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Research Article

Asymptotic Expansions of the Wavelet Transform for Large and Small Values of *b*

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Asymptotic expansions of the wavelet transform for large and small values of the translation parameter b are obtained using asymptotic expansions of the Fourier transforms of the function and the wavelet. Asymptotic expansions of Mexican hat wavelet transform, Morlet wavelet transform, and Haar wavelet transform are obtained as special cases. Asymptotic expansion of the wavelet transform has also been obtained for small values of b when asymptotic expansions of the function and the wavelet near origin are given.

1. Introduction

The wavelet transform of f with respect to the wavelet ψ is defined by

$$(W_\psi f)(b,a) = a^{-1/2}\int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}\,dt, \quad b \in \mathbf{R},\ a > 0, \tag{1.1}$$

provided that the integral exists [1]. Using Fourier transform it can also be expressed as

$$(W_\psi f)(b,a) = \frac{\sqrt{a}}{2\pi}\int_{-\infty}^{\infty} e^{ib\omega}\hat{f}(\omega)\overline{\hat{\psi}}(a\omega)\,d\omega, \tag{1.2}$$

where

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-ix\omega} f(x)\,dx. \tag{1.3}$$

Asymptotic expansion with explicit error term for Mellin convolution

$$I(\lambda) = \int_0^{\infty} f(t)h(\lambda t)dt, \tag{1.4}$$

as $\lambda \to +\infty$, was obtained by Wong [2, pages 740–756]. Let us recall basic results from Wong [2], which will be used in the present investigation.

Assume that

$$\begin{aligned} f(t) &\sim \sum_{s=0}^{\infty} a_s t^{s+\alpha-1}, \quad \text{as } t \longrightarrow 0, \\ &= \sum_{s=0}^{n-1} a_s t^{s+\alpha-1} + f_n(t), \end{aligned} \tag{1.5}$$

where $0 < \alpha \leq 1$ and

$$h(t) \sim e^{\text{ict}} \sum_{s=0}^{\infty} b_s t^{-s-\beta}, \quad \text{as } t \longrightarrow +\infty, \tag{1.6}$$

where c is real and $0 < \beta \leq 1$.

Also assume that

$$f(t) = O\left(t^{-\rho_1}\right), \quad \text{as } t \longrightarrow +\infty, \tag{1.7}$$

where $\beta + \rho_1 > 1$.

$$h(t) = O(t^{\rho_2}), \quad \text{as } t \longrightarrow 0, \tag{1.8}$$

where $\alpha + \rho_2 > 0$.

Asymptotic expansion of (1.4) is given by the following [2, Theorem 3, page 752].

Theorem 1.1. *Assume that* (i) $f^{(n)}(t)$ *is continuous on* $(0, \infty)$, *where* n *is a nonnegative integer;* (ii) $f(t)$ *has an expansion of the form* (1.5), *and the expansion can be differentiated* n *times;* (iii) *as* $t \to \infty$, $f^{(j)}(t)$ *is* $O(t^{-1-\eta})$ *for* $j = 0, 1, \ldots, n$ *and for some* $\eta > 0$; (iv) $f_n(t)$ *has the meaning as given in* (1.5); (v) $h(t)$ *satisfies* (1.8) *and* (1.6) *with* $c \neq 0$. *Then we have*

$$I(\lambda) = \sum_{s=0}^{n-1} a_s M[h; s+\alpha] \lambda^{-s-\alpha} + \delta_n(\lambda), \tag{1.9}$$

where

$$M[h; s+\alpha] = \int_0^\infty t^{s+\alpha-1} h(t) dt, \tag{1.10}$$

and the remainder satisfies

$$\delta_n(\lambda) = \frac{(-1)^n}{\lambda^n} \int_0^\infty f_n^{(n)}(t) h^{(-n)}(\lambda t)\, dt. \tag{1.11}$$

As an application of the above theorem, Wong [2, page 753] has derived the following asymptotic expansion for the Fourier transform for large values of λ:

$$\int_0^\infty f(t) e^{i\lambda t} dt = \sum_{s=0}^{n-1} a_s e^{i\pi(s+\alpha)/2} \Gamma(s+\alpha) \lambda^{-s-\alpha} + \left(\frac{i}{\lambda}\right)^n \int_0^\infty f_n^{(n)}(t) e^{i\lambda t} dt. \tag{1.12}$$

The asymptotic expansion of the wavelet transform (1.2) for large values of dilation parameter a has already been obtained in [3].

The aim of the present paper is to derive asymptotic expansion of the wavelet transform given by (1.2) for large and small values of b. In Section 2 we assume that $\widehat{f}(\omega)$ and $\widehat{\psi}(\omega)$ possess asymptotic expansions of the form (1.5) as $\omega \to 0+$ and derive asymptotic expansion of $(W_\psi^+ f)(b,a)$ as $b \to \infty$ using formula (1.12). Asymptotic expansions of certain special forms of the wavelet transform are obtained in Sections 3–5. In Section 6 we assume that asymptotic expansions of $\widehat{f}(\omega)$ and $\widehat{\psi}(\omega)$ are known as $\omega \to \infty$ and derive asymptotic expansion of $(W_\psi^+ f)(b,a)$ as $b \to 0+$, using Theorem 6.1 due to Wong [4, Theorem 14, page 323]. In Section 7 we assume the asymptotic expansions of $f(t)$ and $\psi(t)$ as $t \to 0+$ and derive asymptotic expansions of $\widehat{f}(\omega)$ and $\widehat{\psi}(\omega)$ as $\omega \to \infty$, using (1.12). These asymptotic expansions of $\widehat{f}(\omega)$ and $\widehat{\psi}(\omega)$ give rise to the asymptotic expansion of $(W_\psi^+ f)(b,a)$ as $b \to 0+$, using Theorem 6.1.

2. Asymptotic Expansion for Large b

Let us rewrite (1.2) in the following form:

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{\sqrt{a}}{2\pi} \left\{ \int_0^\infty e^{ib\omega} \overline{\widehat{\psi}(a\omega)} \widehat{f}(\omega) d\omega + \int_0^\infty e^{-ib\omega} \overline{\widehat{\psi}(-a\omega)} \widehat{f}(-\omega) d\omega \right\} \\ &= \left(W_\psi^+ f\right)(b,a) + \left(W_\psi^- f\right)(b,a) \text{ (say)},\end{aligned} \tag{2.1}$$

where for definiteness we take $b \in \mathbf{R}_+$ and $a \in \mathbf{R}_+$. Now, we consider

$$\left(W_{\psi}^{+} f\right)(b, a) = \frac{\sqrt{a}}{2\pi} \int_0^{\infty} e^{ib\omega} \overline{\hat{\psi}(a\omega)} \hat{f}(\omega) d\omega. \tag{2.2}$$

Assume that

$$\overline{\hat{\psi}(\omega)} \sim \sum_{s=0}^{\infty} a_s \omega^{s+\alpha-1}, \quad \text{as } \omega \longrightarrow 0, \tag{2.3}$$

then for arbitrary but fixed $a \in \mathbf{R}_+$, we have

$$\overline{\hat{\psi}(a\omega)} \sim \sum_{s=0}^{\infty} a_s' \, \omega^{s+\alpha-1}, \quad \text{as } \omega \longrightarrow 0, \tag{2.4}$$

where $a_s' = a_s a^{s+\alpha-1}$.

Next, assume that

$$\hat{f}(\omega) \sim \sum_{r=0}^{\infty} b_r \omega^{r+\beta-1}, \quad \text{as } \omega \longrightarrow 0. \tag{2.5}$$

Then

$$\overline{\hat{\psi}(a\omega)} \hat{f}(\omega) \sim \sum_{s=0}^{\infty} a_s' \, \omega^{s+\alpha-1} \sum_{r=0}^{\infty} b_r \omega^{r+\beta-1} = \omega^{\alpha+\beta-2} \sum_{r=0}^{\infty} c_r \omega^r, \tag{2.6}$$

where

$$c_r = a_0' b_r + \cdots + a_r' b_0 = \sum_{m=0}^{r} a_m' b_{r-m} = \sum_{m=0}^{r} a_m a^{m+\alpha-1} b_{r-m}. \tag{2.7}$$

Now, for fixed $a \in \mathbf{R}_+$, write

$$\phi(\omega) := \overline{\hat{\psi}(a\omega)} \hat{f}(\omega) \sim \sum_{r=0}^{\infty} c_r \omega^{r+\gamma-1}, \tag{2.8}$$

where $\gamma = \alpha + \beta - 1$.

Let us set

$$\phi(\omega) = \sum_{r=0}^{n-1} c_r \omega^{r+\gamma-1} + \phi_n(\omega), \quad \text{as } \omega \longrightarrow 0, \tag{2.9}$$

and assume that (i) $\phi^{(n)}(t)$ is continuous on $(0, \infty)$, where n is a nonnegative integer; (ii) the expansion (2.9) can be differentiated n times; (iii) as $\omega \to \infty, \phi^{(j)}(\omega) = O(\omega^{-1-\eta})$ for $j = 0, 1, \ldots, n$ and for some $\eta > 0$.

Then, by (1.12), for $1 < \alpha + \beta \leq 2$,

$$\begin{aligned}\int_0^\infty e^{ib\omega}\overline{\widehat{\varphi}(a\omega)}\widehat{f}(\omega)d\omega &= \int_0^\infty e^{ib\omega}\phi(\omega)d\omega \\ &= \sum_{r=0}^{n-1} c_r e^{i\pi(r+\gamma)/2}\Gamma(r+\gamma)b^{-r-\gamma} + \left(\frac{i}{b}\right)^n \int_0^\infty e^{ib\omega}\phi_n^{(n)}(\omega)d\omega.\end{aligned} \tag{2.10}$$

Similarly, we get

$$\int_0^\infty e^{-ib\omega}\overline{\widehat{\varphi}(-a\omega)}\widehat{f}(-\omega)d\omega = -\sum_{r=0}^{n-1} c_r e^{i\pi(r+\gamma)/2}\Gamma(r+\gamma)b^{-r-\gamma} + \left(\frac{i}{b}\right)^n \int_0^\infty e^{-ib\omega}\phi_n^{(n)}(-\omega)d\omega. \tag{2.11}$$

Notice that the series expansions in (2.10) and (2.11) are the same but opposite in sign. Therefore, we find asymptotic expansion of $(W_\psi^+ f)(b,a)$ only. From (2.2) and (2.10), we have

$$\begin{aligned}\left(W_\psi^+ f\right)(b,a) &= \frac{\sqrt{a}}{2\pi}\left\{\sum_{r=0}^{n-1} c_r\Gamma(r+\alpha+\beta-1)e^{i\pi\left(r+\alpha+\beta-1\right)/2}b^{-r-\alpha-\beta+1} + \left(\frac{i}{b}\right)^n\int_0^\infty e^{ib\omega}\phi_n^{(n)}(\omega)d\omega\right\} \\ &= \frac{\sqrt{a}}{2\pi}\left\{\sum_{r=0}^{n-1}\sum_{m=0}^{r} a_m b_{r-m} a^{\alpha+m-1}\Gamma(r+\alpha+\beta-1)b^{-r-\alpha-\beta+1}\right. \\ &\qquad \left.\times e^{i\pi(r+\alpha+\beta-1)/2} + \left(\frac{i}{b}\right)^n\int_0^\infty e^{ib\omega}\phi_n^{(n)}(\omega)d\omega\right\}.\end{aligned} \tag{2.12}$$

3. Mexican Hat Wavelet Transform

In this section we choose ψ to be Mexican hat wavelet and derive asymptotic expansion of the corresponding wavelet transform. The Mexican hat wavelet is defined by

$$\psi(t) = \left(1-t^2\right)e^{-t^2/2}, \tag{3.1}$$

then from [1, page 372],

$$\overline{\widehat{\psi}}(\omega) = \sqrt{2\pi}\omega^2 e^{-\omega^2/2}. \tag{3.2}$$

Now, in view of (2.5), we have

$$\begin{aligned}
\phi(\omega) &:= \hat{f}(\omega)\overline{\hat{\varphi}}(a\omega) \\
&\sim \sum_{r=0}^{\infty} b_r \omega^{r+\beta-1}\sqrt{2\pi}a^2\omega^2 e^{-a^2\omega^2/2} \\
&= \sqrt{2\pi}a^2 \sum_{r=0}^{\infty} b_r \omega^{r+\beta-1}\omega^2 \sum_{s=0}^{\infty}\left(-\frac{a^2}{2}\right)^s \frac{\omega^{2s}}{s!} \\
&= \sqrt{2\pi}a^2\omega^{\beta+1} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} b_r \left(-\frac{a^2}{2}\right)^s \frac{\omega^{2s+r}}{s!} \\
&= \sqrt{2\pi}a^2\omega^{\beta+1} \sum_{r=0}^{\infty}\sum_{j=0}^{[r/2]} \frac{b_{r-2j}}{j!}\left(-\frac{a^2}{2}\right)^j \omega^r \\
&= \sum_{r=0}^{\infty} c_r \omega^{r+\beta+1},
\end{aligned} \tag{3.3}$$

where

$$c_r = \sqrt{2\pi}a^2 \sum_{j=0}^{[r/2]} \frac{b_{r-2j}}{j!}\left(-\frac{a^2}{2}\right)^j, \tag{3.4}$$

where $[r/2]$ stands for the greatest positive integer $\leq r/2$. To ensure that $(W_\psi^+ f)(b,a)$ exists for large values of b we also impose the condition that $\hat{f}(u) = O(e^{\sigma u^2})$ for some real number $\sigma > 0$ as $u \to +\infty$. Also, from (3.3) and (2.8) we conclude that in the present case $\alpha = 3$. Therefore, from (2.12), using (3.4) we get

$$\begin{aligned}
\left(W_\psi^+ f\right)(b,a) = \frac{\sqrt{a}}{2\pi}\Bigg\{ &\sqrt{2\pi}a^2 \sum_{r=0}^{n-1}\sum_{j=0}^{[r/2]}\left(-\frac{a^2}{2}\right)^j \frac{b_{r-2j}}{j!}\Gamma(r+\beta+2)b^{-r-\beta-2} \\
&\times e^{i\pi(r+\beta+2)/2} + \left(\frac{i}{b}\right)^n \int_0^{\infty} e^{ib\omega}\phi_n^{(n)}(\omega)d\omega \Bigg\}.
\end{aligned} \tag{3.5}$$

4. Morlet Wavelet Transform

In this section we choose

$$\psi(t) = e^{i\omega_0 t - t^2/2}. \tag{4.1}$$

Then from [1, page 373],

$$\hat{\psi}(\omega) = \sqrt{2\pi}e^{-(\omega-\omega_0)^2/2}. \tag{4.2}$$

Now,

$$\begin{aligned}
\overline{\widehat{\varphi}}(a\omega) &= \sqrt{2\pi}e^{-\omega_o^2/2}e^{a\omega_0\omega}e^{-a^2\omega^2/2} \\
&= \sqrt{2\pi}e^{-\omega_o^2/2}\sum_{r=0}^{\infty}\frac{(a\omega_0\omega)^r}{r!}\sum_{s=0}^{\infty}\left(-\frac{a^2}{2}\right)^s\frac{\omega^{2s}}{s!} \\
&= \sqrt{2\pi}e^{-\omega_o^2/2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(-\frac{a^2}{2}\right)^s\frac{(a\omega_0)^r\omega^{r+2s}}{r!s!} \\
&= \sqrt{2\pi}e^{-\omega_o^2/2}\sum_{r=0}^{\infty}\sum_{j=0}^{[r/2]}\left(-\frac{a^2}{2}\right)^s\frac{(a\omega_0)^{r-2j}\omega^r}{j!(r-2j)!} \\
&= \sum_{r=0}^{\infty}A_r\omega^r,
\end{aligned} \tag{4.3}$$

where

$$A_r = \sqrt{2\pi}e^{-\omega_o^2/2}\sum_{j=0}^{[r/2]}\left(-\frac{a^2}{2}\right)^s\frac{(a\omega_0)^{r-2j}}{j!(r-2j)!}. \tag{4.4}$$

Hence

$$\begin{aligned}
\phi(\omega) &:= \widehat{f}(\omega)\overline{\widehat{\varphi}}(a\omega) \\
&\sim \sum_{r=0}^{\infty}b_r\omega^{r+\beta-1}\sum_{p=0}^{\infty}A_p\omega^p \\
&= \omega^{\beta-1}\sum_{r=0}^{\infty}b_r\omega^r\sum_{p=0}^{\infty}A_p\omega^p \\
&= \sum_{r=0}^{\infty}\left(\sum_{p=0}^{r}A_pb_{r-p}\right)\omega^{r+\beta-1} \\
&= \sum_{r=0}^{\infty}c_r\omega^{r+\beta-1},
\end{aligned} \tag{4.5}$$

where

$$c_r = \sum_{p=0}^{r}A_pb_{r-p}. \tag{4.6}$$

Also, from (2.8) and (4.5) it follows that $\alpha = 1$. Therefore, from (2.12), using (4.6) we get

$$\left(W_{\psi}^{+} f\right)(b, a) = \frac{\sqrt{a}}{2\pi} \left\{ \sum_{r=0}^{n-1} \sum_{p=0}^{r} A_p b_{r-p} \Gamma(r+\beta) b^{-r-\beta} e^{i\pi(r+\beta)/2} + \left(\frac{i}{b}\right)^n \int_0^{\infty} \phi_n^{(n)}(\omega) e^{ib\omega} d\omega \right\}. \tag{4.7}$$

5. Haar Wavelet Transform

The Haar wavelet is defined by

$$\psi(t) = \begin{cases} 1, & 0 \le t < 1/2, \\ -1, & 1/2 \le \mathrm{t} < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

whose Fourier transform [1, page 368] is

$$\widehat{\psi}(\omega) = \frac{i\left(2e^{-i\omega/2} - 1 - e^{-i\omega}\right)}{\omega}. \tag{5.2}$$

Therefore, Haar wavelet transform on half-line is given by

$$\begin{aligned} \left(W_{\psi}^{+} f\right)(b, a) &= \frac{i}{\sqrt{a}2\pi} \int_0^{\infty} e^{ib\omega} \widehat{f}(\omega) \left(\frac{1}{\omega} + \frac{e^{ia\omega}}{\omega} - \frac{2e^{ia\omega/2}}{\omega}\right) d\omega \\ &= \frac{i}{\sqrt{a}2\pi} \left\{ \int_0^{\infty} e^{ib\omega} \frac{\widehat{f}(\omega)}{\omega} d\omega + \int_0^{\infty} e^{ib(\omega+a)} \frac{\widehat{f}(\omega)}{\omega} d\omega - 2\int_0^{\infty} e^{ib(\omega+a/2)} \frac{\widehat{f}(\omega)}{\omega} d\omega \right\}. \end{aligned} \tag{5.3}$$

For $\widehat{f}(\omega)$ possessing asymptotic behavior (2.5), we have

$$\frac{\widehat{f}(\omega)}{\omega} \sim \sum_{r=0}^{\infty} a_r \omega^{r+\beta-2}. \tag{5.4}$$

Then, from (5.3) and (5.4) using formula [2, page 753]

$$M\left[e^{it}; z\right] = e^{iz\pi/2} \Gamma(z), \tag{5.5}$$

we get, for $\beta \geq 1$ and $b \longrightarrow \infty$,

$$\begin{aligned}
&\left(W_{\psi}^{+} f\right)(b, a) \\
&= \frac{i}{\sqrt{a}2\pi} \sum_{r=0}^{\infty} a_r \Gamma(r+\beta-1) e^{i\pi(r+\beta-1)/2} \times \left(b^{-r-\beta+1} + (b+a)^{-r-\beta+1} - 2(b+a/2)^{-r-\beta+1}\right) \\
&= \frac{i}{\sqrt{a}2\pi} \sum_{r=0}^{\infty} a_r \Gamma(r+\beta-1) e^{i\pi(r+\beta-1)/2} \\
&\quad \times \left\{ b^{-r-\beta+1} + \sum_{s=0}^{\infty} \binom{-r-\beta+1}{s} a^{-s} b^{-r-\beta-s+1} - 2\sum_{s=0}^{\infty} \binom{-r-\beta+1}{s} \left(\frac{a}{2}\right)^{-s} b^{-r-\beta-s+1} \right\}.
\end{aligned} \tag{5.6}$$

6. Asymptotic Expansion for Small b

In this section we assume that asymptotic expansions of $\widehat{f}(\omega)$ and $\widehat{\varphi}(\omega)$ as $\omega \longrightarrow \infty$ are known and then derive asymptotic expansion of $(W_{\psi}^{+} f)(b, a)$ as $b \longrightarrow 0+$ for fixed $a > 0$, using the following [4, Therorem 14, page 323].

Theorem 6.1. *Let f be a locally integrable function on $(0, \infty)$ and let $f(t)$ possess an asymptotic expansion of the form*

$$f(t) \sim \sum_{s=0}^{n-1} a_s^{*} t^{-s-\alpha} + f_n(t), \quad \text{as } t \longrightarrow \infty, \tag{6.1}$$

where $0 < \alpha < 1$. Then for small values of ω,

$$\int_0^{\infty} f(t) e^{it\omega} dt = e^{-i\alpha\pi/2} \sum_{s=0}^{n-1} (-i)^{s-1} a_s^{*} \Gamma(1-s-\alpha) \omega^{s+\alpha-1} - \sum_{s=1}^{n} c_s^{*} (-i\omega)^{s-1} + R_n(\omega), \tag{6.2}$$

where

$$\begin{aligned}
c_s^{*} &= \frac{(-1)^s}{(s-1)!} M[f; s], \\
R_n(\omega) &= (-i\omega)^n \int_0^{\infty} e^{i\omega t} f_{n,n}(t) dt
\end{aligned} \tag{6.3}$$

with

$$f_{n,n}(t) = \frac{(-1)^n}{(n-1)!} \int_t^{\infty} (\tau - t)^{n-1} f_n(\tau) d\tau. \tag{6.4}$$

Let

$$\widehat{f}(\omega) \sim \sum_{s=0}^{\infty} b_s \omega^{-s-\alpha} \quad \text{as } \omega \longrightarrow \infty,$$
$$\overline{\widehat{\varphi}(\omega)} \sim \sum_{r=0}^{\infty} a_s \omega^{-r-\beta} \quad \text{as } \omega \longrightarrow \infty. \tag{6.5}$$

Then, writing $a'_r = a^{-r-\alpha}\ a_r$, *we have*

$$\overline{\widehat{\varphi}(a\omega)}\widehat{f}(\omega) \sim \sum_{r=0}^{\infty} a'_r \omega^{-r-\alpha} \sum_{s=0}^{\infty} b_s \omega^{-s-\beta} = \omega^{-\alpha-\beta} \sum_{r=0}^{\infty} c_r \omega^{-r}, \tag{6.6}$$

where

$$c_r = a'_0 b_r + \cdots + a'_r b_0 = \sum_{m=0}^{r} a'_m b_{r-m} = \sum_{m=0}^{r} a_m a^{-m-\alpha} b_{r-m}. \tag{6.7}$$

Now, for fixed $a \in R_+$, write

$$\begin{aligned} \phi(\omega) &:= \overline{\widehat{\varphi}(a\omega)}\widehat{f}(\omega) \\ &\sim \sum_{r=0}^{\infty} c_r \omega^{-r-\alpha-\beta}, \quad \omega \longrightarrow \infty \\ &= \sum_{r=0}^{n-1} c_r \omega^{-r-\alpha-\beta} + \phi_n(\omega), \end{aligned} \tag{6.8}$$

where $0 < \alpha + \beta < 1$.

Then, using (2.2), (6.2), and (6.8), we find asymptotic expansion of wavelet transform for small value of b:

$$\left(W^{+}_{\varphi} f\right)(b,a) = e^{-i\pi(\alpha+\beta)/2} \sum_{s=0}^{n-1} (-i)^{s-1} c_s \Gamma(1-s-\alpha-\beta) b^{s+\alpha+\beta-1} - \sum_{s=1}^{n} D^{*}_s (-ib)^{s-1} + R^{*}_n(b), \tag{6.9}$$

where

$$\begin{aligned} D^{*}_s &= \frac{(-1)^s}{(s-1)!} M[\phi; s], \\ R^{*}_n(\omega) &= (-ib)^n \int_0^{\infty} e^{ibt} \phi_{n,n}(t)\,dt \end{aligned} \tag{6.10}$$

with

$$\phi_{n,n}(t) = \frac{(-1)^n}{(n-1)!}\int_t^{\infty}(\tau - t)^{n-1}\phi_n(\tau)d\tau. \tag{6.11}$$

7. Asymptotic Expansion for Small b Continued

In this section we assume that asymptotic expansions of f and ψ are known, instead of $\widehat{f}(\omega)$ and $\widehat{\psi}(\omega)$ as in previous sections. Then as in [2, page 753] we get asymptotic expansions of $\widehat{f}(\omega)$ and $\widehat{\psi}(\omega)$ as $\omega \to \infty$. On the other hand, in (2.3) and (2.5) their behaviors near the origin were known, that yielded the asymptotic expansion of $(W_{\psi}^{+}f)(b,a)$ as $b \to \infty$. However, in this case, following [4, pages 321–323] we can obtain asymptotic expansion of $(W_{\psi}^{+}f)(b,a)$as $b \to 0+$.

Let

$$\begin{gathered} f(t) \sim \sum_{s=0}^{\infty} b_s' t^{s+\alpha'-1}, \quad \text{as } \ t \longrightarrow 0, \ 0 < \alpha' < 1, \\ \psi(t) \sim \sum_{r=0}^{\infty} a_r' t^{r+\beta'-1}, \quad t \longrightarrow 0. \end{gathered} \tag{7.1}$$

Now, using (1.12)

$$\begin{aligned} \widehat{f}(\omega) &= \int_0^{\infty} e^{-it\omega} f(t)dt \\ &\sim \sum_{s=0}^{\infty} b_s' e^{-i\pi(s+\alpha')/2}\Gamma(s+\alpha')\omega^{-s-\alpha'}, \quad \text{as } \omega \longrightarrow +\infty. \end{aligned} \tag{7.2}$$

Similarly,

$$\begin{aligned} \widehat{\psi}(\omega) &= \int_0^{\infty} e^{-it\omega} \psi(t)dt \\ &\sim \sum_{r=0}^{\infty} a_r' e^{-i\pi(r+\beta')/2}\Gamma(r+\beta')\omega^{-r-\beta'} \quad \text{as } \ \omega \longrightarrow +\infty. \end{aligned} \tag{7.3}$$

Then

$$\phi(\omega) := \widehat{f}(\omega)\overline{\widehat{\psi}}(a\omega) \sim \sum_{r=0}^{\infty} d_r \omega^{-r-\alpha'-\beta'}, \tag{7.4}$$

where

$$d_r = e^{-i\pi(-r+\alpha'-\beta')} \sum_{m=0}^{r} b_m' a^{-(r-m)-\beta'} a_{r-m}' e^{-i\pi 2m}\Gamma(\alpha'+m)\Gamma(\beta'+r-m). \tag{7.5}$$

Let us set

$$\phi(\omega) = \sum_{r=0}^{n-1} d_r \omega^{-r-\alpha'-\beta'} + \phi_n(\omega), \tag{7.6}$$

where $0 < \alpha' + \beta' < 1$.

Assume that $\phi(\omega) = \widehat{f}(\omega)\overline{\widehat{\psi}}(a\omega)$ integrable is locally on $(0, \infty)$. Then applying (6.2) to (2.2) with $\phi(\omega)$ given by (7.6), finally we get

$$\left(W_\psi^+ f\right)(b, a) = e^{-i\pi(\alpha'+\beta')/2} \sum_{s=0}^{n-1} (-i)^{s-1} d_s \Gamma(1 - s - \alpha' - \beta') b^{s+\alpha'+\beta'-1} - \sum_{s=1}^{n} D_s (-ib)^{s-1} + R_n^*(b), \tag{7.7}$$

where

$$\begin{aligned} D_s &= \frac{(-1)^s}{(s-1)!} M[\phi\ ; s], \\ R_n^*(b) &= (-ib)^n \int_0^\infty e^{ibt} \phi_{n,n}(t) dt \end{aligned} \tag{7.8}$$

with

$$\phi_{n,n}(t) = \frac{(-1)^n}{(n-1)!} \int_t^\infty (\tau - t)^{n-1} \phi_n(\tau) d\tau. \tag{7.9}$$

Remark 7.1. We observe that if we assume the asymptotic expansions $f(t)$ and $\psi(t)$ as $t \to \infty$ and derive asymptotic expansions of $\widehat{f}(\omega)$ and $\widehat{\psi}(t)$ as $\omega \to \infty$, then formula (1.2) gives asymptotic expansion of $(W_\psi^+ f)(b, a)$ as $b \to 0+$ for fixed $a > 0$. The aforesaid technique does not yield asymptotic expansion of $(W_\psi^+ f)(b, a)$ as $b \to \infty$ using (1.2). However, if one uses the form (1.1) of the wavelet transform and applies Li and Wong-technique involving a theory of noncommutative convolution [5], asymptotic expansion of $(W_\psi^+ f)(b, a)$ as $b \to \infty$ can be obtained. This gives rise to a complicated form of the asymptotic expansion and needs separate treatment [6].

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References

[1] L. Debnath, *Wavelet Transforms and Their Applications*, Birkhäuser, Boston, Mass, USA, 2002.
[2] R. Wong, "Explicit error terms for asymptotic expansions of Mellin convolutions," *Journal of Mathematical Analysis and Applications*, vol. 72, no. 2, pp. 740–756, 1979.

[3] R. S. Pathak and A. Pathak, “Asymptotic expansion for the wavelet transform with error term,” communicated.
[4] R. Wong, *Asymptotic Approximations of Integrals*, Computer Science and Scientific Computing, Academic Press, Boston, Mass, USA, 1989.
[5] X. Li and R. Wong, “Error bounds for asymptotic expansions of Laplace convolutions,” *SIAM Journal on Mathematical Analysis*, vol. 25, no. 6, pp. 1537–1553, 1994.
[6] R. S. Pathak, “Asymptotic expansion of the wavelet transform,” in *Industrial and Applied Mathematics*, A. H. Siddiqui and K. Ahmad, Eds., pp. 43–50, Narosa, New Delhi, India, 1998.