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Research Article

Commutators and Squares in Free Nilpotent Groups

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Recommended by Howard Bell

In a free group no nontrivial commutator is a square. And in the free group $F_2 = F(x_1, x_2)$ freely generated by x_1, x_2 the commutator $[x_1, x_2]$ is never the product of two squares in F_2, although it is always the product of three squares. Let $F_{2,3} = \langle x_1, x_2 \rangle$ be a free nilpotent group of rank 2 and class 3 freely generated by x_1, x_2. We prove that in $F_{2,3} = \langle x_1, x_2 \rangle$, it is possible to write certain commutators as a square. We denote by $Sq(\gamma)$ the minimal number of squares which is required to write γ as a product of squares in group G. And we define $Sq(G) = \sup\{Sq(\gamma); \gamma \in G'\}$. We discuss the question of when the square length of a given commutator of $F_{2,3}$ is equal to 1 or 2 or 3. The precise formulas for expressing any commutator of $F_{2,3}$ as the minimal number of squares are given. Finally as an application of these results we prove that $Sq(F'_{2,3}) = 3$.

1. Introduction

Schützenberger [1] proved that in a free group the equation

$$[x, y] = z^r, \quad r \geq 2 \tag{1.1}$$

implies $z = 1$; that is, no nontrivial commutator is a proper power. It means that it is impossible to write $[x, y]$ as an rth powers where $r \geq 2$. Lyndon and Newman [2] have shown that in the free group $F_2 = F(x_1, x_2)$ freely generated by x_1, x_2, the commutator $[x_1, x_2]$ is never a product of two squares in F_2, although it is always the product of three squares. In [3] we proved that for an odd integer k, $[x_2, x_1]^k$ is not a product of two squares in F_2, and it is the product of three squares. Put $w = [x_2, x_1]$ and $k = 2n + 1$. We presented the following expression of $[x_2, x_1]^{2n+1}$ as a product of the minimal number of squares:

$$[x_2, x_1]^{2n+1} = \left((w^n x_2 x_1)^{w^n}\right)^2 \left(w^n x_1^{-1}\right)^2 \left(\left(w^{-n} x_2^{-1}\right)^{x_1}\right)^2. \tag{1.2}$$

Recently Abdollahi [4] generalized these results as the following theorem.

Theorem 1.1 (Abdollahi [4]). *Let F be a free group with a basis of distinct elements $x_1, \ldots, x_{2n}$, and N any odd integer. Then there exist elements $u_1, \ldots, u_m$ in F such that*

$$([x_1, x_2] \cdots [x_{2n-1}, x_{2n}])^N = u_1^2 \ldots, u_m^2 \tag{1.3}$$

if and only if $m \geq 2n + 1$.

Definition 1.2. Let G be a group and $\gamma \in G'$. The minimal number of squares which is required to write γ as a product of squares in G is called *the square length of* γ and denoted by $\mathrm{Sq}(\gamma)$. And we define $\mathrm{Sq}(G) = \sup\{\mathrm{Sq}(\gamma);\ \gamma \in G'\}$.

We prove that in the free nilpotent group $F_{2,3} = \langle x_1, x_2 \rangle$ of rank 2 and class 3 freely generated by x_1, x_2 it is possible to write certain nontrivial commutators as a proper power. We consider certain equations over free group $F_{2,3}$. Using this, we find $\mathrm{Sq}[h, g]$ where $h,\ g \in F_{2,3}$. Then we prove that $\mathrm{Sq}(F'_{2,3}) = 3$.

2. Main Results

We will prove the following theorems.

Theorem 2.1. *Let $F_{2,3} = \langle x_1, x_2 \rangle$ be a free nilpotent group of rank 2 and class 3 freely generated by x_1, x_2. Then $\mathrm{Sq}(F'_{2,3}) = 3$.*

An application of Theorem 2.1 is displayed in the next result.

Corollary 2.2. *In a free nilpotent group of rank 2 and class 3, it is possible to find nontrivial solutions for the equation*

$$[x, y] = z^r, \quad r \geq 2. \tag{2.1}$$

We will use the following well-known identities regarding groups which are nilpotent of class 3.

Lemma 2.3. *Let $G = \langle x, y \rangle$ be nilpotent of class 3. Then, for all integers r, s the following hold:*

$$\begin{aligned} [x^r, y] &= [x, y]^r [x, y, x]^{r(r-1)/2}, \\ [x^r, y^s] &= [x, y]^{rs} [x, y, x]^{rs(r-1)/2} [x, y, y]^{rs(s-1)/2}. \end{aligned} \tag{2.2}$$

3. Proofs of the Main Result

Proof of Theorem 2.1. Let h, g be any two elements of $F_{2,3} \setminus \gamma_3(F_{2,3})$. First we study the form of the element $[h, g]$. Since $\gamma_3(F_{2,3})$ lies in the center of $F_{2,3}$ we may express h as $x_1^{r_1} x_2^{r_2} [x_2, x_1]^{\beta}$ and g as $x_1^{s_1} x_2^{s_2} [x_2, x_1]^{\alpha}$. We have shown in [5] that.

$$[h, g] = [x_2, x_1]^{\lambda} [x_2, x_1, x_2]^{\mu} [x_2, x_1, x_1]^{\nu}, \tag{3.1}$$

where

$$\begin{gathered} \lambda = r_2 s_1 - r_1 s_2, \\ \mu = \frac{s_1 r_2 (r_2 - 1)}{2} - \frac{r_1 s_2 (s_2 - 1)}{2} - r_1 r_2 s_2 + r_2 s_1 s_2 + \beta s_2 - \alpha r_2, \\ \nu = \frac{r_2 s_1 (s_1 - 1)}{2} - \frac{s_2 r_1 (r_1 - 1)}{2} + \beta s_1 - \alpha r_1. \end{gathered} \tag{3.2}$$

Now we consider the equation $[h, g] = u^2 (\diamond)$. The element u has a presentation of the following form:

$$u = x_1^{r'_1} x_2^{r'_2} [x_2, x_1]^{\alpha'} [x_2, x_1, x_2]^{\gamma'} [x_2, x_1, x_1]^{\beta'}, \tag{3.3}$$

where $r'_1,\ r'_2,\ \alpha',\ \beta',$ and γ' are unique integer elements.

Lemma 2.3 implies that

$$\begin{aligned} u^2 = {} & x_1^{2r'_1} x_2^{2r'_2} [x_2, x_1]^{2\alpha' + r'_1 r'_2} [x_2, x_1, x_2]^{2\gamma' + \alpha' r'_2 + r'_1 r'_2 (r'_2 - 1)/2 + r'_1 {r'_2}^2} \\ & \times [x_2, x_1, x_1]^{2\beta' + \alpha' r'_1 + r'_1 r'_2 (r'_1 - 1)/2}. \end{aligned} \tag{3.4}$$

Thus equation $(\diamond)$ holds in $F_{2,3}$ if and only if

$$r'_1 = r'_2 = 0, \qquad 2\alpha' = \lambda, \qquad 2\beta' = \nu, \qquad 2\gamma' = \mu. \tag{3.5}$$

In particular the equation $(\diamond)$ has a solution only if $\lambda, \mu,$ and ν are even. Put $c_1 = \alpha r_2 - \beta s_2,\ c_2 = \alpha r_1 - \beta s_1$, then

$$\alpha = \frac{\begin{vmatrix} c_1 & -s_2 \\ c_2 & -s_1 \end{vmatrix}}{\begin{vmatrix} r_2 & -s_2 \\ r_1 & -s_1 \end{vmatrix}} = \frac{s_1 c_1 - s_2 c_2}{2\alpha'}, \qquad \beta = \frac{\begin{vmatrix} r_2 & c_1 \\ r_1 & c_2 \end{vmatrix}}{-2\alpha'} = \frac{r_1 c_1 - r_2 c_2}{2\alpha'}. \tag{3.6}$$

Hence we need $s_1 c_1 - s_2 c_2$ and $r_1 c_1 - r_2 c_2$ to be even. We have the following two cases.

Case 1. If $r_1 s_2 = 2k$, for some integer k, then $r_2 s_1 = 2\alpha' + 2k$, and hence $r_2 s_1 \equiv_2 0$. And we have

$$c_1 = -\alpha' + \alpha'(r_2 + s_2) - kr_2 + (\alpha' + k)s_2 - 2\gamma',$$
$$c_2 = \alpha' + (\alpha' + k)s_1 - kr_1 - 2\beta'. \tag{3.7}$$

Further,

$$0 \equiv_2 s_1 c_1 + s_2 c_2 \equiv_2 \alpha'(s_1 + s_1 s_2 + s_2),$$
$$0 \equiv_2 r_1 c_1 + r_2 c_2 \equiv_2 \alpha'(r_1 + r_1 r_2 + r_2). \tag{3.8}$$

Now if α' is an odd integer, then we have

$$0 \equiv_2 r_1 + r_1 r_2 + r_2 \equiv_2 s_1 + s_1 s_2 + s_2. \tag{3.9}$$

It follows that r_1, r_2, s_1, and s_2 are all even. Hence $\lambda = r_2 s_1 - r_1 s_2$ is divisible by 4. But $\lambda = 2\alpha'$ implies that $\alpha' \equiv_2 0$, a contradiction. Hence in Case 1 we have $\alpha' \equiv_2 0$ and $\lambda \equiv_4 0$.

Now $r_1 s_2 = 2k$, and $r_2 s_1 = 2\alpha' + 2k$ imply that

$$\mu = \alpha' r_2 - kr_2 - \alpha' + ks_2 + 2\alpha' s_2 + \beta s_2 - \alpha r_2 = 2\gamma',$$
$$\nu = \alpha' s_1 + ks_1 - \alpha' - kr_1 + \beta s_1 - \alpha r_1 = 2\beta'. \tag{3.10}$$

Hence we have

$$\mu \equiv_2 r_2(k + \alpha) + s_2(k + \beta),$$
$$\nu \equiv_2 r_1(k + \alpha) + s_1(k + \beta). \tag{3.11}$$

And we have the following cases.

Subcase 1.1. If $r_1 \equiv_2 r_2 \equiv_2 s_1 \equiv_2 s_2 \equiv_2 0$, then it is clear that for any integer numbers α and β we have;

$$\lambda \equiv_4 0, \qquad \mu \equiv_2 \nu \equiv_2 0. \tag{3.12}$$

And the equation $(\diamond)$ has solution.

Subcase 1.2. If $r_1 \equiv_2 r_2 \equiv_2 s_1 \equiv_2 0$ and $s_2 \equiv_2 1$, then $r_1 s_2 \equiv_4 \lambda \equiv_4 0$. We have the following two cases.

(1.2.1) If $r_1 \equiv_4 0$, then we have $\lambda \equiv_4 0$. Also from $r_1 s_2 = 2k$, it follows that $k \equiv_2 0$. Now if we choose $\beta \equiv_2 0$, then from (3.11) it follows that $\mu \equiv_2 0$ and $\nu \equiv_2 0$ for any $\alpha \in \mathbb{Z}$. And in this case the equation $(\diamond)$ has a solution.

(1.2.2) If $r_1 \equiv_4 2$, then $\lambda \equiv_4 2$, and the equation $(\diamond)$ has no solution.

Hence in Subcase 1.2 if $r_1 \equiv_4 0$, $r_2 \equiv_2 s_1 \equiv_2 0$, $s_2 \equiv_2 1$, and $\beta \equiv_2 0$, for any $\alpha \in \mathbb{Z}$ the equation $(\diamond)$ has a solution.

Subcase 1.3. If $r_1 \equiv_2 r_2 \equiv_2 s_2 \equiv_2 0$ and $s_1 \equiv_2 1$, then $s_1 r_2 \equiv_4 \lambda \equiv_4 0$. We have two cases.

(1.3.1) If $r_2 \equiv_4 0$, then $\lambda \equiv_4 0$. Since $r_1 s_2 = 2k$, and $r_1 \equiv_2 s_2 \equiv_2 0$, hence $k \equiv_2 0$. Now if we identify $\beta \equiv_2 0$, then from (3.11) it follows that $\mu \equiv_2 0$ and $\nu \equiv_2 0$. And the equation $(\diamond)$ has a solution.

(1.3.2) If $r_2 \equiv_4 2$, then $\lambda \equiv_4 2$, and the equation $(\diamond)$ has no solution.

Hence in Subcase 1.3 if $r_1 \equiv_2 s_2 \equiv_2 0$, $r_2 \equiv_4 0$, and $\beta \equiv_2 0$, for any $\alpha \in \mathbb{Z}$ the equation $(\diamond)$ has a solution.

Subcase 1.4. If $r_1 \equiv_2 r_2 \equiv_2 0$ and $s_1 \equiv_2 s_2 \equiv_2 1$, then we have the following two cases.

(1.4.1) If $r_1 \equiv_4 0$, then $\lambda \equiv_4 s_1 r_2 \equiv_4 0$. Now $s_1 \equiv_2 1$ implies $r_2 \equiv_4 2$. If we choose $\beta \equiv_2 0$, then for any $\alpha \in \mathbb{Z}$ the equation $(\diamond)$ has a solution. Hence if $r_1 \equiv_4 r_2 \equiv_4 0$, $s_1 \equiv_2 s_2 \equiv_2 1$, and $\beta \equiv_2 0$, then for any $\alpha \in \mathbb{Z}$, the equation $(\diamond)$ has a solution.

(1.4.2) If$r_1 \equiv_4 2$. Since $\lambda \equiv_4 s_1 r_2 - r_1 s_2 \equiv_4 0$, hence $r_2 \equiv_4 2$. If we identify $\beta \equiv_2 1$, for any $\alpha \in \mathbb{Z}$ then $\mu \equiv_2 \nu \equiv_2 0$. And the equation $(\diamond)$ has a solution.

Subcase 1.5. If $r_1 \equiv_2 s_1 \equiv_2 r_2 \equiv_2 0$, and $r_2 \equiv_2 1$, we have the following two cases.

(1.5.1) If $s_1 \equiv_4 0$, then $\lambda \equiv_4 0$. Since $r_1 s_2 = 2k$, hence $k \equiv_2 0$. If we identify $\alpha \equiv_2 0$, for any $\beta \in \mathbb{Z}$, then $\mu \equiv_2 \nu \equiv_2 0$. And the equation $(\diamond)$ has a solution.

(1.5.2) If $s_1 \equiv_4 2$, then $\lambda \equiv_4 2$. And the equation $(\diamond)$ has no solution. Hence in this case only if $s_1 \equiv_4 0$, the equation $(\diamond)$ has a solution.

Subcase 1.6. If $r_1 \equiv_2 s_1 \equiv_2 0$ and $r_2 \equiv_2 s_2 \equiv_2 1$, then similar to Case 4, if $r_1 \equiv_4 s_1 \equiv_4 0$ or $r_1 \equiv_4 s_1 \equiv_4 2$ then $\lambda \equiv_4 0$. And for any $\alpha \equiv_2 \beta$, $\mu \equiv_2 \nu \equiv_2 0$, the equation $(\diamond)$ has a solution.

Subcase 1.7. If $r_1 \equiv_2 s_2 \equiv_2 0$ and $r_2 \equiv_2 s_1 \equiv_2 1$, then $\lambda \equiv_2 1$. Hence the equation $(\diamond)$ has no solution.

Subcase 1.8. If $r_1 \equiv_2 0$ and $r_2 \equiv_2 s_2 \equiv_2 s_1 \equiv_2 1$, then $\lambda \equiv_2 1$. Hence the equation $(\diamond)$ has no solution.

Subcase 1.9. If $r_1 \equiv_2 1$ and $r_2 \equiv_2 s_2 \equiv_2 s_1 \equiv_2 0$, we have two cases.

(1.9.1) If $s_2 \equiv_4 0$, then $\lambda \equiv_4 0$. Since $r_1 s_2 = 2k$, hence $k \equiv_2 0$. If we identify $\alpha \equiv_2 0$, for any $\beta \in \mathbb{Z}$, then $\mu \equiv_2 \nu \equiv_2 0$. And the equation $(\diamond)$ has a solution.

(1.9.2) If $s_2 \equiv_4 2$, then $\lambda \equiv_4 2$. And the equation $(\diamond)$ has no solution.

Subcase 1.10. If $r_1 \equiv_2 s_2 \equiv_2 1$ and $r_2 \equiv_2 s_1 \equiv_2 0$, then $r_1 s_2 \equiv_2 1$. And the equation $(\diamond)$ has no solution.

Subcase 1.11. If $r_1 \equiv_2 s_1 \equiv_2 1$ and $r_2 \equiv_2 s_2 \equiv_2 0$, then similar to Subcase 1.6, if $r_2 \equiv_4 s_2 \equiv_4 0$ or $r_2 \equiv_4 s_2 \equiv_4 2$ then $\lambda \equiv_4 0$. And for any $\alpha \equiv_2 \beta$, $\mu \equiv_2 \nu \equiv_2 0$, the equation $(\diamond)$ has a solution.

Subcase 1.12. If $r_1 \equiv_2 s_1 \equiv_2 s_2 \equiv_2 1$ and $r_2 \equiv_2 0$, then $r_1 s_2 \equiv_2 1$. And the equation $(\diamond)$ has no solution.

Subcase 1.13. If $r_1 \equiv_2 r_2 \equiv_2 1$ and $s_1 \equiv_2 s_2 \equiv_2 0$, then we have two cases.

(1.13.1) If $s_1 \equiv_4 0$, then $\lambda \equiv_4 0$ implies $s_2 \equiv_2 0$. If we identify $\alpha \equiv_2 0$, for any $\beta \in \mathbb{Z}$, the equation $(\diamond)$ has a solution.

(1.13.2) If $s_1 \equiv_4 2$, then $s_2 \equiv_4 2$. And if $\alpha \equiv_2 1$, for any $\beta \in \mathbb{Z}$, the equation $(\diamond)$ has a solution.

Subcase 1.14. If $r_1 \equiv_2 r_2 \equiv_2 s_2 \equiv_2 1$ and $s_1 \equiv_2 0$, then $r_1 s_2 \equiv_2 1$. In this case the equation $(\diamond)$ has no solution.

Subcase 1.15. If $r_1 \equiv_2 r_2 \equiv_2 s_1 \equiv_2 1$ and $s_1 \equiv_2 0$, then $r_2 s_1 \equiv_2 1$. In this case the equation $(\diamond)$ has no solution.

Case 2. If $r_1 s_2 \equiv_2 1$. Since $\lambda = s_1 r_2 - r_1 s_2 \equiv_2 0$, hence $r_1 \equiv_2 r_2 \equiv_2 s_1 \equiv_2 s_2 \equiv_2 1$. If we identify $\alpha \equiv_2 \beta$, then $\mu \equiv_2 \nu \equiv_2 0$. In this case the equation $(\diamond)$ has a solution.

Hence we show that in the following twelve cases the equation $(\diamond)$ has solution. And $\mathrm{Sq}[h, g] = 1$.

(1) $r_1 \equiv_2 s_1 \equiv_2 r_2 \equiv_2 s_2 \equiv_2 0$, for all α, β.

(2) $s_1 \equiv_2 r_2 \equiv_2 0,\ s_2 \equiv_2 1,\ r_1 \equiv_4 0$, for all $\alpha,\ \beta \equiv_2 0$.

(3) $r_1 \equiv_2 s_2 \equiv_2 0,\ s_1 \equiv_2 1,\ r_2 \equiv_4 0$, for all $\alpha,\ \beta \equiv_2 0$.

(4) $s_1 \equiv_2 s_2 \equiv_2 1,\ r_1 \equiv_4 r_2 \equiv_2 0$, for all $\alpha,\ \beta \equiv_2 0$.

(5) $s_1 \equiv_2 s_2 \equiv_2 1,\ r_1 \equiv_4 r_2 \equiv_4 2$, for all $\alpha,\ \beta \equiv_2 0$.

(6) $r_1 \equiv_2 s_2 \equiv_2 0,\ r_2 \equiv_2 1,\ s_1 \equiv_4 0,\ \alpha \equiv_2 0$, for all β.

(7) $r_1 \equiv_2 s_1 \equiv_2 1,\ r_2 \equiv_2 s_2 \equiv_2 0,\ \alpha \equiv_2 \beta$.

(8) $r_1 \equiv_2 s_1 \equiv_2 0,\ r_2 \equiv_2 s_2 \equiv_2 1,\ \alpha \equiv_2 \beta$.

(9) $r_1 \equiv_2 1,\ r_2 \equiv_2 s_1 \equiv_2 0,\ s_2 \equiv_4 0,\ \alpha \equiv_2 0$, for all β.

(10) $r_1 \equiv_2 r_2 \equiv_2 1,\ s_1 \equiv_4 s_2 \equiv_4 0,\ \alpha \equiv_2 0$, for all β.

(11) $r_1 \equiv_2 r_2 \equiv_2 1,\ s_1 \equiv_4 s_2 \equiv_4 2,\ \alpha \equiv_2 1$, for all β.

(12) $r_1 \equiv_2 s_1 \equiv_2 r_2 \equiv_2 s_2 \equiv_2 1,\ \alpha \equiv_2 \beta$.

And more precisely we have

$$[h, g] = \left([x_2, x_1]^{\lambda/2}[x_2, x_1, x_2]^{\mu/2}[x_2, x_1, x_1]^{\nu/2}\right)^2. \tag{3.13}$$

Now in the following ten cases the equation $(\diamond)$ has no solution.

(13) $r_2 \equiv_2 s_1 \equiv_2 0,\ s_2 \equiv_2 1,\ r_1 \equiv_4 2$.

(14) $r_1 \equiv_2 s_2 \equiv_2 0,\ s_1 \equiv_2 1,\ r_2 \equiv_4 2$.

(15) $r_1 \equiv_2 s_2 \equiv_2 0,\ r_2 \equiv_2 1,\ s_1 \equiv_4 2$.

(16) $r_2 \equiv_2 s_1 \equiv_2 0,\ r_1 \equiv_2 1,\ s_2 \equiv_4 2$.

(17) $r_1 \equiv_2 s_2 \equiv_2 0,\ r_2 \equiv_2 s_1 \equiv_2 1$.

(18) $r_1 \equiv_2 s_2 \equiv_2 1,\ r_2 \equiv_2 s_1 \equiv_2 0$.

(19) $r_1 \equiv_2 s_1 \equiv_2 s_2 \equiv_2 1,\ r_2 \equiv_2 0$.

(20) $r_1 \equiv_2 r_2 \equiv_2 s_2 \equiv_2 1,\ s_1 \equiv_2 0$.

(21) $r_1 \equiv_2 s_1 \equiv_2 r_2 \equiv_2 1,\ s_2 \equiv_2 0$.

(22) $r_2 \equiv_2 s_1 \equiv_2 s_2 \equiv_2 1,\ r_1 \equiv_2 0$.

We consider the equation $[h,g] = u_1^2 u_2^2$ ($\Diamond$). Suppose that the equation ($\Diamond$) has a nontrivial solution(u_1, u_2). The elements u_1 and u_2 have a representation of the following forms:

$$\begin{aligned} u_1 &= x_1^{r_{11}} x_2^{r_{21}} [x_2, x_1]^{\alpha_1} [x_2, x_1, x_1]^{\beta_1} [x_2, x_1, x_2]^{\gamma_1}, \\ u_2 &= x_1^{r_{12}} x_2^{r_{22}} [x_2, x_1]^{\alpha_2} [x_2, x_1, x_1]^{\beta_2} [x_2, x_1, x_2]^{\gamma_2}, \end{aligned} \tag{3.14}$$

where $r_{ij}, \alpha_i, \beta_i,$ and γ_i are unique integer numbers. By applying Lemma 2.3 one obtains

$$\begin{aligned} u_i^2 &= x_1^{2r_{1i}} x_2^{2r_{2i}} [x_2, x_1]^{2\alpha_i + r_{1i} r_{2i}} \\ &\quad \times [x_2, x_1, x_1]^{2\beta_i + \alpha_i r_{1i} + r_{1i} r_{2i}((r_{1i}-1)/2)} \\ &\quad \times [x_2, x_1, x_2]^{2\gamma_i + \alpha_i r_{2i} + r_{1i} r_{2i} (\frac{r_{2i}-1}{2}) + r_{1i} r_{2i}^2}. \end{aligned} \tag{3.15}$$

Hence

$$\begin{aligned} u_1^2 u_2^2 = x_1^{2(r_{11}+r_{12})} & x_2^{2(r_{21}+r_{122})} [x_2, x_1]^{2(\alpha_1+\alpha_2) + r_{11} r_{21} + r_{12} r_{22} + 4 r_{21} r_{12}} \\ & \times [x_2, x_1, x_2]^{n_1 + n_2 + 2k_1 r_{22} + 4 r_{21} r_{12}((2r_{21}-1)/2) + 8 r_{21} r_{12} r_{22}} \\ & \times [x_2, x_1, x_1]^{m_1 + m_2 + 2k_1 r_{12} + 4 r_{21} r_{12}((2r_{12}-1)/2)}, \end{aligned} \tag{3.16}$$

where for $i = 1, 2$,

$$\begin{aligned} k_i &= 2\alpha_i + r_{1i} r_{2i}, \\ m_i &= 2\beta_i + \alpha_i r_{1i} + r_{1i} r_{2i} \left(\frac{r_{1i}-1}{2} \right), \\ n_i &= 2\gamma_i + \alpha_i r_{2i} + r_{1i} r_{2i} \left(\frac{r_{2i}-1}{2} \right) + r_{1i} r_{2i}^2. \end{aligned} \tag{3.17}$$

Hence equation ($\Diamond$) holds if

$$\begin{aligned} r_{11} &= -r_{12}, \qquad r_{21} = -r_{21}, \\ \lambda &= 2(\alpha_1 + \alpha_2) - 2r_{11} r_{21}, \\ \mu &= 2(\gamma_1 + \gamma_2) + r_{21}(\alpha_1 - \alpha_2) - 2k_1 r_{21} + r_{11} r_{21}(-4r_{21} + 1), \\ \nu &= 2(\beta_1 + \beta_2) + r_{11}(\alpha_1 - \alpha_2) - 2k_1 r_{11} + r_{11} r_{21}(4r_{11} + 1). \end{aligned} \tag{3.18}$$

Note that second equation gives $\lambda \equiv_2 0$; hence equation ($\Diamond$) has nontrivial solution only if $\lambda \equiv_2 0$. In particular in the cases from (17) to (22), since λ is odd, the equation has no solution and $\mathrm{Sq}[h,g] = 3$.

Finally it remains to consider the cases from (13) to (16). In these cases we have $\lambda \equiv_4 2$. And we prove that if $\nu \equiv_2 1$, then $\mu \equiv_2 0$. It is clear that $\nu \equiv_2 1$ implies $m_1 + m_2 \equiv_2 1$. Hence $r_{11}(\alpha_1 + \alpha_2 + r_{21}) \equiv_2 1$. In particular $r_{11} \equiv_2 1$ and $\alpha_1 + \alpha_2 + r_{21} \equiv_2 1$. Now we have

$$\begin{aligned}\mu &\equiv_2 n_1 + n_2 \equiv_2 \alpha_1 r_{21} + r_{11} r_{21}\left(\frac{r_{21}-1}{2}\right) + r_{11} r_{21}^2 \\ &\quad + \alpha_2 r_{22} + r_{12} r_{22}\left(\frac{r_{22}-1}{2}\right) + r_{12} r_{22}^2 \\ &\equiv_2 (1 + r_{12}) r_{12} \equiv_2 0.\end{aligned} \tag{3.19}$$

Now in the cases from (13) and (15), we have $\nu \equiv_2 1$. Hence $\mu \equiv_2 0$. And if we identify:

$$\begin{gathered} r_{11} = -s_1 + 1, \qquad r_{12} = s_1 - 1, \qquad r_{22} = -r_{21} = 0, \\ \alpha_1 = \beta_2 = \gamma_2 = 0, \qquad \alpha_2 = \frac{\lambda}{2}, \qquad \beta_1 = \frac{\nu + r_{11}\alpha_2}{2}, \qquad \gamma_1 = \frac{\mu}{2}. \end{gathered} \tag{3.20}$$

then for the elements

$$\begin{gathered} u_1 = x_1^{-s+1} [x_2, x_1, x_1]^{(\nu + r_{11}(\lambda/2))/2} [x_2, x_1, x_2]^{\mu/2}, \\ u_2 = x_1^{s_1 - 1} [x_2, x_1]^{\lambda/2}. \end{gathered} \tag{3.21}$$

we have $[h,g] = u_1^2 u_2^2$. It covers the cases from (13) and (15).

Now we consider the cases from (14) and (16). Since in these cases $\mu \equiv_2 1$, hence $\nu \equiv_2 0$. If we identify

$$\begin{gathered} r_{11} = r_{12} = 0, \qquad r_{21} = 1, \qquad r_{22} = -1, \\ \alpha_1 = \beta_2 = \gamma_1 = 0, \qquad \alpha_2 = \frac{\lambda}{2}, \qquad \beta_1 = \frac{\nu}{2}, \qquad \gamma_1 = \frac{\mu + \alpha_2}{2}. \end{gathered} \tag{3.22}$$

then for the elements

$$\begin{gathered} u_1 = x_2 [x_2, x_1, x_1]^{\nu/2}, \\ u_2 = x_2^{-1} [x_2, x_1]^{\lambda/2} [x_2, x_1, x_2]^{(\mu + \lambda/2)/2}. \end{gathered} \tag{3.23}$$

one obtains $[h,g] = u_1^2 u_2^2$. And the equation ($\Diamond$) satisfies.

In particular in the cases from (13) to (16), we have $\mathrm{Sq}[h,g]$=2. This completes the proof. □

As an immediate consequence of Theorem 2.1, we obtain the exact value of the $\mathrm{Sq}(F'_{2,3})$.

The proof of Corollary 2.2 is based on our previous result [5] which we summarize here.

Theorem 3.1 (Rhemtulla-Akhavan[5]). *Let $F_{2,3} = \langle x_1, x_2 \rangle$ be a free nilpotent group of rank 2 and class 3 freely generated by x_1, x_2. Then any element of $F'_{2,3}$ can be expressed as a product of at most two commutators.*

We will also use the fact that if a, b, and c are any elements of a group G, then

$$a^2[b,c] = \left(a^2b^{-1}c^{-1}\right)^2\left(aba^{-1}c^{-1}a^{-1}\right)^2(ac)^2. \qquad (\dagger)$$

Proof of Corollary 2.2. Let $\zeta = [x,y][w,z]$ be any element of $F'_{2,3}$. We may write

$$\begin{aligned}[x,y] &= [x_2,x_1]^{\lambda}[x_2,x_1,x_2]^{\mu}[x_2,x_1,x_1]^{\nu},\\ [z,w] &= [x_2,x_1]^{\lambda'}[x_2,x_1,x_2]^{\mu'}[x_2,x_1,x_1]^{\nu'},\end{aligned} \qquad (3.24)$$

where $\lambda, \lambda', \mu, \mu', \nu$, and ν' are suitable integer numbers. Since $\gamma_3(F_{2,3})$ lies in the center of $F_{2,3}$ and $F'_{2,3}$ is abelian, we may express ζ as

$$\zeta = [x_2,x_1]^{\lambda+\lambda'}\left[x_2,x_1,x_2^{\mu+\mu'}x_1^{\nu+\nu'}\right]. \qquad (3.25)$$

There are two cases:

(1) $\lambda + \lambda' \equiv_2 0$,

(2) $\lambda + \lambda' \equiv_2 1$.

Case 1. By ($\dagger$), we may write ζ as a product of three squares.

Case 2. We may write

$$\zeta = [x_2,x_1]^{\lambda+\lambda'-1}[x_1,x_2]^{x_2^{\mu+\mu'}x_1^{\nu+\nu'}}. \qquad (3.26)$$

Since $\lambda + \lambda' - 1$ is even, ($\dagger$) yields $\mathrm{Sq}(\zeta) \le 3$. In Theorem 2.1 we produce elements of square length equal to three. This shows that $\mathrm{Sq}(F'_{2,3}) = 3$ and completes the proof. □

Note. Let $G = \langle x_1, x_2 \rangle$ be a free nilpotent group of rank 2 and class $c \ge 3$ freely generated by x_1, x_2. Now $F_{2,3}$ is a quotient of G. Since the equations ($\diamond$) and ($\Diamond$) do not hold in the cases from (17) to (22) in $F_{2,3}$, these equations should not hold in G. And similarly since the equation ($\diamond$) does not hold in the cases from (13) to (16) in $F_{2,3}$, hence these equations will not hold in G.

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