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Research Article

Uniform in Time Description for Weak Solutions of the Hopf Equation with Nonconvex Nonlinearity

Antonio Olivas Martinez and Georgy A. Omel'yanov

Departamento de Matematicas, Universidad de Sonora, Calle Rosales y Blvd Luis Encinas, s/n, 83000, Hermosillo, Sonora, Mexico

Correspondence should be addressed to Georgy A. Omel'yanov, omel@hades.mat.uson.mx

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We consider the Riemann problem for the Hopf equation with concave-convex flux functions. Applying the weak asymptotics method we construct a uniform in time description for the Cauchy data evolution and show that the use of this method implies automatically the appearance of the Oleinik E-condition.

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1. Introduction

It is well known that the uniqueness problem for weak solutions of hyperbolic quasilinear systems remains unsolved up to now in the case of arbitrary jump amplitudes. Moreover, the approach which has been used successfully for shocks with sufficiently small amplitudes [1, 2] cannot be extended to the general case. On the other hand, there is a possibility to construct the unique stable solution passing to parabolic regularization. However, the vanishing viscosity method cannot be used effectively for nontrivial vector problems. Indeed, in the essentially nonintegrable case we, obviously, do not have the exact solution. Moreover, any traditional asymptotic method does not serve for the problem of nonlinear wave interaction since it leads to the appearance of a chain of partial differential equations, the first of them is nonlinear and, in fact, coincides with the original equation.

We are of opinion that a progress in this problem can be achieved in the framework of the weak asymptotics method; see, for example, [3–5]. In this method the approximated solutions are sought in the same form as in the Whitham method modified for nonlinear waves with localized fast variation [6, 7] (for the original Whitham method for rapidly oscillating waves see [8]). At the same time, the discrepancy in the weak asymptotics method is assumed to be small in the sense of the space of functionals $\mathfrak{D}'_x$ over test functions depending only on the "space" variable x. This somehow trivial modification allows us to reduce the problem of describing interaction of nonlinear waves to solving some systems of ordinary differential equations (instead of solving partial differential equations). Respectively, the main characteristics of the solution (the trajectory of the limiting singularity motion, etc.) can be found by this method, whereas the shape of the real solution cannot be found.

Applications of the weak asymptotics method allowed among other to investigate the interaction of solitons for nonintegrable versions of the KdV and sine-Gordon equations [9–11], to describe uniformly in time the confluence of the shock waves for the Hopf equation with convex nonlinearities [4], as well as to construct uniform in time asymptotics for the Riemann problem for isothermal gas dynamics [12–14] and delta-shock solutions for the so-called pressureless gas dynamics [15, 16]. However, it should be necessary to verify the method application to each new type of problems.

As for the uniqueness problem, we are not ready now to consider the vector case; so we are going to simulate it and to investigate the Riemann problem for the scalar conservation law with nonconvex nonlinearity:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad t > 0, \ x \in \mathbb{R}^1, \tag{1.1}$$

$$u|_{t=0} = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.2}$$

Furthermore, the structure of the uniform in time asymptotics for a regularization of the problem (1.1), (1.2) with an arbitrary $f(u)$ can be very complicated. On the other hand, it is clear that we can define a sequence of time intervals and consider the asymptotics u_ε for each time interval as a combination of local interacting solutions. Almost without loss of generality we can suppose that the local solutions correspond to convex or concave-convex parts of the nonlinearity $f(u_\varepsilon)$. That is why, in view of the result [4], we restrict ourselves to the concave-convex case; that is, we will suppose that

$$uf''(u) > 0 \quad (u \neq 0), \quad f''(0) = 0, \quad f'''(0) \neq 0, \quad \lim_{|u| \to \infty} f'(u) = \infty. \tag{1.3}$$

For definiteness we assume also that

$$u_- > 0 > u_+. \tag{1.4}$$

Let us recall that the solution of the initial-value problem is called stable if it depends continuously on the initial data (see, e.g., [2]). Obviously, the stable solution to the problem (1.1)–(1.4) is well known (see, e.g., [17]) and it can be constructed using the characteristics method for (1.1) with regularized initial data. In particular, the stable solution will be the shock wave with amplitude $u_- - u_+$ if and only if the Oleinik E-condition

$$\frac{f(u) - f(u_-)}{u - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geq \frac{f(u_+) - f(u)}{u_+ - u} \tag{1.5}$$

is satisfied for any $u \in [u_+, u_-]$.

The same shock wave presents an example of nonstable weak solutions if the condition (1.5) is violated. Let us note that this nonadmissible shock wave looks as if it is stable if $f'(u_-) > f'(u_+)$.

Technically, our result consists of obtaining uniform in time asymptotic solutions for a regularization of the problem (1.1), (1.2). However, we consider as the main result the fact that the weak asymptotics method allows to construct the admissible limiting solution without any additional conditions. In particular, we obtain automatically the Oleinik E-condition for the shock wave solution.

The structure of the asymptotics construction is the following. Firstly we pass from the initial step function to a sequence of step functions such that each jump corresponds to a stable solution (in fact, to a shock wave or a centered rarefaction). Here we take into account the fact that weak asymptotics similarly to exact weak solution is not unique in the unstable case. At the same time, describing the collision of stable waves, we obtain automatically the stable scenario of interaction. Therefore, this passage from the Riemann problem to the problem of interaction of stable waves can be treated as a "regularization." For our model example it means the transformation of the problem (1.1), (1.2) to the following "regularization":

$$\begin{gathered}
\frac{\partial u_\Delta}{\partial t} + \frac{\partial f(u_\Delta)}{\partial x} = 0, \quad t > 0, \ x \in \mathbb{R}^1, \\
u_\Delta|_{t=0} = \overline{u} + (u_- - \overline{u})H(x_1^0 - x) + (u_+ - \overline{u})H(x - x_2^0),
\end{gathered} \tag{1.6}$$

where $\Delta = x_2^0 - x_1^0 > 0$ is the "regularization" parameter, $H(x)$ is the Heaviside function, and $\overline{u} \in (u_+, u_-)$. We choose the intermediate state $\overline{u} < 0$ such that the left jump (at the point $x = x_1^0$) corresponds for $t \ll \Delta$ to the stable shock wave, whereas the right jump (at the point $x = x_2^0$) corresponds to the centered rarefaction. Let us note that the problem (1.6) with $\Delta = \text{const}$ is of interest by itself.

Next, we pass from (1.6) to the parabolic regularization:

$$\begin{gathered}
\frac{\partial u_{\Delta\varepsilon}}{\partial t} + \frac{\partial f(u_{\Delta\varepsilon})}{\partial x} = \varepsilon \frac{\partial^2 u_{\Delta\varepsilon}}{\partial x^2}, \quad t > 0, \ x \in \mathbb{R}^1, \\
u_{\Delta\varepsilon}|_{t=0} = \overline{u} + (u_- - \overline{u})\omega\left(\frac{(x_1^0 - x)}{\varepsilon}\right) + (u_+ - \overline{u})\omega\left(\frac{(x - x_2^0)}{\varepsilon}\right),
\end{gathered} \tag{1.7}$$

where $\omega(x/\varepsilon)$ is a regularization of the Heaviside function with the parameter $\varepsilon \ll \Delta$. The contents of Sections 2 and 3 are the construction of the weak asymptotic solution to the problem (1.7).

Finally, in conclusion, we consider the limiting solution both for $\varepsilon \to 0$ and for $\Delta \to 0$.

Completing this section let us formalize the concept of the weak asymptotics.

Definition 1.1. Let $u_{\Delta\varepsilon} = u_{\Delta\varepsilon}(t, x)$ be a function that belongs to $C^\infty([0,T] \times \mathbb{R}_x^1)$ for each $\varepsilon = \text{const} > 0$ and to $C([0,T]; \mathfrak{D}'(\mathbb{R}_x^1))$ uniformly in $\varepsilon \in [0, \text{const}]$. One says that $u_{\Delta\varepsilon}(t, x)$ is a weak asymptotic mod $\mathcal{O}_{\mathfrak{D}'}(\varepsilon)$ solution of (1.7) if the relation

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_{\Delta\varepsilon}\psi dx - \int_{-\infty}^{\infty} f(u_{\Delta\varepsilon})\frac{\partial\psi}{\partial x}dx = \mathcal{O}(\varepsilon) \tag{1.8}$$

holds uniformly in $t \in (0, T]$ for any test function $\psi = \psi(x) \in \mathfrak{D}(\mathbb{R}_x^1)$.

Here and below the estimate $\mathcal{O}(\varepsilon^k)$ is understood in the $C([0,T])$ sense: $|\mathcal{O}(\varepsilon^k)| \leq C_T\varepsilon^k$ for $t \in [0, T]$.

Definition 1.2. A function $g(t, x, \varepsilon)$ is said to be of the value $\mathcal{O}_{\mathfrak{D}'}(\varepsilon^k)$ if the relation

$$(g, \psi) = \int_{-\infty}^{\infty} g(t, x, \varepsilon)\psi(x)dx = \mathcal{O}\left(\varepsilon^k\right) \tag{1.9}$$

holds for any test function $\psi = \psi(x) \in \mathfrak{D}(\mathbb{R}_x^1)$.

It is very important to note that the viscosity term in (1.7) has the value $\mathcal{O}_{\mathfrak{D}'}(\varepsilon)$ and disappears in (1.8). The same is true for any parabolic regularization of the form $\varepsilon(b(u))_{xx}$. Thus, we see that the weak asymptotic mod $\mathcal{O}_{\mathfrak{D}'}(\varepsilon)$ solution does not depend on the dissipative terms. In what follows we will omit the subindex Δ for u.

2. Construction of the Asymptotic Solution for the First Interaction

2.1. Asymptotic Ansatz

To present the asymptotic ansatz for the problem (1.7) let us consider the possible scenario of the initial data (1.6) evolution. Our choice of $\overline{u}$ in (1.6) implies that

$$f(u) < f(\overline{u}) + \frac{f(u_-) - f(\overline{u})}{u_- - \overline{u}}(u - \overline{u}) \quad \forall u \in (\overline{u}, u_-), \tag{2.1}$$

$$f(u) > f(\overline{u}) + \frac{f(\overline{u}) - f(u_+)}{\overline{u} - u_+}(u - \overline{u}) \quad \forall u \in (u_+, \overline{u}). \tag{2.2}$$

Thus, the problem (1.6) solution should be the superposition of noninteracting shock wave and centered rarefaction during a sufficiently small time interval, namely,

$$\begin{aligned} u = \overline{u} &+ (u_- - \overline{u})H(\varphi_{10}(t) - x) + \left\{r\left(\frac{x - x_2^0}{t}\right) - \overline{u}\right\}H(x - \varphi_{20}(t)) \\ &+ \left\{u_+ - r\left(\frac{x - x_2^0}{t}\right)\right\}H(x - \varphi_{30}(t)), \end{aligned} \tag{2.3}$$

where $t \ll \Delta$, $\varphi_{10}(t)$ is the shock wave phase:

$$\varphi_{10}(t) = x_1^0 + s_{10}t, \quad s_{10} \overset{\text{deff}}{=} \frac{f(u_-) - f(\overline{u})}{u_- - \overline{u}}, \tag{2.4}$$

$\varphi_{k0} = \varphi_{k0}(t)$ for $k = 2, 3$ are the characteristics:

$$\varphi_{20} = x_2^0 + f'(\overline{u})t, \qquad \varphi_{30} = x_2^0 + f'(u_+)t, \tag{2.5}$$

and $r = r((x - x_2^0)/t)$ is the centered rarefaction with the support between $x = \varphi_{20}$ and $x = \varphi_{30}$:

$$r \in C^\infty \text{ is such that } f'(r(z)) = z. \tag{2.6}$$

Assumption (2.1) implies the intersection of the shock wave trajectory φ_{10} with the characteristic φ_{20} at some time instant $t_1^* = \mathcal{O}(\Delta)$. Accordingly, the interaction between the shock and the singularity of the type $(x - \varphi_{20})_+^\lambda$, $0 < \lambda < 1$ (i.e., with the left border of the rarefaction) has to occur, which will result in the appearance of a shock wave with a variable amplitude. Furthermore, this shock wave can interact with the right border of the rarefaction wave. So, generally speaking, the asymptotic ansatz needs to contain two fast variables. However, the distance between the characteristics $x = \varphi_{20}(t)$ and $x = \varphi_{30}(t)$ at the first critical time t_1^* is greater than a constant for $\Delta = \text{const}$. Thus, the shock wave trajectory can intersect the characteristic $x = \varphi_{30}(t)$ only at a second critical time instant t_2^* such that $t_2^* - t_1^* \geq \text{const} > 0$. Therefore, we can investigate the interaction process by stages.

Let us consider the first evolution stage for the solution of the problem (1.7). We present the asymptotic ansatz as a natural regularization of (2.3):

$$u_\varepsilon = \overline{u} + (u_- - \overline{u})\omega_1 + (R - \overline{u})\omega_2 + (u_+ - R)\omega_3, \tag{2.7}$$

where $R = R(x, t, \varepsilon) \in C^\infty(\mathbb{R}^1 \times \mathbb{R}_+^1 \times [0, 1])$ is a function such that

$$R(x, t, \varepsilon) = \begin{cases} \overline{u} & \text{if } x < \varphi_{20} - c\varepsilon, \\ r\left(\dfrac{x - x_2^0}{t}\right) & \text{if } \varphi_{20} < x < \varphi_{30}, \\ u_+ & \text{if } x > \varphi_{30} + c\varepsilon \end{cases} \tag{2.8}$$

with a constant $c > 0$,

$$\omega_1 = \omega\left(\frac{-x + \varphi_1}{\varepsilon}\right), \qquad \omega_2 = \omega\left(\frac{x - \varphi_2}{\varepsilon}\right), \qquad \omega_3 = \omega\left(\frac{x - \varphi_{30}}{\varepsilon}\right), \tag{2.9}$$

and $\omega(z/\varepsilon)$ is the Heaviside function regularization.

Furthermore, the phases $\varphi_k = \varphi_k(\tau, t)$, $k = 1, 2$, are assumed to be smooth functions such that

$$\varphi_k(\tau, t) \longrightarrow \varphi_{k0}(t) \quad \text{as } \tau \longrightarrow +\infty, \qquad \varphi_k(\tau, t) \longrightarrow \varphi_{k1}(t) \quad \text{as } \tau \longrightarrow -\infty \tag{2.10}$$

exponentially fast, where τ denotes the "fast time":

$$\tau = \frac{\psi_0(t)}{\varepsilon}, \quad \psi_0(t) = \varphi_{20}(t) - \varphi_{10}(t). \tag{2.11}$$

To simplify the formulas we also suppose that

$$\varphi_{11}(t) = \varphi_{21}(t). \tag{2.12}$$

We assume that ω tends to its limiting values

$$0 = \lim_{\eta \to -\infty} \omega(\eta), \qquad 1 = \lim_{\eta \to \infty} \omega(\eta) \tag{2.13}$$

at an exponential rate. Moreover, since the limiting as $\varepsilon \to 0$ solution does not depend on the choice of ω, let

$$\omega'_\eta > 0, \quad \omega(\eta) + \omega(-\eta) = 1. \tag{2.14}$$

The first assumption (2.10) implies that the ansatz (2.7) describes the two noninteracting waves (2.3) for $t \leq t_1^* - c\varepsilon^\alpha$, $\alpha \in (0,1)$. The second assumptions (2.10) and (2.12) imply that the ansatz (2.7) describes the union of the shock and the rarefaction waves for $t \geq t_1^* + c\varepsilon^\alpha$.

2.2. *Preliminary Calculations*

To determine the asymptotics (2.7) we should calculate weak expansions of u_ε and $f(u_\varepsilon)$. Almost trivial calculations show that

$$u_\varepsilon = u_- - (u_- - \overline{u})H_1 + (R - \overline{u})H_2 + (u_+ - R)H_3 + \mathcal{O}_{\mathfrak{D}'}(\varepsilon), \tag{2.15}$$

where

$$H_k = H(x - \varphi_k) \quad \text{for } k = 1,2, \qquad H_3 = H(x - \varphi_{30}). \tag{2.16}$$

Next, we have to calculate the weak expansion for the nonlinear term.

Lemma 2.1. *Under the assumptions mentioned above the following relation holds:*

$$\begin{aligned} f(u_\varepsilon) = f(u_-) &- (u_- - \overline{u})B_1 H_1 + \left\{ (R_2 - \overline{u})B_2 - f(R_2) + f(R) \right\} H_2 \\ &+ \left\{ f(u_+) - f(R) \right\} H_3 + \mathcal{O}_{\mathfrak{D}'}(\varepsilon), \end{aligned} \tag{2.17}$$

where B_i are the following convolutions:

$$\begin{aligned} B_1 &= \int_{-\infty}^{\infty} \omega'(\eta) f'(\overline{u} + (u_- - \overline{u})\omega(\eta) + (R_1 - \overline{u})\omega(-\eta - \sigma))d\eta, \\ B_2 &= \int_{-\infty}^{\infty} \omega'(\eta) f'(\overline{u} + (u_- - \overline{u})\omega(-\eta - \sigma) + (R_2 - \overline{u})\omega(\eta))d\eta \end{aligned} \tag{2.18}$$

with the properties

$$\begin{gathered} \lim_{\sigma \to +\infty} B_1 = \frac{f(u_-) - f(\overline{u})}{u_- - \overline{u}}, \qquad \lim_{\sigma \to -\infty} B_1 = \frac{f(R_1 + u_- - \overline{u}) - f(\overline{u})}{u_- - \overline{u}}, \\ \lim_{\sigma \to +\infty} B_2 = f'(\overline{u}), \qquad \lim_{\sigma \to -\infty} B_2 = \frac{f(u_- + R_2 - \overline{u}) - f(u_-)}{R_2 - \overline{u}}, \end{gathered} \tag{2.19}$$

$\sigma = \sigma(\tau, t, \varepsilon)$ characterizes the distance between the trajectories φ_1 and φ_2, namely,

$$\sigma = \frac{\varphi_2 - \varphi_1}{\varepsilon}, \tag{2.20}$$

and $R_k = R(\varphi_k, t, \varepsilon)$ for $k = 1, 2$, $R_3 = R(\varphi_{30}, t, \varepsilon)$.

Sketch of the Proof

For each $\psi(x) \in \mathfrak{D}(\mathbb{R}^1)$ we have

$$\begin{aligned} (f(u_\varepsilon), \psi) &= -\int_{-\infty}^{\infty} f(u_\varepsilon) \frac{d\phi(x)}{dx} dx \\ &= f(u_-) \int_{-\infty}^{\infty} \psi(x) dx + \int_{-\infty}^{\infty} \frac{\partial u_\varepsilon}{\partial x} f'(u_\varepsilon) \phi(x) dx, \end{aligned} \tag{2.21}$$

where $\phi(x) = \int_x^\infty \psi(x') dx'$.

Next, the derivative $\partial u_\varepsilon / \partial x$ contains terms of value $\mathcal{O}(1/\varepsilon)$, say $\omega'((\varphi_1 - x)/\varepsilon)/\varepsilon$ and the term $(\omega_2 - \omega_3) R'_x$. To calculate the first term we change the variable, say $\eta = (\varphi_1 - x)/\varepsilon$, and apply the Taylor expansion. Therefore,

$$\begin{aligned} -\int_{-\infty}^{\infty} \frac{1}{\varepsilon} \omega'\left(\frac{\varphi_1 - x}{\varepsilon}\right) f'(u_\varepsilon) \phi(x) dx &= \int_{-\infty}^{\infty} \omega'(\eta)\, f'(u_\varepsilon) \phi(x) \big|_{x = \varphi_1 - \varepsilon\eta} d\eta \\ &= B_1 \phi(\varphi_1) + \mathcal{O}(\varepsilon). \end{aligned} \tag{2.22}$$

Finally, we note that

$$\begin{aligned} \omega_2 - \omega_3 &= H(x - \varphi_2) - H(x - \varphi_3) + \mathcal{O}_{\mathfrak{D}'}(\varepsilon), \\ u_\varepsilon \big|_{x \in [\varphi_2, \varphi_{30}]} &= \overline{u} + (R - \overline{u})\omega_2 = R + \mathcal{O}_{\mathfrak{D}'}(\varepsilon). \end{aligned} \tag{2.23}$$

Thus,

$$\int_{-\infty}^{\infty} R'_x(\omega_2-\omega_3)f'(u_\varepsilon)\phi(x)dx = \int_{\varphi_2}^{\varphi_{30}} R'_x f'(R)\phi(x)dx + \mathcal{O}(\varepsilon)$$
$$= \phi(x)f(R)\Big|_{x=\varphi_2}^{x=\varphi_{30}} + \int_{\varphi_2}^{\varphi_{30}} f(R)\psi(x)dx + \mathcal{O}(\varepsilon). \tag{2.24}$$

This implies the formula (2.17).

To calculate the limiting values (2.19) of the convolutions B_i it is enough to use the stabilization properties (2.13) of the function $\omega(\eta)$.

Remark 2.2. The convolutions B_i are the functions of σ, τ, and t. At the same time we can treat B_i as functions of σ, τ, and ε. Indeed, let us denote by x_1^* the intersection point of the trajectories $x=\varphi_{10}(t)$ and $x=\varphi_{20}(t)$, that is, $x_1^*=\varphi_{10}(t_1^*)=\varphi_{20}(t_1^*)$. Then, by virtue of (2.4) and (2.5)

$$\varphi_{10}(t) = x_1^* + s_{10}(t-t_1^*), \qquad \varphi_{20} = x_1^* + f'(\overline{u})(t-t_1^*). \tag{2.25}$$

Consequently,

$$\tau = \frac{\varphi_0'}{\varepsilon}(t-t_1^*), \quad \varphi_0' \overset{\text{deff}}{=} f'(\overline{u}) - s_{10}, \tag{2.26}$$

$$B_i(\sigma,\tau,t)\big|_{t=t_1^*+\varepsilon\tau/\varphi_0'} \overset{\text{deff}}{=} \widetilde{B}_i(\sigma,\tau,\varepsilon). \tag{2.27}$$

Substituting the expressions (2.15) and (2.17) into the left-hand side of (1.8), we derive our main relation for obtaining the parameters of the asymptotic solution (2.7):

$$(u_- - \overline{u})\left\{\frac{d\varphi_1}{dt} - B_1\right\}\delta(x-\varphi_1) - (R_2-\overline{u})\left\{\frac{d\varphi_2}{dt} - B_2\right\}\delta(x-\varphi_2)$$
$$+\left\{\frac{\partial R}{\partial t} + \frac{\partial f(R)}{\partial x}\right\}\left(H(x-\varphi_2) - H(x-\varphi_{30})\right) = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{2.28}$$

2.3. Analysis of the Singularity Dynamics

Let us consider the system that is obtained by setting equal to zero the coefficients of the δ functions in relation (2.28), namely,

$$\frac{d\varphi_k}{dt} = B_k, \quad k=1,2. \tag{2.29}$$

Before the interaction ($\tau \to +\infty$) the first assumption (2.10) for $k=1,2$ implies $\sigma \to \tau \to +\infty$. Therefore, the limiting relations (2.19) verify the concordance of (2.29) with our definition (2.4) and (2.5) of φ_{10} and φ_{20}.

To find the limiting behavior of φ_k after the interaction ($\tau \to -\infty$) let us reduce the system (2.29) to a scalar equation. In view of (2.20) and (2.26)

$$\frac{d(\varphi_2 - \varphi_1)}{dt} = \varphi_0' \frac{d\sigma}{d\tau}. \tag{2.30}$$

Hence, by subtracting one equation in (2.29) from the other we obtain

$$\varphi_0' \frac{d\sigma}{d\tau} = \widetilde{B}_2 - \widetilde{B}_1 \stackrel{\text{deff}}{=} F(\sigma, \tau, \varepsilon), \tag{2.31}$$

where we take into account Remark 2.2. Using the first assumption (2.10) again we complete (2.31) with the condition

$$\lim_{\tau \to +\infty} \frac{\sigma}{\tau} = 1. \tag{2.32}$$

To study this problem let us analyze the function $F(\sigma, \tau, \varepsilon)$.

Lemma 2.3. *The value* $\sigma = 0$ *is the unique critical point for the problems* (2.31) *and* (2.32) *and is achieved for* $\tau \to -\infty$.

Proof. First we calculate

$$\begin{aligned} F|_{\sigma=0} &= \int_{-\infty}^{\infty} \{ f'(u_- + (R_2 - u_-)\omega(\eta)) - f'(R_1 + (u_- - R_1)\omega(\eta)) \} \\ &\quad \times \omega'(\eta) d\eta \big|_{\sigma=0} = \frac{f(R_2) - f(u_-)}{R_2 - u_-} - \frac{f(u_-) - f(R_1)}{u_- - R_1} \bigg|_{\sigma=0} = 0 \end{aligned} \tag{2.33}$$

since $\sigma = 0$ implies $\varphi_1 = \varphi_2$. Next we note that the assumption (2.1) implies the inequality: $F|_{\sigma \to +\infty} = \varphi_0' < 0$.

Let us consider now the function F for $|\sigma|$ bounded by a constant. Since $\varphi_2 - \varphi_1 = \sigma\varepsilon = \mathcal{O}(\varepsilon)$ for such values of σ, we can conclude that $R_k - \overline{u} = \mathcal{O}(\varepsilon)$, $k = 1, 2$. Therefore, with accuracy $\mathcal{O}(\varepsilon)$

$$F(\sigma, \tau, \varepsilon) = \int_{-\infty}^{\infty} \omega'(\eta - \sigma) f'(u_- - (u_- - \overline{u})\omega(\eta)) d\eta - \frac{f(u_-) - f(\overline{u})}{u_- - \overline{u}}. \tag{2.34}$$

In fact, the integral in the right-hand side of (2.34) is the average of f' with the kernel ω'. For concave-convex functions f the derivative $f'(u_- - (u_- - \overline{u})\omega(\eta))$ decreases monotonically from $f'(u_-) > 0$ to its minimal value $f'(0) < 0$ when η goes form $-\infty$ to the value $\eta = \eta_0$ where η_0 is such that $u_- - (u_- - \overline{u})\omega(\eta_0) = 0$. Next, when η goes form η_0 to $+\infty$, the derivative increases monotonically from $f'(0)$ to the limiting value $f'(\overline{u}) < 0$. At the same time, $\omega'(\eta - \sigma) > 0$ is a soliton-type exponentially vanishing function concentrated around the point $\eta = \sigma$. This implies that the behavior of the integral as a function of σ is the same as the behavior of $f'(u_- - (u_- - \overline{u})\omega(\eta))$ as the function of η. Therefore, the integral diagram has the unique solution of the equation $F(\sigma,\cdot,\cdot) = \text{const}$ for any nonnegative const $< f'(u_-)$. Thus, the equation $F(\sigma,\cdot,\cdot) = 0$ has the unique solution $\sigma = 0$; moreover $F'_\sigma|_{\sigma=0} < 0$.

Furthermore,

$$\begin{aligned}\frac{\partial F(\sigma,\tau,\varepsilon)}{\partial\tau}\bigg|_{\sigma=0} &= R'_{2_\tau}\int_{-\infty}^{\infty}\omega(\eta)\omega'(\eta)f''(\overline{u}+(u_- - \overline{u})\omega(-\eta)+(R_2-\overline{u})\omega(\eta))d\eta \\ &\quad - R'_{1_\tau}\int_{-\infty}^{\infty}\omega(-\eta)\omega'(\eta)f''(\overline{u}+(u_- - \overline{u})\omega(\eta)+(R_1-\overline{u})\omega(-\eta))\,d\eta\Big|_{\sigma=0} = 0\end{aligned} \tag{2.35}$$

since $R_1 = R_2$ and $R'_{1_\tau} = R'_{2_\tau}$ for $\sigma = 0$.

By induction we obtain the equality

$$\frac{d^m F(\sigma(\tau),\tau,\varepsilon)}{d\tau^m}\bigg|_{\sigma=0} = 0 \quad \forall m \in \mathbb{N} \tag{2.36}$$

which implies the statement of Lemma 2.3. □

Consequently, φ_1 and φ_2 converge after the first interaction that confirms the *a priori* supposition (2.12). To obtain the limiting trajectory $x = \varphi_{11} = \varphi_{21}$ of the shock wave, it is enough to pass to the limit $\tau \to -\infty$ in one of the equalities (2.29). Obviously, we obtain the following equation:

$$\frac{d\varphi_{11}}{dt} = \frac{f(u_-) - f(r)}{u_- - r}\bigg|_{x=\varphi_{11}}. \tag{2.37}$$

Let us come back to the relation (2.28). Defining φ_k in accordance with (2.29), we transform (2.28) to the following form:

$$\left\{\frac{\partial R}{\partial t} + \frac{\partial f(R)}{\partial x}\right\}(H(x-\varphi_2) - H(x-\varphi_{30})) = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{2.38}$$

For each test function ψ we have

$$\begin{aligned}&\left(\left\{\frac{\partial R}{\partial t} + \frac{\partial f(R)}{\partial x}\right\}(H(x-\varphi_2) - H(x-\varphi_{30})),\psi\right) \\ &\quad = \sum_{\pm}\int_{\Omega_\pm}\left\{\frac{\partial R}{\partial t} + \frac{\partial f(R)}{\partial x}\right\}\psi(x)dx + \int_{\varphi_{20}}^{\varphi_{30}}\left\{\frac{\partial R}{\partial t} + \frac{\partial f(R)}{\partial x}\right\}\psi(x)dx,\end{aligned} \tag{2.39}$$

where

$$\Omega_- = \{x : \varphi_2 < x < \varphi_{20}\}, \qquad \Omega_+ = \{x : \varphi_{20} < x < \varphi_2\}. \tag{2.40}$$

For $\varphi_{20} < x < \varphi_{30}$ the function R coincides with the centered rarefaction r, thus

$$\frac{\partial r}{\partial t} + \frac{\partial f(r)}{\partial x} = 0, \tag{2.41}$$

and the last integral in (2.39) is equal to zero. For $x \in \Omega_{\pm}$ we note that, according to definition (2.8), either $R = \text{const}$ or $|\varphi_2(\tau, t) - \varphi_{20}(t)| \leq c\varepsilon$, $c = \text{const}$. Since R'_t and R'_x are bounded uniformly in $t > 0$, we conclude that the first integrals in (2.39) have the value $\mathcal{O}(\varepsilon)$.

This completes the construction of the asymptotic solution (2.7).

Obviously, for $t \in (0, t_1^* - c_1\varepsilon^\alpha]$, $c_1 > 0$, $\alpha \in (0,1)$, the formula (2.7) is transformed to the form

$$\begin{aligned} u_\varepsilon = \overline{u} &+ (u_- - \overline{u})\omega\left(\frac{\varphi_{10}(t) - x}{\varepsilon}\right) + (R - \overline{u})\omega\left(\frac{x - \varphi_{20}(t)}{\varepsilon}\right) \\ &+ (u_+ - R)\omega\left(\frac{x - \varphi_{30}(t)}{\varepsilon}\right), \end{aligned} \tag{2.42}$$

which is the limit of (2.7) as $\tau \to +\infty$, $\sigma \to +\infty$.

For $t \in [t_1^* + c_2\varepsilon^\alpha, t_1^* + c_3\varepsilon^\alpha]$, $c_3 > c_2 > 0$, $\alpha \in (0,1)$, the formula (2.7) is transformed to the form

$$u_\varepsilon = u_- + (R - u_-)\omega\left(\frac{x - \varphi_{11}(t)}{\varepsilon}\right) + (u_+ - R)\omega\left(\frac{x - \varphi_{30}(t)}{\varepsilon}\right), \tag{2.43}$$

which is the limit of (2.7) as $\tau \to -\infty$, $\sigma \to 0$. This implies the following.

Lemma 2.4. *The weak asymptotic* $\operatorname{mod} \mathcal{O}_{\mathcal{D}'}(\varepsilon)$ *solution* (2.7) *describes uniformly in time the evolution of the problem* (1.7) *solution from the state* (2.42) *to the state* (2.43) *when t increases from* 0 *to* $t_1^* + c\varepsilon^\alpha$.

Clearly, passing to the limit as $\varepsilon \to 0$ we obtain the well-known result for the stable scenario of the collision of the shock wave and the centered rarefaction, when the shock wave enters into the rarefaction domain and propagates with variables velocity and amplitude (see (2.37) and (2.41)).

3. The Shock Wave Propagation over the Centered Rarefaction

Let us consider the evolution of the problem (1.7) solution for $t > t_1^*$. The behavior of (2.37) solution is well known (see, e.g., [18]): the trajectory $x = \varphi_{11}$ crosses all the characteristics $X = f'(u)t + x_2^0$ if

$$\frac{f(u_-) - f(u)}{u_- - u} > f'(u) \quad \text{for } u \in (\widetilde{u}, \overline{u}] \tag{3.1}$$

and tends to the characteristic $X = f'(\tilde{u})t + x_2^0$ with $\tilde{u}$ such that

$$\frac{f(u_-) - f(\tilde{u})}{u_- - \tilde{u}} = f'(\tilde{u}). \tag{3.2}$$

If $u_+ < \tilde{u}$, the resulting solution for the problem (1.7) will be a combination of the smoothed shock wave (with amplitude $u_- - \tilde{u}$ and the front trajectory $\varphi_{11} = f'(\tilde{u})t + x_2^0$) and the regularization for the centered rarefaction (defined near the domain bounded by the characteristics $\widetilde{X} = f'(\tilde{u})t + x_2^0$ and $X_+ = f'(u_+)t + x_2^0$). Obviously, $u \equiv u_-$ for $x < \varphi_{11}(t)$ and $u \equiv u_+$ for $x \geq X_+(t)$. Therefore, we obtain the following.

Theorem 3.1. *Let $u_+ < \tilde{u}$. Then the weak asymptotic* mod $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ *solution* (2.7) *describes uniformly in time the evolution of the initial data* (1.7) *into the described above regularization for the combination of the shock wave and the centered rarefaction.*

If $u_+ > \tilde{u}$, there occurs the collision of the shock wave and the weak singularity of the $(x - \varphi_{30})_-^\lambda$ type, $0 < \lambda < 1$ (in the limit as $\varepsilon \to 0$). To describe this collision let us construct again a weak asymptotic mod $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ solution. In a similar way to (2.7) we write

$$u_\varepsilon = u_- + (R - u_-)\omega_1 + (u_+ - R)\omega_3, \tag{3.3}$$

where $R = R(x, t, \varepsilon)$ is defined in (2.8) and

$$\omega_k = \omega\left(\frac{x - \varphi_k}{\varepsilon}\right), \quad k = 1, 3. \tag{3.4}$$

We suppose that the phases $\varphi_k = \varphi_k(\tau_1, t)$ are smooth functions such that

$$\varphi_1(\tau_1, t) \longrightarrow \varphi_{11}(t), \quad \varphi_3(\tau_1, t) \longrightarrow \varphi_{30}(t) \quad \text{as } \tau_1 \longrightarrow +\infty, \tag{3.5}$$

$$\varphi_1(\tau_1, t) \longrightarrow \overline{\varphi}(t), \quad \varphi_3(\tau_1, t) \longrightarrow \varphi_{31}(t) \quad \text{as } \tau_1 \longrightarrow -\infty, \tag{3.6}$$

exponentially fast, where the "fast time" τ_1 is defined as follows:

$$\tau_1 = \frac{\psi_1(t)}{\varepsilon}, \quad \psi_1(t) = \varphi_{30}(t) - \varphi_{11}(t). \tag{3.7}$$

To simplify the formulas we also suppose that

$$\overline{\varphi}(t) = \varphi_{31}(t). \tag{3.8}$$

The assumptions (3.5), (3.6), and (3.8) imply that the ansatz (3.3) coincides with the solution described in Section 2 as $\tau_1 \to +\infty$ and tends to the shock wave as $\tau_1 \to -\infty$.

Repeating the analysis of Section 2 we obtain the following statement.

Lemma 3.2. *Under the assumptions mentioned above the following relations hold:*

$$
\begin{aligned}
u_\varepsilon &= u_- + (R - u_-)H(x-\varphi_1) + (u_+ - R)H(x-\varphi_3) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),\\
f(u_\varepsilon) &= f(u_-) + \{(R_1 - u_-)C_1 - f(R_1) + f(R)\}H_1\\
&\quad + \{(u_+ - R_3)C_3 + f(R_3) - f(R)\}H_3 + \mathcal{O}_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{3.9}
$$

where C_i are the convolutions

$$
\begin{aligned}
C_1 &= \int_{-\infty}^{\infty} \omega'(\eta) f'(u_- + (R_1 - u_-)\omega(\eta) + (u_+ - R_1)\omega(\eta - \sigma_1))\,d\eta,\\
C_3 &= \int_{-\infty}^{\infty} \omega'(\eta) f'(u_- + (R_3 - u_-)\omega(\eta + \sigma_1) + (u_+ - R_3)\omega(\eta))\,d\eta
\end{aligned}
\tag{3.10}
$$

with the properties

$$
\begin{aligned}
\lim_{\sigma\to+\infty} C_1 &= \frac{f(u_-) - f(R_1)}{u_- - R_1}, \qquad \lim_{\sigma\to-\infty} C_1 = \frac{f(u_+ + u_- - R_1) - f(u_+)}{u_- - R_1},\\
\lim_{\sigma\to+\infty} C_3 &= f'(u_+), \qquad \lim_{\sigma\to-\infty} C_3 = \frac{f(u_- + u_+ - R_3) - f(u_-)}{u_+ - R_3},
\end{aligned}
\tag{3.11}
$$

$\sigma_1 = \sigma_1(\tau_1, t, \varepsilon)$ characterizes the distance between the trajectories φ_1 and φ_3, namely,

$$
\sigma_1 = \frac{(\varphi_3 - \varphi_1)}{\varepsilon},
\tag{3.12}
$$

and $R_k = R(\varphi_k, t, \varepsilon)$ for $k = 1, 3$.

Substituting the expressions (3.9) into the left-hand side of (1.8) we derive the following relation for obtaining the asymptotic parameters:

$$
\begin{aligned}
&-(R_1 - u_-)\left\{\frac{d\varphi_1}{dt} - C_1\right\}\delta(x-\varphi_1) - (u_+ - R_3)\left\{\frac{d\varphi_3}{dt} - C_3\right\}\delta(x-\varphi_3)\\
&\quad + \left\{\frac{\partial R}{\partial t} + \frac{\partial f(R)}{\partial x}\right\}(H_1 - H_3) = \mathcal{O}_{\mathcal{D}'}(\varepsilon).
\end{aligned}
\tag{3.13}
$$

To calculate the trajectories φ_1 and φ_3 we set the coefficients of the δ-functions in relation (3.13) equal to zero, namely,

$$
\frac{d\varphi_k}{dt} = C_k, \quad k = 1, 3.
\tag{3.14}
$$

Lemma 3.3. *Under the assumption $u_+ > \tilde{u}$, system* (3.14) *describes the confluence of the trajectories φ_1 and φ_3.*

Proof. Before the interaction ($\tau_1 \to +\infty$) $\sigma_1 \to +\infty$, so that we obtain again the Rankine-Hugoniot condition (2.37) for φ_{11}. Moreover, we obtain the second formula in (2.5) for the characteristic φ_{30}.

Subtracting the above relations we pass to the equation

$$\frac{d(\varphi_3 - \varphi_1)}{dt} = \varphi_1' \frac{d\sigma_1}{d\tau_1} = C_3 - C_1 \overset{\text{deff}}{=} F_1(\sigma_1, \tau_1, \varepsilon), \tag{3.15}$$

where we put t in terms of τ_1 and ε.

Suppositions (3.5) complete equation (3.15) with the condition

$$\lim_{\tau_1 \to +\infty} \frac{\sigma_1}{\tau_1} = 1. \tag{3.16}$$

The last step of the proof is similar to Lemma 2.3 verification of the following statement.

Lemma 3.4. *The value $\sigma_1 = 0$ is the unique critical point for the problems* (3.15) *and* (3.16) *and is achieved for $\tau_1 \to -\infty$.* □

Consequently, φ_1 and φ_3 converge after the second interaction that confirms the *a priori* supposition (3.8). Passing in (3.14) to the limit $\tau_1 \to -\infty$ we find the Rankine-Hugoniot condition

$$\frac{d\overline{\varphi}}{dt} = \frac{f(u_-) - f(u_+)}{u_- - u_+} \tag{3.17}$$

for the limiting trajectory $x = \overline{\varphi} = \varphi_{31}$ of the shock wave with the amplitude $u_- - u_+$. Thus, the supposition $u_+ \in (\widetilde{u}, \overline{u})$ is explicitly the stability condition for the limiting shock wave.

Finally we note that the relation

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial f(u_\varepsilon)}{\partial x} = \mathcal{O}_{\mathcal{D}'}(\varepsilon), \quad \text{for } \varphi_1 < x < \varphi_3 \tag{3.18}$$

can be proved in a similar way as in Section 2.

Summarizing the above arguments we obtain the following assertion.

Theorem 3.5. *Let $u_+ > \widetilde{u}$. Then the weak asymptotic* mod $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ *solutions* (2.7) *and* (3.3) *describes uniformly in time the evolution of the initial data* (1.6) *to the smoothed shock wave with amplitude $u_- - u_+$.*

4. Conclusion

Concluding all the result we obtain the following uniform in time description of the problem (1.7) solution: the front φ_1 of the smoothed shock wave and the left front φ_2 of the smoothed centered rarefaction merge during the time interval $(t_1^* - c\varepsilon^\alpha, t_1^* + c\varepsilon^\alpha)$, $0 < \alpha < 1$, in accordance with (2.29). If $u_+ < \tilde{u}$, then the further evolution of the front $\varphi_{11} \equiv \varphi_{21}$ is described by (2.37) whereas the right front of the rarefaction wave remains the characteristic φ_{30}. In the case $u_+ = \tilde{u}$ the trajectory φ_{11} tends to φ_{30} as $t \to \infty$. If $u_+ > \tilde{u}$, then the trajectories φ_{11} and φ_3 merge during the time interval $(t_2^* - c\varepsilon^\alpha, t_2^* + c\varepsilon^\alpha)$ in accordance with (3.14) and the resulting trajectory for $t \geq t_2^* + c\varepsilon^\alpha$ coincides with the shock wave front (3.17).

The condition $u_+ > \tilde{u}$, in view of (2.37) and the assumption (1.3), is equivalent to the inequality

$$f(u) \leq f(u_+) + \frac{f(u_+) - f(u_-)}{u_+ - u_-}(u - u_+) \quad \forall u \in [u_+, u_-], \tag{4.1}$$

which is explicitly the Oleinik E-condition.

In the limit as $\varepsilon \to 0$ but $\Delta = \text{const}$ all the trajectories loose the smoothness remaining continuous. However, the condition (4.1) does not depend on ε; so it remains valid for limiting solution.

To calculate the limit as $\Delta \to 0$ it is enough to note that $t_1^* = \mathcal{O}(\Delta)$ and $|\varphi_{30} - \varphi_{20}||_{t=t_1^*} = \mathcal{O}(\Delta)$. Therefore, the problems (1.1) and (1.2) solution will be, in accordance with the condition (4.1), either the shock wave with amplitude $u_- - u_+$ or the union of the shock wave (with amplitude $u_- - \tilde{u}$) and the centered rarefaction (with support between the characteristics $f'(\tilde{u})t$ and $f'(u_+)t$).

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