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Research Article

Weak and Strong Forms of ω-Continuous Functions

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We introduce weak and strong forms of ω-continuous functions, namely, θ-ω-continuous and strongly θ-ω-continuous functions, and investigate their fundamental properties.

1. Introduction

In 1943, Fomin [1] introduced the notion of θ-continuity. In 1968, the notions of θ-open subsets, θ-closed subsets, and θ-closure were introduced by Veličko [2]. In 1989, Hdeib [3] introduced the notion of ω-continuity. The main purpose of the present paper is to introduce and investigate fundamental properties of weak and strong forms of ω-continuous functions. Throughout this paper, (X,τ) and (Y,σ) stand for topological spaces (called simply spaces) with no separation axioms assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Let (X,τ) be a space and A a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. However, A is said to be ω-closed [4] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X,τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The family of all ω-open sets of X containing $x \in X$ is denoted by $\omega O(X,x)$. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\mathrm{Cl}(A)$ and $\omega\mathrm{Int}(A)$. Several characterizations of ω-closed subsets were provided in [5–8].

A point x of X is called a θ-cluster points of A if $\mathrm{Cl}(U) \cap A \neq \phi$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. A subset A is said to be θ-closed [2] if $A = \mathrm{Cl}_\theta(A)$. The complement of a θ-closed set is said to be θ-open. A point x of X is called an ω-θ-cluster point of A if $\omega\mathrm{Cl}(U) \cap A \neq \phi$ for every ω-open set U of X containing x. The set of all ω-θ-cluster points of A is called the ω-θ-closure of A and is denoted by $\omega\mathrm{Cl}_\theta(A)$. A subset A is said to be ω-θ-closed if $A = \omega\mathrm{Cl}_\theta(A)$. The complement of a ω-θ-closed set is said to be ω-θ-open. The ω-θ-interior of A is defined by the union of all ω-θ-open sets contained in A and is denoted by $\omega\mathrm{Int}_\theta(A)$.

2. θ-ω-Continuous Functions

We begin by recalling the following definition. Next, we introduce a relatively new notion.

Definition 2.1. A function $f : X \to Y$ is said to be ω-continuous (see [3]) (resp., almost weakly ω-continuous (see [9])) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq V$ (resp., $f(U) \subseteq \mathrm{Cl}(V)$).

Definition 2.2. A function $f : X \to Y$ is said to be θ-ω-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$.

Next, several characterizations of θ-ω-continuous functions are obtained.

Theorem 2.3. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is θ-ω-continuous;*

(2) *$\omega\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_\theta(B))$ for every subset B of Y;*

(3) *$f(\omega\mathrm{Cl}_\theta(A)) \subseteq \mathrm{Cl}_\theta(f(A))$ for every subset A of X.*

Proof. (1)⇒(2) Let B be any subset of Y. Suppose that $x \notin f^{-1}(\mathrm{Cl}_\theta(B))$. Then $f(x) \notin \mathrm{Cl}_\theta(B)$ and there exists an open set V containing $f(x)$ such that $\mathrm{Cl}(V) \cap B = \phi$. Since f is θ-ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. Therefore, we have $f(\omega\mathrm{Cl}(U)) \cap B = \phi$ and $\omega\mathrm{Cl}(U) \cap f^{-1}(B) = \phi$. This shows that $x \notin \omega\mathrm{Cl}_\theta(f^{-1}(B))$. Thus, we obtain $\omega\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_\theta(B))$.

(2)⇒(1) Let $x \in X$ and V be an open set of Y containing $f(x)$. Then we have $\mathrm{Cl}(V) \cap (Y - \mathrm{Cl}(V)) = \phi$ and $f(x) \notin \mathrm{Cl}_\theta(Y - \mathrm{Cl}(V))$. Hence, $x \notin f^{-1}(\mathrm{Cl}_\theta(Y - \mathrm{Cl}(V)))$ and $x \notin \omega\mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. There exists $U \in \omega O(X, x)$ such that $\omega\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \phi$ and hence $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. Therefore, f is θ-ω-continuous.

(2)⇒(3) Let A be any subset of X. Then we have $\omega\mathrm{Cl}_\theta(A) \subseteq \omega\mathrm{Cl}_\theta(f^{-1}(f(A))) \subseteq f^{-1}(\mathrm{Cl}_\theta(f(A)))$ and hence $f(\omega\mathrm{Cl}_\theta(A)) \subseteq \mathrm{Cl}_\theta(f(A))$.

(3)⇒(2) Let B be a subset of Y. We have $f(\omega\mathrm{Cl}_\theta(f^{-1}(B))) \subseteq \mathrm{Cl}_\theta(f(f^{-1}(B))) \subseteq \mathrm{Cl}_\theta(B)$ and hence $\omega\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_\theta(B))$. □

Proposition 2.4. *A subset U of a space X is ω-θ-open in X if and only if for each $x \in U$, there exists an ω-open set V with $x \in V$ such that $\omega\mathrm{Cl}(V) \subseteq U$.*

Proof. Suppose that U is ω-θ-open in X. Then $X - U$ is ω-θ-closed. Let $x \in U$. Then $x \notin \omega\mathrm{Cl}_\theta(X - U) = X - U$, and so there exists an ω-open set V with $x \in V$ such that $\omega\mathrm{Cl}(V) \cap (X - U) = \phi$. Thus $\omega\mathrm{Cl}(V) \subseteq U$. Conversely, assume that U is not ω-θ-open. Then $X - U$ is not ω-θ-closed, and so there exists $x \in \omega\mathrm{Cl}_\theta(X - U)$ such that $x \notin X - U$. Since

$x \in U$, by hypothesis, there exists an ω-open set V with $x \in V$ such that $\omega\mathrm{Cl}(V) \subseteq U$. Thus $\omega\mathrm{Cl}(V) \cap (X - U) = \phi$. This is a contradiction since $x \in \omega\mathrm{Cl}_\theta(X - U)$. □

Theorem 2.5. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is θ-ω-continuous;*

(2) *$f^{-1}(V) \subseteq \omega\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$ for every open set V of Y;*

(3) *$\omega\mathrm{Cl}_\theta(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y.*

Proof. (1)⇒(2) Suppose that V is any open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. Therefore, $x \in U \subseteq \omega\mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}(V))$. This shows that $x \in \omega\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq \omega\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$.

(2)⇒(3) Suppose that V is any open set of Y and $x \notin f^{-1}(\mathrm{Cl}(V))$. Then $f(x) \notin \mathrm{Cl}(V)$ and there exists an open set W containing $f(x)$ such that $W \cap V = \phi$; hence $\mathrm{Cl}(W) \cap V = \phi$. Therefore, we have $f^{-1}(\mathrm{Cl}(W)) \cap f^{-1}(V) = \phi$. Since $x \in f^{-1}(W)$, by (2) $x \in \omega\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(W)))$, there exists $U \in \omega O(X, x)$ such that $\omega\mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}(W))$. Thus we have $\omega\mathrm{Cl}(U) \cap f^{-1}(V) = \phi$ and hence $x \notin \omega\mathrm{Cl}_\theta(f^{-1}(V))$. This shows that $\omega\mathrm{Cl}_\theta(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$.

(3)⇒(1) Suppose that $x \in X$ and V are any open set of Y containing $f(x)$. Then $V \cap (Y - \mathrm{Cl}(V)) = \phi$ and $f(x) \notin \mathrm{Cl}(Y - \mathrm{Cl}(V))$. Therefore $x \notin f^{-1}(\mathrm{Cl}(Y - \mathrm{Cl}(V)))$ and by (3) $x \notin \omega\mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. There exists $U \in \omega O(X, x)$ such that $\omega\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \phi$. Therefore, we obtain $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. This shows that f is θ-ω-continuous. □

A subset A of X is said to be regular open (resp., regular closed) (see [10]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp., $A = \mathrm{Cl}(\mathrm{Int}(A))$). Also, the family of all regular open (resp., regular closed) sets of X is denoted by $\mathrm{RO}(X)$ (resp., $\mathrm{RC}(X)$).

Theorem 2.6. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is θ-ω-continuous;*

(2) *$\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))] \subseteq f^{-1}(\mathrm{Cl}_\theta(B))$ for every subset B of Y;*

(3) *$\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))] \subseteq f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y;*

(4) *$\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(K))] \subseteq f^{-1}(K)$ for every closed set K of Y;*

(5) *$\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(R))] \subseteq f^{-1}(R)$ for every regular closed set R of Y.*

Proof. (1)⇒(2) This follows immediately from Theorem 2.5(3) with $V = \mathrm{Int}(\mathrm{Cl}_\theta(B))$.

(2)⇒(3) This is obvious since $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of Y.

(3)⇒(4) For any closed set K of Y, $\mathrm{Int}(K) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K)))$ and by (3)

$$\begin{aligned}\omega\mathrm{Cl}_\theta\left(f^{-1}(\mathrm{Int}(K))\right) &= \omega\mathrm{Cl}_\theta\left(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K))))\right) \\ &\subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(K))) \\ &\subset f^{-1}(K).\end{aligned} \tag{2.1}$$

(4)⇒(5) This is obvious.

(5)⇒(1) Let V be any open set of Y. Since $\mathrm{Cl}(V)$ is regular closed, by (5) $\omega\mathrm{Cl}_\theta(f^{-1}(V))) \subset \omega\mathrm{Cl}_\theta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset f^{-1}(\mathrm{Cl}(V))$. It follows from Theorem 2.5 that f is θ-ω-continuous. □

Definition 2.7. A subset A of a space X is said to be semi-open (see [11]) (resp., preopen (see [12]), β-open (see [13])) if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ (resp., $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

Theorem 2.8. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is θ-ω-continuous;*

(2) $\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))] \subseteq f^{-1}(\mathrm{Cl}(G))$ *for every β-open set G of Y;*

(3) $\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))] \subseteq f^{-1}(\mathrm{Cl}(G))$ *for every semi-open set G of Y.*

Proof. (1)⇒(2) This is obvious by Theorem 2.6(5) since $\mathrm{Cl}(G)$ is regular closed for every β-open set set G.

(2)⇒(3) This is obvious since every semi-open set is β-open.

(3)⇒(1) This follows immediately from Theorem 2.5(3) and since every open set is semi-open. □

Theorem 2.9. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is θ-ω-continuous;*

(2) $\omega\mathrm{Cl}_\theta[f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))] \subseteq f^{-1}(\mathrm{Cl}(G))$ *for every preopen set G of Y;*

(3) $\omega\mathrm{Cl}_\theta[f^{-1}(G)] \subseteq f^{-1}(\mathrm{Cl}(G))$ *for every preopen set G of Y;*

(4) $f^{-1}(G) \subset \omega\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(G)))$ *for every preopen set G of Y.*

Proof. (1)⇒(2) The proof follows from Theorem 2.8 (2) since every preopen set is β-open.

(2)⇒(3) This is obvious by the definition of a preopen set.

(3)⇒(4) Let G be any preopen set of Y. Then, by (3) we have

$$\begin{aligned} X - \omega\mathrm{Int}_\theta\left(f^{-1}(\mathrm{Cl}(G))\right) &= \omega\mathrm{Cl}_\theta\left(X - f^{-1}(\mathrm{Cl}(G))\right) \\ &= \omega\mathrm{Cl}_\theta\left(f^{-1}(Y - \mathrm{Cl}(G))\right) \\ &\subset f^{-1}(\mathrm{Cl}(Y - \mathrm{Cl}(G))) \\ &= f^{-1}(Y - \mathrm{Int}(\mathrm{Cl}(G))) \\ &\subset f^{-1}(Y - G) \\ &= X - f^{-1}(G). \end{aligned} \tag{2.2}$$

Therefore, we obtain $f^{-1}(G) \subset \omega\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(G)))$.

(4)⇒(1) This is obvious by Theorem 2.5 since every open set is preopen. □

Definition 2.10. A function $f : X \to Y$ is said to be almost ω-continuous if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq V$.

Lemma 2.11. *For a function $f : X \to Y$, the following assertions are equivalent:*

(1) *f is almost ω-continuous;*

(2) *for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \mathrm{Int}(\mathrm{Cl}(V))$;*

(3) *$f^{-1}(F) \in \omega C(X)$ for every $F \in \mathrm{RC}(Y)$;*

(4) *$f^{-1}(V) \in \omega O(X)$ for every $V \in \mathrm{RO}(Y)$.*

Proposition 2.12. *For a function $f : X \to Y$, the following properties hold:*

(1) *if f is almost ω-continuous, then it is θ-ω-continuous;*

(2) *if f is θ-ω-continuous, then it is almost weakly ω-continuous.*

Proof. (1) Suppose that $x \in X$ and V is any open set of Y containing $f(x)$. Since f is almost ω-continuous, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ is ω-open and $f^{-1}(\mathrm{Cl}(V))$ is ω-closed in X by Lemma 2.11. Now, set $U = f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$. Then we have $U \in \omega O(X, x)$ and $\omega\mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}(V))$. Therefore, we obtain $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. This shows that f is θ-ω-continuous.

(2) The proof follows immediately from the definition. □

Example 2.13. Let X be an uncountable set and let A, B, and C be subsets of X such that each of them is uncountable and the family $\{A, B, C\}$ is a partition of X. We define the topology $\tau = \{\phi, X, \{A\}, \{B\}, \{A, B\}\}$. Then, the function $f : (X, \tau) \to (X, \tau)$ defined by $f(A) = A$, $f(B) = C$, and $f(C) = A$ is θ-ω-continuous (and almost weakly ω-continuous) but is not almost ω-continuous since for $x_c \in C \subseteq X$, A is regular open and $f(x_c) \in A$ but there is not open set U_{x_c} containing x_c such that $f(U_{x_c}) \subseteq A$.

Question. Is the converse of Proposition 2.12(2) true?

It is clear that, for a subset A of a space X, $x \in \omega\mathrm{Cl}(A)$ if and only if for any ω-open set U containing x, $U \cap A \neq \phi$.

Lemma 2.14. *For an ω-open set U in a space X, $\omega\mathrm{Cl}(U) = \omega\mathrm{Cl}_\theta(U)$.*

Proof. By definition, $\omega\mathrm{Cl}(U) \subseteq \omega\mathrm{Cl}_\theta(U)$. Let $x \in \omega\mathrm{Cl}_\theta(U)$. Then for any ω-open set V containing x, $\omega\mathrm{Cl}(V) \cap U \neq \phi$. Let $z \in \omega\mathrm{Cl}(V) \cap U$. Then $U \cap V \neq \phi$ and $x \in \omega\mathrm{Cl}(U)$. Thus $\omega\mathrm{Cl}_\theta(U) \subseteq \omega\mathrm{Cl}(U)$. □

Definition 2.15. A topological space X is said to be ω-regular (resp., ω^*-regular) if for each ω-closed (resp., closed) set F and each point $x \in X - F$, there exist disjoint ω-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 2.16. *A topological space X is ω-regular (resp., ω^*-regular) if and only if for each $U \in \omega O(X)$ (resp., $U \in O(X)$) and each point $x \in U$, there exists $V \in \omega O(X, x)$ such that $x \in V \subseteq \omega\mathrm{Cl}(V) \subseteq U$.*

Proposition 2.17. *Let X be an ω-regular space. Then $f : X \to Y$ is θ-ω-continuous if and only if it is almost weakly ω-continuous.*

Proof. Suppose that f is almost weakly ω-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \mathrm{Cl}(V)$. Since X is

ω-regular, by Lemma 2.16 there exists $W \in \omega O(X, x)$ such that $x \in W \subseteq \omega\mathrm{Cl}(W) \subseteq U$. Therefore, we obtain $f(\omega\mathrm{Cl}(W)) \subseteq \mathrm{Cl}(V)$. This shows that f is θ-ω-continuous. □

Theorem 2.18. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is θ-ω-continuous if and only if f is θ-ω-continuous.*

Proof

Necessity. Suppose that g is θ-ω-continuous. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is θ-ω-continuous, there exists $U \in \omega O(X, x)$ such that $g(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. Therefore, we obtain $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. This shows that f is θ-ω-continuous.

Sufficiency. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subseteq W$. Since f is θ-ω-continuous, there exists $U_2 \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U_2)) \subseteq \mathrm{Cl}(V)$. Let $U = U_1 \cap U_2$, then $U \in \omega O(X, x)$. Therefore, we obtain $g(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(U_1) \times f(\omega\mathrm{Cl}(U_2)) \subseteq \mathrm{Cl}(U_1) \times \mathrm{Cl}(V) \subseteq \mathrm{Cl}(W)$. This shows that g is θ-ω-continuous.

3. Strongly θ-ω-Continuous Functions

We introduce the following relatively new definition.

Definition 3.1 (see [14]). A function $f : X \to Y$ is said to be strongly θ-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Cl}(U)) \subseteq V$.

Definition 3.2. A function $f : X \to Y$ is said to be strongly θ-ω-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U)) \subseteq V$.

Clearly, the following holds and none of its implications is reversible:

$$
\begin{tikzcd}
\omega\text{-continuous} \arrow[rr] \arrow[dr] & & \text{almost weakly } \omega\text{-continuous} \\
& \text{almost } \omega\text{-continuous} \arrow[ur] \arrow[dr] & \\
\text{strongly } \theta\text{-}\omega\text{-continuous} \arrow[uu] \arrow[ur] \arrow[rr] & & \theta\text{-}\omega\text{-continuous} \arrow[uu]
\end{tikzcd}
$$

Remark 3.3. Strong θ-ω-continuity is stronger than ω-continuity and is weaker than strong θ-continuity. Strong θ-ω-continuity and continuity are independent of each other as the following examples show.

Example 3.4. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$, and $\sigma = \{\phi, X, \{c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ as follows: $f(a) = a$, $f(b) = f(c) = c$. Then f is strongly θ-ω-continuous but it is not continuous.

Example 3.5. Let X be an uncountable set and let A, B, and C be subsets of X such that each of them is uncountable and the family $\{A, B, C\}$ is a partition of X. We defined the topology $\tau = \{\phi, X, \{A\}, \{B\}, \{A, B\}\}$ and $\sigma = \{\phi, X, \{A\}, \{A, B\}\}$. Then, the identity function $f : (X, \tau) \to (X, \sigma)$ is continuous (and ω-continuous) but is not strongly θ-ω-continuous.

Next, several characterizations of strongly θ-ω-continuous functions are obtained.

Theorem 3.6. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is strongly θ-ω-continuous;*

(2) *$f^{-1}(V)$ is ω-θ-open in X for every open set V of Y;*

(3) *$f^{-1}(F)$ is ω-θ-closed in X for every closed set F of Y;*

(4) *$f(\omega\mathrm{Cl}_\theta(A)) \subseteq \mathrm{Cl}(f(A))$ for every subset A of X;*

(5) *$\omega\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}(B))$ for every subset B of Y.*

Proof. (1)⇒(2) Let V be any open set of Y. Suppose that $x \in f^{-1}(V)$. Since f is strongly θ-ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U)) \subseteq V$. Therefore, we have $x \in U \subseteq \omega\mathrm{Cl}(U) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is ω-θ-open in X.

(2)⇒(3) This is obvious.

(3)⇒(4) Let A be any subset of X. Since $\mathrm{Cl}(f(A))$ is closed in Y, by (3) $f^{-1}(\mathrm{Cl}(f(A)))$ is ω-θ-closed, and we have $\omega\mathrm{Cl}_\theta(A) \subseteq \omega\mathrm{Cl}_\theta(f^{-1}(f(A))) \subseteq \omega\mathrm{Cl}_\theta(f^{-1}(\mathrm{Cl}(f(A)))) = f^{-1}(\mathrm{Cl}(f(A)))$. Therefore, we obtain $f(\omega\mathrm{Cl}_\theta(A)) \subseteq \mathrm{Cl}(f(A))$.

(4)⇒(5) Let B be any subset of Y. By (4), we obtain $f(\omega\mathrm{Cl}_\theta(f^{-1}(B))) \subseteq \mathrm{Cl}(f(f^{-1}(B))) \subseteq \mathrm{Cl}(B)$ and hence $\omega\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}(B))$.

(5)⇒(1) Let $x \in X$ and V be any open neighborhood of $f(x)$. Since $Y - V$ is closed in Y, we have $\omega\mathrm{Cl}_\theta(f^{-1}(Y - V)) \subseteq f^{-1}(\mathrm{Cl}(Y - V)) = f^{-1}(Y - V)$. Therefore, $f^{-1}(Y - V)$ is ω-θ-closed in X and $f^{-1}(V)$ is an ω-θ-open set containing x. There exists $U \in \omega O(X, x)$ such that $\omega\mathrm{Cl}(U) \subseteq f^{-1}(V)$ and hence $f(\omega\mathrm{Cl}(U)) \subseteq V$. This shows that f is strongly θ-ω-continuous. □

Theorem 3.7. *Let Y be a regular space. Then, for a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is almost weakly ω-continuous;*

(2) *f is ω-continuous;*

(3) *f strongly θ-ω-continuous.*

Proof. (1)⇒(2) Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W such that $f(x) \in W \subseteq \mathrm{Cl}(W) \subseteq V$. Since f is almost weakly ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \mathrm{Cl}(W) \subseteq V$. Therefore f is ω-continuous.

(2)⇒(3) Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W such that $f(x) \in W \subseteq \mathrm{Cl}(W) \subseteq V$. Since f is ω-continuous, $f^{-1}(W)$ is ω-open and $f^{-1}(\mathrm{Cl}(W))$ is ω-closed. Set $U = f^{-1}(W)$, then since $x \in f^{-1}(W) \subseteq f^{-1}(\mathrm{Cl}(W))$, $U \in \omega O(X, x)$ and $\omega\mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}(W))$. Consequently, we have $f(\omega\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(W) \subseteq V$.

(3)⇒(1) The proof follows immediately from the definition. □

Corollary 3.8. *Let Y be a regular space. Then, for a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is strongly θ-ω-continuous;*

(2) *f is ω-continuous;*

(3) *f is almost ω-continuous;*

(4) *f is θ-ω-continuous;*

(5) *f is almost weakly ω-continuous.*

Theorem 3.9. *A space X is ω^*-regular if and only if, for any space Y, any continuous function $f : X \to Y$ is strongly θ-ω-continuous.*

Proof

Sufficiency. Let $f : X \to X$ be the identity function. Then f is continuous and strongly θ-ω-continuous by our hypothesis. For any open set U of X and any points x of U, we have $f(x) = x \in U$ and there exists $G \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(G)) \subseteq U$. Therefore, we have $x \in G \subseteq \omega\mathrm{Cl}(G) \subseteq U$. It follows from Lemma 2.16, that is, X is ω^*-regular.

Necessity. Suppose that $f : X \to Y$ is continuous and X is ω^*-regular. For any $x \in X$ and any open neighborhood V of $f(x)$, $f^{-1}(V)$ is an open set of X containing x. Since X is ω^*-regular, there exists $U \in \omega O(X)$ such that $x \in U \subseteq \omega\mathrm{Cl}(U) \subseteq f^{-1}(V)$ by Lemma 2.16. Therefore, we have $f(\omega\mathrm{Cl}(U)) \subseteq V$. This shows that f is strongly θ-ω-continuous.

Theorem 3.10. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. If g is strongly θ-ω-continuous, then f is strongly θ-ω-continuous and X is ω^*-regular.*

Proof. Suppose that g is strongly θ-ω-continuous. First, we show that f is strongly θ-ω-continuous. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is strongly θ-ω-continuous, there exists $U \in \omega O(X, x)$ such that $g(\omega\mathrm{Cl}(U)) \subseteq X \times V$. Therefore, we obtain $f(\omega\mathrm{Cl}(U)) \subseteq V$. Next, we show that X is ω^*-regular. Let U be any open set of X and $x \in U$. Since $g(x) \in U \times Y$ and $U \times Y$ is open in $X \times Y$, there exists $G \in \omega O(X, x)$ such that $g(\omega\mathrm{Cl}(G)) \subseteq U \times Y$. Therefore, we obtain $x \in G \subseteq \omega\mathrm{Cl}(G) \subseteq U$ and hence X is ω^*-regular. □

Proposition 3.11. *Let X be an ω-regular space. Then $f : X \to Y$ is strongly θ-ω-continuous if and only if f is ω-continuous.*

Proof. Suppose that f is ω-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. By the ω-continuity of f, we have $f^{-1}(V) \in \omega O(X, x)$ and hence there exists $U \in \omega O(X, x)$ such that $\omega\mathrm{Cl}(U) \subseteq f^{-1}(V)$. Therefore, we obtain $f(\omega\mathrm{Cl}(U)) \subseteq V$. This shows that f is strongly θ-ω-continuous. □

Theorem 3.12. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. If f is strongly θ-ω-continuous and X is ω-regular, then g is strongly θ-ω-continuous.*

Proof. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subseteq W$. Since f is strongly θ-ω-continuous, there exists $U_2 \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U_2)) \subseteq V$. Since X is ω-regular and $U_1 \cap U_2 \in \omega O(X, x)$, there exists $U \in \omega O(X, x)$ such that $x \in U \subseteq \omega\mathrm{Cl}(U) \subseteq U_1 \cap U_2$ (by Lemma 2.16). Therefore, we obtain $g(\omega\mathrm{Cl}(U)) \subseteq U_1 \times f(\omega\mathrm{Cl}(U_2)) \subseteq U_1 \times V \subseteq W$. This shows that g is strongly θ-ω-continuous. □

Theorem 3.13. *Suppose that the product of two ω-open sets of X is ω-open. If $f : X \to Y$ is strongly θ-ω-continuous injection and Y is Hausdorff, then $E = \{(x, y) : f(x) = f(y)\}$ is ω-θ-closed in $X \times X$.*

Proof. Suppose that $(x, y) \notin E$. Then $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open sets V and U containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \phi$. Since f is strongly θ-ω-continuous, there exist $G \in \omega O(X, x)$ and $H \in \omega O(X, y)$ such that $f(\omega\mathrm{Cl}(G)) \subseteq V$ and $f(\omega\mathrm{Cl}(H)) \subseteq U$. Set $D = G \times H$. It follows that $(x, y) \in D \in \omega O(X \times Y)$ and $\omega\mathrm{Cl}(G \times H) \cap E \subseteq [\omega\mathrm{Cl}(G) \times \omega\mathrm{Cl}(H)] \cap E = \phi$. By Proposition 2.4, E is ω-θ-closed in $X \times X$. □

Definition 3.14 (see [9]). A space X is said to be ω-T_2-space (resp., *ω-Urysohn*) if for each pair of distinct points x and y in X, there exist $U \in \omega O(X, x)$ and $V \in \omega O(X, y)$ such that $U \cap V = \phi$ (resp., $\omega\mathrm{Cl}(U) \cap \omega\mathrm{Cl}(V) = \phi$).

Theorem 3.15. *If $f : X \to Y$ is strongly θ-ω-continuous injection and Y is T_0-space (resp., Hausdorff), then X is ω-T_2-space (resp., ω-Urysohn).*

Proof. (1) Suppose that Y is T_0-space. Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$ and there exists either an open neighborhood V of $f(x)$ not containing $f(y)$ or an open neighborhood W of $f(y)$ not containing $f(x)$. If the first case holds, then there exists $U \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U)) \subseteq V$. Therefore, we obtain $f(y) \notin f(\omega\mathrm{Cl}(U))$ and hence $X - \omega\mathrm{Cl}(U) \in \omega O(X, y)$. If the second case holds, then we obtain a similar result. Therefore, X is ω-T_2.

(2) Suppose that Y is Hausdorff. Let x and y be any distinct points of X. Then $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open sets V and U containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \phi$. Since f is strongly θ-ω-continuous, there exist $G \in \omega O(X, x)$ and $H \in \omega O(X, y)$ such that $f(\omega\mathrm{Cl}(G)) \subseteq V$ and $f(\omega\mathrm{Cl}(H)) \subseteq U$. It follows that $f(\omega\mathrm{Cl}(G)) \cap f(\omega\mathrm{Cl}(H)) = \phi$, hence $\omega\mathrm{Cl}(G) \cap \omega\mathrm{Cl}(H) = \phi$. This shows that X is *ω-Urysohn*. □

A subset K of a space X is said to be ω-closed relative to X if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of K by ω-open sets of X, there exists a finite subset Λ_0 of Λ such that $K \subseteq \cup\{\omega\mathrm{Cl}(V_\alpha) : \alpha \in \Lambda_0\}$.

Theorem 3.16. *Let $f : X \to Y$ be strongly θ-ω-continuous and K ω-closed relative to X, then $f(K)$ is a compact set of Y.*

Proof. Suppose that $f : X \to Y$ is a strongly θ-ω-continuous function and K is ω-closed relative to X. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $f(K)$. For each point $x \in K$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is strongly θ-ω-continuous, there exists $U_x \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U_x)) \subseteq V_{\alpha(x)}$. The family $\{U_x : x \in K\}$ is a cover of K by ω-open sets of X and hence there exists a finite subset K_* of K such that $K \subseteq \bigcup_{x \in K_*} \omega\mathrm{Cl}(U_x)$. Therefore, we obtain $f(K) \subseteq \bigcup_{x \in K_*} V_{\alpha(x)}$. This shows that $f(K)$ is compact. □

Recall that a subset A of a space X is quasi H-closed relative to X if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by open sets of X, there exist a finite subset Λ_0 of Λ such that $A \subseteq \cup\{\mathrm{Cl}(V_\alpha) : \alpha \in \Lambda_0\}$. A space X is said to be quasi H-closed (see [15]) if X is quasi H-closed relative to X.

Theorem 3.17. *Let $f : X \to Y$ be θ-ω-continuous and $K\omega$-closed relative to X, then $f(K)$ is quasi H-closed relative to Y.*

Proof. Suppose that $f : X \to Y$ is a θ-ω-continuous function and K is ω-closed relative to X. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $f(K)$. For each point $x \in K$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ-ω-continuous, there exists $U_x \in \omega O(X, x)$ such that $f(\omega\mathrm{Cl}(U_x)) \subseteq \mathrm{Cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by ω-open sets of X and hence there exists a finite subset K_* of K such that $K \subseteq \bigcup_{x\in K_*} \omega\mathrm{Cl}(U_x)$. Therefore, we obtain $f(K) \subseteq \bigcup_{x\in K_*} \mathrm{Cl}(V_{\alpha(x)})$. This shows that $f(K)$ is quasi H-closed relative to Y. □

Definition 3.18 (see [9]). A function $f : X \to Y$ is said to be pre-ω-open if $f(U) \in \omega O(Y)$ for every $U \in \omega O(X)$.

Proposition 3.19. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let $g \circ f : X \to Z$ be strongly θ-ω-continuous. If $f : X \to Y$ is pre-ω-open and bijective, then g is strongly θ-ω-continuous.*

Proof. Let $y \in Y$ and W be any open set of Z containing $g(y)$. Since f is bijective, $y = f(x)$ for some $x \in X$. Since $(g \circ f)$ is strongly θ-ω-continuous, there exists $U \in \omega O(X, x)$ such that $(g\circ f)(\omega\mathrm{Cl}(U)) \subseteq W$. Since f is pre-ω-open and bijective, the image $f(A)$ of an ω-closed set A of X is ω-closed in Y. Therefore, we have $\omega\mathrm{Cl}(f(U)) \subseteq f(\omega\mathrm{Cl}(U))$ and hence $g(\omega\mathrm{Cl}(f(U))) \subseteq (g \circ f)(\omega\mathrm{Cl}(U)) \subseteq W$. Since $f(U) \in \omega O(Y, y)$, g is strongly θ-ω-continuous. □

Definition 3.20 (see [16]). A function $f : X \to Y$ is said to be ω-irresolute if $f^{-1}(V) \in \omega O(X)$ for each $V \in \omega O(Y)$.

Lemma 3.21. *If $f : X \to Y$ is ω-irresolute and V is ω-θ-open in Y, then $f^{-1}(V)$ is ω-θ-open in X.*

Proof. Let V be an ω-θ-open set of Y and $x \in f^{-1}(V)$. There exists $W \in \omega O(Y)$ such that $f(x) \in W \subseteq \omega\mathrm{Cl}(W) \subseteq V$. Since f is ω-irresolute, we have $f^{-1}(W) \in \omega O(X)$ and $f^{-1}(\omega\mathrm{Cl}(W)) \in \omega C(X)$. Therefore, we obtain $x \in f^{-1}(W) \subseteq \omega\mathrm{Cl}(f^{-1}(W)) \subseteq f^{-1}(\omega\mathrm{Cl}(W)) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is ω-θ-open in X. □

Theorem 3.22. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold.*

(1) *If f is strongly θ-ω-continuous and g is continuous, then the composition $g \circ f$ is strongly θ-ω-continuous.*

(2) *If f is ω-irresolute and g is strongly θ-ω-continuous, then the composition $g \circ f$ is strongly θ-ω-continuous.*

Proof. (1) This is obvious from Theorem 3.6.

(2) This follows immediately from Theorem 3.6 and Lemma 3.21. □

Theorem 3.23 (see [3]). *For any space X, the following are equivalent:*

(1) *X is Lindelöf;*

(2) *every ω-open cover of X has a countable subcover.*

Definition 3.24 (see [17]). A space X is said to be nearly Lindelöf if every regular open cover of X has a countably subcover.

Proposition 3.25. *Let $f : X \to Y$ be an almost ω-continuous surjection. If X is Lindelöf, then Y is nearly Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a regular open cover of Y. Since f is almost ω-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an ω-open cover of X. Since X is Lindelöf, by Theorem 3.23 there exists a countable subcover $\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}$ of X. Hence $\{V_{\alpha_n} : n \in \mathbb{N}\}$ is a countable subcover of Y. □

Definition 3.26 (see [18]). A topological space X is said to be almost Lindelöf if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Lambda$ such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}(U_{\alpha_n})$.

Theorem 3.27. *Let $f : X \to Y$ be an almost weakly ω-continuous surjection. If X is Lindelöf, then Y is almost Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Let $x \in X$ and $V_{\alpha(x)}$ be an open set in Y such that $f(x) \in V_{\alpha(x)}$. Since f is almost weakly ω-continuous, there exists an ω-open set $U_{\alpha(x)}$ of X containing x such that $f(U_{\alpha(x)}) \subseteq \mathrm{Cl}(V_{\alpha(x)})$. Now $\{U_{\alpha(x)} : x \in X\}$ is an ω-open cover of the Lindelöf space X. So by Theorem 3.23, there exists a countable subset $\{U_{\alpha(x_n)} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}}(U_{\alpha(x_n)})$. Thus $Y = f(\bigcup_{n\in\mathbb{N}}(U_{\alpha(x_n)})) \subseteq \bigcup_{n\in\mathbb{N}} f(U_{\alpha(x_n)}) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{Cl}(V_{\alpha(x_n)})$. This shows that Y is almost Lindelöf. □

We notice that a subspace A of a space X is Lindelöf if and only if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by open set of X, there exists a countable subset Λ_0 of Λ such that $\{V_\alpha : \alpha \in \Lambda_0\}$ covers A.

Definition 3.28 (see [4]). A function $f : X \to Y$ is said to be ω-closed if the image of every closed subset of X is ω-closed in Y.

Theorem 3.29. *If $f : X \to Y$ is an ω-closed surjection such that $f^{-1}(y)$ is a Lindelöf subspace for each $y \in Y$ and Y is Lindelöf, then X is Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Since $f^{-1}(y)$ is a Lindelöf subspace for each $y \in Y$, there exists a countable subset $\Lambda(y)$ of Λ such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in \Lambda(y)\}$. Let $U(y) = \cup\{U_\alpha : \alpha \in \Lambda(y)\}$ and $V(y) = Y - f(X - U(y))$. Since f is ω-closed, $V(y)$ is an ω-open set containing y such that $f^{-1}(V(y)) \subseteq U(y)$. Then $\{V(y) : y \in Y\}$ is an ω-open cover of the Lindelöf space Y. By Theorem 3.23, there exist countable points of Y, says, $y_1, y_2, \ldots, y_n, \ldots$ such that $Y = \bigcup_{n\in\mathbf{N}} V(y_n)$. Therefore, we have $X = f^{-1}(\bigcup_{n\in\mathbf{N}} V(y_n)) = \bigcup_{n\in\mathbf{N}} f^{-1}(V(y_n)) \subseteq \bigcup_{n\in\mathbf{N}}(U(y_n)) = \bigcup_{n\in\mathbf{N}}(\cup\{U_\alpha : \alpha \in \Lambda(y_n)\}) = \cup\{U_\alpha : \alpha \in \Lambda(y_n), n \in \mathbf{N}\}$. This shows that X is Lindelöf. □

Theorem 3.30 (see [3]). *Let f be an ω-continuous function from a space X onto a space Y. If X is Lindelöf, then Y is Lindelöf.*

Corollary 3.31. *Let $f : X \to Y$ be an ω-closed and ω-continuous surjection such that $f^{-1}(y)$ is a Lindelöf subspace for each $y \in Y$. Then X is Lindelöf if and only if Y is Lindelöf.*

Proof. Let X be Lindelöf. It follows from Theorem 3.30 that Y is Lindelöf. The converse is an immediate consequence of Theorem 3.29. □

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