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Research Article

Common Fixed Point Theorem of Two Mappings Satisfying a Generalized Weak Contractive Condition

M. Abbas and M. Ali Khan

Department of Mathematics, Lahore University of Management Sciences, Lahore 54792, Pakistan

Correspondence should be addressed to M. Abbas, mujahid@lums.edu.pk

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Existence of common fixed point for two mappings which satisfy a generalized weak contractive condition is established. As a consequence, a common fixed point result for mappings satisfying a contractive condition of integral type is obtained. Our results generalize, extend, and unify several well-known comparable results in literature.

1. Introduction and Preliminaries

Let X be a metric space and $T : C \to C$ a mapping. Recall that T is contraction if $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$, where $0 \leq k < 1$. A point $x \in C$ is a fixed point of T provided $Tx = x$. If a map T satisfies $F(T) = F(T^n)$ for each $n \in N$, where $F(T)$ denotes the set of all fixed points of T, then it is said to have property P. Banach contraction principle which gives an answer on existence and uniqueness of a solution of an operator equation $Tx = x$ is the most widely used fixed point theorem in all of analysis. Branciari [1] obtained a fixed point theorem for a mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. Akgun and Rhoades [2] have shown that a map satisfying a Meir- Keeler type contractive condition of integral type has a property P. Rhoades and Abbas [3] extended [4, Theorem 1] for mappings satisfying contractive condition of integral type. They also studied several results for maps which have property P, defined on a metric space satisfying generalized contractive conditions of integral type. Rhoades [5] proved two fixed point theorems involving more general contractive condition of integral type (see, also [6, 7]). If maps S and T satisfy $F(S) \cap F(T) = F(S^n) \cap F(T^n)$ for each $n \in N$, then they are said to have property Q. Jeong and Rhoades [8] studied the property Q for pairs of maps satisfying a number of contractive conditions.

Recently Dutta and Choudhury [9] gave a generalization of Banach contraction principle, which in turn generalize [4, Theorem 1] and corresponding result of [10]. Sessa [11] defined the concept of weakly commuting to obtain common fixed point for pairs of maps. Jungck generalized this idea, first to compatible mappings [12] and then to weakly compatible mappings [13]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. The aim of this paper is to present a common fixed point theorem for weakly compatible maps satisfying a generalized weak contractive condition which is more general than the corresponding contractive condition of integral type. Our results substantially extend, improve, and generalize comparable results in literature [3, 14, 15].

The following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a set, and f, g selfmaps of X. A point x in X is called a coincidence point of f and g if and only if $fx = gx$. We will call $w = fx = gx$ a point of coincidence of f and g.

Definition 1.2. Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Lemma 1.3 (see [16]). *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence w (say), then w is the unique common fixed point of f and g.*

2. A Common Fixed Point Theorem

Set $F = \{\phi : R^+ \to R^+ : \phi$ is a Lebesgue integrable mapping which is summable and nonnegative and satisfies $\int_0^{\varepsilon}\phi(t)dt > 0$, for each $\varepsilon > 0\}$ and $G = \{\psi : [0,\infty] \to [0,\infty] : \psi$ is continuous and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$.

The following is the main result of this paper.

Theorem 2.1. *Let f, g be two self maps of a metric space (X, d) satisfying*

$$\psi(d(fx, fy)) \le \psi(d(gx, gy)) - \varphi(d(gx, gy)) \tag{2.1}$$

for all $x, y \in X$, where $\psi, \varphi \in G$. If range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$, $n = 0, 1, 2, \dots$. Suppose for any $n, g(x_n) \neq g(x_{n+1})$, since, otherwise, f and g have a point of coincidence. From (2.1), we have

$$\begin{aligned}\psi(d(gx_{n+1}, gx_n)) &= \psi(d(fx_n, fx_{n-1})) \\ &\le \psi(d(gx_n, gx_{n-1})) - \varphi(d(gx_n, gx_{n-1})) \\ &< \psi(d(gx_n, gx_{n-1})),\end{aligned} \tag{2.2}$$

that is, $\psi(d(gx_{n+1}, gx_n)) < \psi(d(gx_n, gx_{n-1}))$, and hence

$$d(gx_n, gx_{n+1}) \leq d(gx_n, gx_{n-1}). \tag{2.3}$$

It follows that $\{d(gx_n, gx_{n+1})\}$ is monotone decreasing sequence of numbers and consequently there exists $r \geq 0$ such that $d(gx_n, gx_{n+1}) \to r$ as $n \to \infty$. Suppose that $r > 0$, then

$$\begin{aligned} 0 < \psi(r) &\leq \psi(d(gx_{n+1}, gx_n)) = \psi(d(fx_n, fx_{n-1})) \\ &\leq \psi(d(gx_n, gx_{n-1})) - \varphi(d(gx_n, gx_{n-1})), \end{aligned} \tag{2.4}$$

which on taking limit as $n \to \infty$ yields

$$\psi(r) \leq \psi(r) - \varphi(r) < \psi(r), \tag{2.5}$$

which is a contradiction. Therefore $r = 0$. Now we prove that $\{gx_n\}$ is a Cauchy sequence. If not, then there exist some $\varepsilon > 0$ and subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ of $\{gx_n\}$ with $k < n_k < m_k$ such that $d(gx_{n_k}, gx_{m_k}) \geq 3\varepsilon$ for each k. As $d(gx_{n_k+1}, gx_{n_k}) \to 0$ as $k \to \infty$, for large enough k, we have $d(gx_{n_k+1}, gx_{n_k}) < \varepsilon$ and $d(gx_{m_k+1}, gx_{m_k}) < \varepsilon$. Thus we obtain

$$\begin{aligned} d(gx_{n_k+1}, gx_{m_k}) &\geq d(gx_{n_k}, gx_{m_k}) - d(gx_{n_k+1}, gx_{n_k}) > \varepsilon, \\ d(gx_{n_k+1}, gx_{m_k-1}) &\geq d(gx_{n_k}, gx_{m_k}) - d(gx_{m_k-1}, gx_{m_k}) - d(gx_{n_k+1}, gx_{n_k}) \\ &> \varepsilon. \end{aligned} \tag{2.6}$$

We may assume that n_k are even and m_k are odd and that $d(gx_{n_k}, gx_{m_k}) > \varepsilon$ for all k. Put

$$r_k = \min\{m_k : d(gx_{n_k}, gx_{m_k}) > \varepsilon\}. \tag{2.7}$$

Now,

$$\varepsilon < d(gx_{n_k}, gx_{r_k}) \leq d(gx_{n_k}, gx_{r_k-2}) + d(gx_{r_k-2}, gx_{r_k-1}) + d(gx_{r_k-1}, gx_{r_k}) \tag{2.8}$$

implies that $d(gx_{n_k}, gx_{r_k}) \to \varepsilon$ as $k \to \infty$. Furthermore

$$\begin{aligned} &d(gx_{n_k}, gx_{r_k}) - d(gx_{n_k}, gx_{n_k+1}) - d(gx_{r_k}, gx_{r_k+1}) \\ &\quad \leq d(gx_{n_k+1}, gx_{r_k+1}) \leq d(gx_{n_k}, gx_{r_k}) + d(gx_{n_k}, gx_{n_k+1}) + d(gx_{r_k}, gx_{r_k+1}) \end{aligned} \tag{2.9}$$

gives $d(gx_{n_k+1}, gx_{r_k+1}) \to \varepsilon$, as $k \to \infty$. Therefore

$$\begin{aligned}\psi(d(gx_{n_k+1}, gx_{r_k+1})) &= \psi(d(fx_{n_k}, fx_{r_k})) \\ &\leq \psi(d(gx_{n_k}, gx_{r_k})) - \varphi(d(gx_{n_k}, gx_{r_k})).\end{aligned} \tag{2.10}$$

Taking limit as $k \to \infty$ yields

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon), \tag{2.11}$$

which is a contradiction. Hence $\{gx_n\}$ is a Cauchy sequence. From completeness of $g(X)$, there exists a point q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find p in X such that $g(p) = q$. Now

$$\begin{aligned}\psi(d(gx_{n+1}, fp)) &= \psi(d(fx_n, fp)) \\ &\leq \psi(d(gx_n, gp)) - \varphi(d(gx_n, gp))\end{aligned} \tag{2.12}$$

on taking limit as $n \to \infty$ implies

$$\psi(d(q, fp)) \leq \psi(0) - \varphi(0), \tag{2.13}$$

$\psi(d(q, fp)) = 0$, and $f(p) = q$. Hence q is the point of coincidence of f and g. Assume that there is another point of coincident r in X such that $r \neq q$. Then there exists s in X such that $f(s) = g(s) = r$. Using (2.1), we have

$$\begin{aligned}\psi(d(gp, gs)) &= \psi(d(fp, fs)) \\ &\leq \psi(d(gp, gs)) - \varphi(d(gp, gs)) \\ &< \psi(d(gp, gs)),\end{aligned} \tag{2.14}$$

which is a contradiction which proves the uniqueness of point of coincidence; the result now follows from Lemma 1.3 □

Corollary 2.2. *Let f, g be two self maps of a metric space (X, d) satisfying*

$$\int_0^{\psi(d(fx,fy))} \phi(t)dt \leq \int_0^{\psi(d(gx,\, gy))} \phi(t)dt - \int_0^{\varphi(d(gx,gy))} \phi(t)dt \tag{2.15}$$

for all $x, y \in X$, where $\phi \in F$ and $\psi, \varphi \in G$. If range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Define $\Phi : R^+ \to R^+$ by $\Phi(x) = \int_0^x \phi(t)dt$, then $\Phi \in G$ and (2.15) becomes

$$\Phi(\psi(d(fx, fy))) \leq \Phi(\psi(d(gx, gy))) - \Phi(\varphi(d(gx, gy))), \tag{2.16}$$

which further can be written as

$$\psi_1(d(fx, fy)) \leq \psi_1(d(gx, gy)) - \varphi_1(d(gx, gy)), \tag{2.17}$$

where $\psi_1 = \Phi \circ \psi$ and $\varphi_1 = \Phi \circ \varphi \in G$. Clearly $\psi_1, \varphi_1 \in G$. Hence by Theorem 2.1 f and g have unique common fixed point. □

Now we present two examples in the support of Theorem 2.1.

Example 2.3. Let X = $[0,1] \cup \{2,3,4,\ldots\}$,

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0,1],\ x \neq y, \\ x + y, & \text{if at least one of } x \text{ or } y \notin [0,1],\ x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{2.18}$$

Then (X, d) is a complete metric space [17]. Consider $f : X \to X$, and $\psi, \varphi \in G$ as given in [9]:

$$fx = \begin{cases} x - \dfrac{1}{2}x^2, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } x > 1, \end{cases} \tag{2.19}$$

$$\psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{if } t \geq 1, \end{cases} \tag{2.20}$$

$$\varphi(t) = \begin{cases} \dfrac{1}{2}t^2, & \text{if } 0 \leq t \leq 1, \\ \dfrac{1}{2}, & \text{if } t \geq 1. \end{cases} \tag{2.21}$$

Let $g : X \to X$ be defined as

$$gx = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ x + 1, & \text{if } x > 1. \end{cases} \tag{2.22}$$

Assume that $x > y$ and discuss the following cases.

When $x \in [0,1]$, then

$$\begin{aligned}\varphi(d(fx,fy)) &= \left(x-\frac{1}{2}x^2\right)-\left(y-\frac{1}{2}y^2\right)\\ &\leq (x-y)-\frac{1}{2}\,(x-y)^2\\ &= \varphi(d(gx,gy))-\varphi(d(gx,gy)).\end{aligned} \tag{2.23}$$

Taking x in $\{3,4,\ldots\}$, and y in $[0,1]$, we obtain

$$\begin{aligned}\varphi(d(fx,fy)) &= \left(x-1+y-\frac{1}{2}y^2\right)^2\\ &\leq (x+y-1)^2,\\ \varphi(d(gx,gy)) &= (x+y+1)^2,\\ \varphi(d(gx,gy)) &= \frac{1}{2}.\end{aligned} \tag{2.24}$$

Hence

$$\varphi(d(fx,fy)) \leq \varphi(d(gx,gy))-\varphi(d(gx,gy)). \tag{2.25}$$

Now, when $x \in \{3,4,\ldots\}$, and $y \notin [0,1]$, then

$$\begin{aligned}\varphi(d(fx,fy)) &= (x-1+y-1)^2\\ &< (x+y-1)^2,\\ \varphi(d(gx,gy)) &= (x+y+2)^2,\\ \varphi(d(gx,gy)) &= \frac{1}{2}.\end{aligned} \tag{2.26}$$

Obviously (2.31) holds. Finally when $x = 2$, we have $y \in [0,1]$, $fx = 1$, and

$$d(fx,fy) = 1-\left(y-\frac{1}{2}y^2\right) \leq 1, \tag{2.27}$$

so that $\psi(d(fx, fy)) \leq 1$, then

$$\begin{aligned}\psi(d(gx, gy)) - \varphi(d(gx, gy)) &= (3 + y)^2 - \frac{1}{2} \\ &> 1 \geq \psi(d(fx, fy)).\end{aligned} \tag{2.28}$$

Thus all conditions of Theorem 2.1 are satisfied. Moreover f and g have a unique common fixed point.

Example 2.4. Let $X = [0, 1]$ and $f, g : X \to X$ be given as

$$f(x) = \frac{2}{5}x^2 + \frac{3}{5}, \qquad g(x) = \frac{2}{3}x^2 + \frac{1}{3}. \tag{2.29}$$

Consider $\psi, \varphi \in G$ as $\psi(t) = (1/2)t$ and $\varphi(t) = (1/10)t$. Then we have

$$\begin{aligned}\psi(d(fx, fy)) &= \frac{2}{10}\left|x^2 - y^2\right| \\ &\leq \frac{1}{2}\frac{2}{3}\left|x^2 - y^2\right| - \frac{1}{10}\frac{2}{3}\left|x^2 - y^2\right| \\ &= \psi(d(gx, gy)) - \varphi(d(gx, gy)).\end{aligned} \tag{2.30}$$

Note that $x = 1$ is the unique coincidence point of f and g, and f and g are commuting at $x = 1$. Hence all conditions of Theorem 2.1 are satisfied. Moreover, $x = 1$ is the unique common fixed point of f and g.

Following theorem can be viewed as generalization and extension of [3, Theorem 3].

Theorem 2.5. *Let f be a self map of a complete metric space (X, d) satisfying*

$$\int_0^{\psi(d(fx,fy))} \phi(t)dt \leq \int_0^{\psi(d(x,y))} \phi(t)dt - \int_0^{\varphi(d(x,y))} \phi(t)dt \tag{2.31}$$

for all $x, y \in X$, where $\phi \in F$ and $\psi, \varphi \in G$. Then f has a unique fixed point. Moreover f has property P.

Proof. Existence and uniqueness of fixed point of f follows from Corollary 2.2. Now we prove that f has property P. Let $u \in F(f^n)$. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. We claim that $fu = u$. If not, then, by (2.31),

$$\begin{aligned}\int_0^{\psi(d(u,fu))} \phi(t)dt &= \int_0^{\psi(d(f^n u,f(f^n u)))} \phi(t)dt = \int_0^{\psi(d(f(f^{n-1}u),f(f^n u)))} \phi(t)dt \\ &\le \int_0^{\psi(d(f^{n-1}u,f^n u))} \phi(t)dt - \int_0^{\varphi(d(f^{n-1}u,f^n u))} \phi(t)dt \\ &\le \int_0^{\psi(d(f^{n-1}u,f^n u))} \phi(t)dt = \int_0^{\psi(d(f(f^{n-2}u),f(f^{n-1}u)))} \phi(t)dt \\ &\le \int_0^{\psi(d(f^{n-2}u,f^{n-1}u))} \phi(t)dt - \int_0^{\varphi(d(f^{n-2}u,f^{n-1}u))} \phi(t)dt \\ &\le \int_0^{\psi(d(f^{n-2}u,f^{n-1}u))} \phi(t)dt.\end{aligned} \tag{2.32}$$

Continuing this process we arrive at

$$\begin{aligned}\int_0^{\psi(d(u,fu))} \phi(t)dt &\le \int_0^{\psi(d(u,fu))} \phi(t)dt - \int_0^{\varphi(d(u,fu))} \phi(t)dt \\ &< \int_0^{\psi(d(u,fu))} \phi(t)dt,\end{aligned} \tag{2.33}$$

which is a contradiction. Hence the result follows. □

Remarks 2.6. Existence and uniqueness of fixed point of f in above theorem also follows from [9, Theorem 1].

Remarks 2.7. (a) It is noted that if maps f and g involved in Theorem 2.1 are commuting, then they have property Q.

(b) Suzuki [18] observed that Branciari [1, Theorem 1] is a particular case of Meir-Keeler fixed point theorem [19]. We pose an open problem to see if a link exists between the contractive conditions (2.15) and the Meir-Keeler condition.

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