$$

Note that those means are homogeneous of order 1, that is, $W_{r,s}(p,q;tx,ty) = tW_{r,s}(p,q;x,y)$, $t>0$, symmetric in r,s, $W_{r,s}(p,q;x,y) = W_{s,r}(p,q;x,y)$ but are not symmetric in x,y unless $p=q=1/2$.

1.2. Multivariable Case

A natural generalization of weighted Stolarsky means to the multivariable case gives

$$W_{r,s}(\mathbf{p};\mathbf{x}) = \begin{cases} \left(\dfrac{r^2\left(\sum p_i x_i^s - \left(\prod x_i^{p_i}\right)^s\right)}{s^2\left(\sum p_i x_i^r - \left(\prod x_i^{p_i}\right)^r\right)}\right)^{1/(s-r)}, & rs(s-r)\neq 0; \\ \left(\dfrac{2}{s^2}\dfrac{\sum p_i x_i^s - \left(\prod x_i^{p_i}\right)^s}{\sum p_i\log^2 x_i - \left(\sum p_i \log x_i\right)^2}\right)^{1/s}, & r=0,\ s\neq 0; \\ \exp\left(\dfrac{-2}{s}+\dfrac{\sum p_i x_i^s \log x_i - \left(\sum p_i \log x_i\right)\left(\prod x_i^{p_i}\right)^s}{\sum p_i x_i^s - \left(\prod x_i^{p_i}\right)^s}\right), & r=s\neq 0; \\ \exp\left(\dfrac{\sum p_i \log^3 x_i - \left(\sum p_i \log x_i\right)^3}{3\left(\sum p_i\log^2 x_i - \left(\sum p_i \log x_i\right)^2\right)}\right), & r=s=0, \end{cases} \tag{1.15}$$

where $\mathbf{x} = (x_1,x_2,\dots,x_n)\in\mathbb{R}_+^n$, $n\geq 2$, $\mathbf{p}$ is an arbitrary positive weight sequence associated with $\mathbf{x}$ and $W_{r,s}(\mathbf{p};\mathbf{x}_0) = a$ for $\mathbf{x}_0 = (a,a,\dots,a)$.

We also write $\sum(\cdot), \prod(\cdot)$ instead of $\sum_1^n(\cdot), \prod_1^n(\cdot)$.

The above formulae are obtained by an appropriate limit process, implying continuity. For example, applying

$$t^s = 1 + s\log t + \frac{s^2}{2}\log^2 t + \frac{s^3}{6}\log^3 t + o\left(s^3\right) \quad (s \longrightarrow 0), \tag{1.16}$$

we get

$$\begin{aligned}
W_{0,0}(\mathbf{p};\mathbf{x}) &= \lim_{s\to 0} W_{s,0}(\mathbf{p};\mathbf{x}) = \lim_{s\to 0}\left(\frac{2}{s^2}\frac{\sum p_i x_i^s - \left(\prod x_i^{p_i}\right)^s}{\sum p_i \log^2 x_i - \left(\sum p_i \log x_i\right)^2}\right)^{1/s} \\
&= \lim_{s\to 0}\left(\frac{2}{s^2\left(\sum p_i \log^2 x_i - \left(\sum p_i \log x_i\right)^2\right)}\right. \\
&\quad \times \left(\left(\sum p_i + s\sum p_i \log x_i + \left(\frac{s^2}{2}\right)\sum p_i \log^2 x_i + \left(\frac{s^3}{6}\right)\sum p_i \log^3 x_i\right)\right. \\
&\quad - \left(\sum p_i + s\log\left(\prod x_i^{p_i}\right) + \left(\frac{s^2}{2}\right)\log^2\left(\prod x_i^{p_i}\right)\right. \\
&\quad \left.\left.\left. + \left(\frac{s^3}{6}\right)\log^3\left(\prod x_i^{p_i}\right)\right) + o\left(s^3\right)\right)\right)^{1/s} \\
&= \lim_{s\to 0}\left(1 + \frac{\sum p_i \log^3 x_i - \left(\sum p_i \log x_i\right)^3}{3\left(\sum p_i \log^2 x_i - \left(\sum p_i \log x_i\right)^2\right)} s(1+o(1))\right)^{1/s} \\
&= \exp\left(\frac{\sum p_i \log^3 x_i - \left(\sum p_i \log x_i\right)^3}{3\left(\sum p_i \log^2 x_i - \left(\sum p_i \log x_i\right)^2\right)}\right).
\end{aligned} \tag{1.17}$$

Remark 1.1. Analogously to the former considerations, one can define a class of Stolarsky means in n variables $E_{r,s}(\mathbf{x};n)$ as

$$E_{r,s}(\mathbf{x};n) := W_{nr,ns}(\mathbf{p}_0,\mathbf{x}), \tag{1.18}$$

where $\mathbf{p}_0 = \{1/n\}_1^n$.

Therefore,

$$E_{r,s}(\mathbf{x};n)=\left(\frac{r^2}{s^2}\frac{\sum_1^n x_i^{ns}-n\prod_1^n x_i^s}{\sum_1^n x_i^{nr}-n\prod_1^n x_i^r}\right)^{1/n(s-r)},\quad rs(r-s)\neq 0. \tag{1.19}$$

Details are left to the readers.

2. Results

The following basic assertion is of importance.

Proposition 2.1. *The expressions $W_{r,s}(\mathbf{p};\mathbf{x})$ are actual means, that is, for arbitrary weight sequence $\mathbf{p}$ one has*

$$\min\{x_1,x_2\dots,x_n\}\le W_{r,s}(\mathbf{p};\mathbf{x})\le \max\{x_1,x_2,\dots,x_n\}. \tag{2.1}$$

Our main result is contained in the following.

Proposition 2.2. *The means $W_{r,s}(\mathbf{p},\mathbf{x})$ are monotone increasing in both variables r and s.*

Passing to the continuous variable case, we get the following definition of the class $\overline{W}_{r,s}(p,x)$.

Assuming that all integrals exist,

$$\overline{W}_{r,s}(\mathbf{p},\mathbf{x})=\begin{cases}\left(\dfrac{r^2\left(\int p(t)x^s(t)dt-\exp\left(s\int p(t)\log x(t)dt\right)\right)}{s^2\left(\int p(t)x^r(t)dt-\exp\left(r\int p(t)\log x(t)dt\right)\right)}\right)^{1/(s-r)}, & rs(s-r)\neq 0;\\[2ex] \left(\dfrac{2}{s^2}\dfrac{\int p(t)x^s(t)dt-\exp\left(s\int p(t)\log x(t)dt\right)}{\int p(t)\log^2 x(t)dt-\left(\int p(t)\log x(t)dt\right)^2}\right)^{1/s}, & r=0,\ s\neq 0;\\[2ex] \exp\left(\dfrac{-2}{s}+\dfrac{\int p(t)x^s(t)\log x(t)dt-\left(\int p(t)\log x(t)dt\right)\exp\left(s\int p(t)\log x(t)dt\right)}{\int p(t)x^s(t)dt-\exp\left(s\int p(t)\log x(t)dt\right)}\right), & r=s\neq 0;\\[2ex] \exp\left(\dfrac{\int p(t)\log^3 x(t)dt-\left(\int p(t)\log x(t)dt\right)^3}{3\left(\int p(t)\log^2 x(t)dt-\left(\int p(t)\log x(t)dt\right)^2\right)}\right), & r=s=0,\end{cases} \tag{2.2}$$

where $x(t)$ is a positive integrable function and $p(t)$ is a nonnegative function with $\int p(t)dt=1$.

From our former considerations, a very applicable assertion follows.

Proposition 2.3. *$\overline{W}_{r,s}(\mathbf{p},\mathbf{x})$ is monotone increasing in either r or s.*

3. Applications

3.1. Applications in Analysis

As an illustration of the above, we give the following proposition.

Proposition 3.1. *The function $w(s)$, defined by*

$$w(s) := \begin{cases} \left(\dfrac{12}{(\pi s)^2}\left(\Gamma(1+s) - e^{-\gamma s}\right)\right)^{1/s}, & s \neq 0; \\ \exp\left(-\gamma - \dfrac{4\xi(3)}{\pi^2}\right), & s = 0, \end{cases} \tag{3.1}$$

is monotone increasing for $s \in (-1, \infty)$.

In particular, for $s \in (-1, 1)$, one has

$$\Gamma(1-s)e^{-\gamma s} + \Gamma(1+s)e^{\gamma s} - \frac{\pi s}{\sin(\pi s)} \le 1 - \frac{(\pi s)^4}{144}, \tag{3.2}$$

where $\Gamma(\cdot), \xi(\cdot), \gamma$ stands for the Gamma function, Zeta function, and Euler's constant, respectively.

3.2. Applications in Probability Theory

For a random variable X and an arbitrary probability distribution with support on $(-\infty, +\infty)$, it is well known that

$$Ee^X \ge e^{EX}. \tag{3.3}$$

Denoting the central moment of order k by $\mu_k = \mu_k(X) := E(X - EX)^k$, we improve this inequality to the following propsositions.

Proposition 3.2. *For an arbitrary probability law with support on $\mathbb{R}$, one has*

$$Ee^X \ge \left(1 + \left(\frac{\mu_2}{2}\right)\exp\left(\frac{\mu_3}{3\mu_2}\right)\right)e^{EX}. \tag{3.4}$$

Proposition 3.3. *One also has that*

$$\left(\frac{Ee^{sX} - e^{sEX}}{s^2\sigma_X^2/2}\right)^{1/s} \tag{3.5}$$

is monotone increasing in s.

3.3. Shifted Stolarsky Means

Especially interesting is studying the *shifted Stolarsky means* E^*, defined by

$$E^*_{r,s}(x,y) := \lim_{p\to 0^+} W_{r,s}(p,q;x,y). \tag{3.6}$$

Their analytic continuation to the whole (r,s) plane is given by

$$E^*_{r,s}(x,y) = \begin{cases} \left(\dfrac{r^2(x^s - y^s(1+s\log(x/y)))}{s^2(x^r - y^r(1+r\log(x/y)))}\right)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0; \\ \left(\dfrac{2}{s^2}\dfrac{x^s - y^s(1+s\log(x/y))}{\log^2(x/y)}\right)^{1/s}, & s(x-y)\neq 0,\ r=0; \\ \exp\left(\dfrac{-2}{s} + \dfrac{(x^s-y^s)\log x - sy^s\log y\log(x/y)}{x^s - y^s(1+s\log(x/y))}\right), & s(x-y)\neq 0,\ r=s; \\ x^{1/3}y^{2/3}, & r=s=0; \\ x, & x=y. \end{cases} \tag{3.7}$$

Main results concerning the means E^* are contained in the following propositions.

Proposition 3.4. *Means $E^*_{r,s}(x,y)$ are monotone increasing in either r or s for each fixed $x,y\in\mathbb{R}^+$.*

Proposition 3.5. *Means $E^*_{r,s}(x,y)$ are monotone increasing in either x or y for each $r,s\in\mathbb{R}$.*

A well known result of Qi ([5]) states that the means $E_{r,s}(x,y)$ are logarithmically concave for each fixed $x,y>0$ and $r,s\in[0,+\infty)$; also, they are logarithmically convex for $r,s\in(-\infty,0]$.

According to this, we propose the following proposition.

Open Question

Is there any compact interval $I, I\subset\mathbb{R}$ such that the means $E^*_{r,s}(x,y)$ are logarithmically convex (concave) for $r,s\in I$ and each $x,y\in\mathbb{R}^+$?

A partial answer to this problem is given in what follows.

Proposition 3.6. *On any interval I which includes zero and $r, s \in I$,*

(i) $E^*_{r,s}(x, y)$ are not logarithmically convex (concave);

(ii) $W_{r,s}(p, q; x, y)$ are logarithmically convex (concave) if and only if $p = q = 1/2$.

4. Proofs

For the proof of Proposition 2.1, we apply the following assertion on Jensen functionals $J_f(\mathbf{p}, \mathbf{x})$ from [6].

Theorem 4.1. *Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions. Assume that g is strictly convex and ϕ is a continuous and strictly monotonic function on I. Then the expression*

$$\phi^{-1}\left(\frac{J_n(\mathbf{p}, \mathbf{x}; f)}{J_n(\mathbf{p}, \mathbf{x}; g)}\right) \quad (n \geq 2) \tag{4.1}$$

represents a mean value of the numbers $x_1, \ldots, x_n$, that is,

$$\min\{x_1, \ldots, x_n\} \leq \phi^{-1}\left(\frac{J_n(\mathbf{p}, \mathbf{x}; f)}{J_n(\mathbf{p}, \mathbf{x}; g)}\right) \leq \max\{x_1, \ldots, x_n\} \tag{4.2}$$

if and only if the relation

$$f''(t) = \phi(t) g''(t) \tag{4.3}$$

holds for each $t \in I$.

Recall that the Jensen functional $J_n(\mathbf{p}, \mathbf{x}; f)$ is defined on an interval $I, I \subseteq \mathbb{R}$ by

$$J_n(\mathbf{p}, \mathbf{x}; f) := \sum_1^n p_i f(x_i) - f\left(\sum_1^n p_i x_i\right), \tag{4.4}$$

where $f : I \to \mathbb{R}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$, and $\mathbf{p} = \{p_i\}_1^n$ is a positive weight sequence.

The famous Jensen's inequality asserts that

$$J_n(\mathbf{p}, \mathbf{x}; f) \geq 0, \tag{4.5}$$

whenever f is a (strictly) convex function on I, with the equality case if and only if $x_1 = x_2 = \cdots = x_n$.

Proof of Proposition 2.1. Define the auxiliary function $h_s(x)$ by

$$h_s(x) := \begin{cases} \dfrac{e^{sx} - sx - 1}{s^2}, & s \neq 0; \\ \dfrac{x^2}{2}, & s = 0. \end{cases} \tag{4.6}$$

Since

$$h'_s(x) = \begin{cases} \dfrac{e^{sx} - 1}{s}, & s \neq 0; \\ x, & s = 0, \end{cases} \tag{4.7}$$

$$h''_s(x) = e^{sx}, \quad s \in \mathbb{R},$$

we conclude that $h_s(x)$ is a continuously twice differentiable convex function on $\mathbb{R}$.

Denoting $f(t) := h_s(t)$, $g(t) := h_r(t)$, we realize that the condition (4.3) of Theorem 4.1 is fulfilled with $\phi(t) = e^{(s-r)t}$. Hence, applying Theorem 4.1, we obtain that $\log W_{r,s}(\mathbf{p}, e^{\mathbf{x}})$ represents a mean value, which is equivalent to the assertion of Proposition 2.1. □

Proof of Proposition 2.2. We prove first a global theorem concerning log-convexity of the Jensen's functional with a parameter, which can be very usable (cf. [7]).

Theorem 4.2. *Let $f_s(x)$ be a twice continuously differentiable function in x with a parameter s. If $f''_s(x)$ is* log-*convex in s for $s \in I := (a,b)$; $x \in K := (c,d)$, then the Jensen functional*

$$J_f(w, x; s) = J(s) := \sum w_i f_s(x_i) - f_s\left(\sum w_i x_i\right), \tag{4.8}$$

is log-*convex in s for $s \in I$, $x_i \in K$, $i = 1, 2, \ldots$, where $w = \{w_i\}$ is any positive weight sequence.*

At the beginning, we need some preliminary lemmas.

Lemma 4.3. *A positive function f is* log-*convex on I if and only if the relation*

$$f(s)u^2 + 2f\left(\frac{s+t}{2}\right)uw + f(t)w^2 \geq 0 \tag{4.9}$$

holds for each real u, w and $s, t \in I$.

This assertion is nothing more than the discriminant test for the nonnegativity of second-order polynomials. Other well known assertions are the following (cf [8, pages 74, 97-98]) lemmas.

Lemma 4.4 (Jensen's inequality). *If $g(x)$ is twice continuously differentiable and $g''(x) \geq 0$ on K, then $g(x)$ is convex on K and the inequality*

$$\sum w_i g(x_i) - g\left(\sum w_i x_i\right) \geq 0 \tag{4.10}$$

holds for each $x_i \in K$, $i = 1, 2, \ldots$, and any positive weight sequence $\{w_i\}$, $\sum w_i = 1$.

Lemma 4.5. *For a convex f, the expression*

$$\frac{f(s) - f(r)}{s - r} \tag{4.11}$$

is increasing in both variables.

Proof of Theorem 4.2. Consider the function $F(x)$ defined as

$$F(x) = F(u, v, s, t; x) := u^2 f_s(x) + 2uv f_{(s+t)/2}(x) + v^2 f_t(x), \tag{4.12}$$

where $u, v \in \mathbb{R}$; $s, t \in I$ are real parameters independent of the variable $x \in K$.

Since

$$F''(x) = u^2 f_s''(x) + 2uv f_{(s+t)/2}''(x) + v^2 f_t''(x), \tag{4.13}$$

and by assuming $f_s''(x)$ is log-convex in s, it follows from Lemma 4.3 that $F''(x) \geq 0$, $x \in K$.

Therefore, by Lemma 4.4, we get

$$\sum w_i F(x_i) - F\left(\sum w_i x_i\right) \geq 0, \quad x_i \in K, \tag{4.14}$$

which is equivalent to

$$u^2 J(s) + 2uv J\left(\frac{s+t}{2}\right) + v^2 J(t) \geq 0. \tag{4.15}$$

According to Lemma 4.3 again, this is possible only if $J(s)$ is log-convex and the proof is done. □

Now, the proof of Proposition 2.2 easily follows.

From the above, we see that $h_s(x)$ is twice continuously differentiable and that $h_s''(x)$ is a log-convex function for each real s, x.

Applying Theorem 4.2, we conclude that the form

$$\Phi_h(w,x;s) = \Phi(s) := \begin{cases} \dfrac{\sum w_i e^{sx_i} - e^{s\sum w_i x_i}}{s^2}, & s \neq 0, \\ \dfrac{\sum w_i x_i^2 - \left(\sum w_i x_i\right)^2}{2}, & s = 0, \end{cases} \quad (4.16)$$

is log-convex in s.

By Lemma 4.5, with $f(s) = \log \Phi(s)$, we find out that

$$\frac{\log \Phi(s) - \log \Phi(r)}{s - r} = \log\left(\frac{\Phi(s)}{\Phi(r)}\right)^{1/(s-r)} \quad (4.17)$$

is monotone increasing either in s or r. Therefore, by changing variable $x_i \to \log x_i$, we finally obtain the proof of Proposition 2.2. □

Proof of Proposition 2.3. The assertion of Proposition 2.3 follows from Proposition 2.2 by the standard argument (cf. [8, pages 131–134]). Details are left to the reader. □

Proof of Proposition 3.1. The proof follows putting $x(t) = t$, $p(t) = e^{-t}, t \in (0, +\infty)$ and applying Proposition 2.2. with $r = 0$. Corresponding integrals are

$$\int_0^\infty e^{-t} \log t = -\gamma; \qquad \int_0^\infty e^{-t} \log^2 t = \gamma^2 + \frac{\pi^2}{6}; \qquad \int_0^\infty e^{-t} \log^3 t = -\gamma^3 - \frac{\gamma\pi^2}{2} - 2\xi(3), \quad (4.18)$$

with

$$\Gamma(1-s)\Gamma(1+s) = \frac{\pi s}{\sin(\pi s)}. \quad (4.19)$$

□

Proof of Proposition 3.2. By Proposition 2.3, we get

$$W_{0,1}(\mathbf{p}, e^{\mathbf{x}}) \geq W_{0,0}(\mathbf{p}, e^{\mathbf{x}}), \quad (4.20)$$

that is,

$$\frac{Ee^X - e^{EX}}{\mu_2/2} \geq \exp\left(\frac{EX^3 - (EX)^3}{3\mu_2}\right). \quad (4.21)$$

Using the identity $EX^3 - (EX)^3 = \mu_3 + 3\mu_2 EX$, we obtain the proof of Proposition 3.2. □

Proof of Proposition 3.3. This assertion is straightforward consequence of the fact that $W_{0,s}(\mathbf{p}, e^{\mathbf{x}})$ is monotone increasing in s. □

Proof of Proposition 3.4. Direct consequence of Proposition 2.2. □

Proof of Proposition 3.5. This is left as an easy exercise to the readers. □

Proof of Proposition 3.6. We prove only part (ii). The proof of (i) goes along the same lines.

Suppose that $0 \in (a,b) := I$ and that $E_{r,s}(p,q;x,y)$ are log-convex (concave) for $r,s \in I$ and any fixed $x,y \in \mathbb{R}^+$. Then there should be an s, $s > 0$ such that

$$F_s(p,q;x,y) := W_{0,s}(p,q;x,y)W_{0,-s}(p,q;x,y) - (W_{0,0}(p,q;x,y))^2 \tag{4.22}$$

is of constant sign for each $x,y > 0$.

Substituting $(x/y)^s := e^w$, $w \in \mathbb{R}$, after some calculations, we get that the above is equivalent to the assertion that $F(p,q;w)$ is of constant sign, where

$$F(p,q;w) := pe^w + q - e^{pw} - e^{(2/3)(1+p)w}\left(pe^{-w} + q - e^{-pw}\right). \tag{4.23}$$

Developing in power series in w, we get

$$F(p,q;w) = \frac{1}{1620}pq(1+p)(2-p)(1-2p)w^5 + O\left(w^6\right). \tag{4.24}$$

Therefore, $F(p,q;w)$ can be of constant sign for each $w \in \mathbb{R}$ only if $p = 1/2(= q)$.

Suppose now that I is of the form $I := [0,a)$ or $I := (-a,0]$, $a > 0$. Then there should be an $s, s \neq 0, s \in I$ such that

$$W_{0,0}(p,q;x,y)W_{0,2s}(p,q;x,y) - (W_{0,s}(p,q;x,y))^2 \tag{4.25}$$

is of constant sign for each $x,y \in \mathbb{R}^+$.

Proceeding as before, this is equivalent to the assertion that $G(p,q;w)$ is of constant sign with

$$G(p,q;w) := p^3q^3w^6e^{(2/3)(p+1)w}\left(pe^{2w} + q - e^{2pw}\right) - \left(pe^w + q - e^{pw}\right)^4. \tag{4.26}$$

However,

$$G(p,q;w) = \frac{2}{405}p^4q^4(1+p)(1+q)(q-p)w^{11} + O\left(w^{12}\right). \tag{4.27}$$

Hence, we conclude that $G(p,q;w)$ can be of constant sign for sufficiently small w, $w \in \mathbb{R}$ only if $p = q = 1/2$. Combining this with Feng Qi theorem, the assertion from Proposition 3.6 follows. □

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