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Research Article

Automorphisms of Right-Angled Coxeter Groups

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If (W, S) is a right-angled Coxeter system, then $\mathrm{Aut}(W)$ is a semidirect product of the group $\mathrm{Aut}^{\circ}(W)$ of symmetric automorphisms by the automorphism group of a certain groupoid. We show that, under mild conditions, $\mathrm{Aut}^{\circ}(W)$ is a semidirect product of $\mathrm{Inn}(W)$ by the quotient $\mathrm{Out}^{\circ}(W) = \mathrm{Aut}^{\circ}(W)/\mathrm{Inn}(W)$. We also give sufficient conditions for the compatibility of the two semidirect products. When this occurs there is an induced splitting of the sequence $1 \to \mathrm{Inn}(W) \to \mathrm{Aut}(W) \to \mathrm{Out}(W) \to 1$ and consequently, all group extensions $1 \to W \to G \to Q \to 1$ are trivial.

1. Introduction

A Coxeter group W is determined by its diagram Γ. It is known that in certain cases, W determines Γ as well (see, e.g., [1, 2]). This is the case for right-angled Coxeter groups [3, 4], where the only relations are $s^2 = 1$ for all generators s and $st = ts$ for some pairs of generators s and t. For right-angled Coxeter groups, it is convenient to consider the Coxeter diagram (rather than the classical Coxeter graph): the presence of an edge with endpoints s and t means that s and t commute in W.

The properties of a right-angled Coxeter group W depend almost exclusively on the combinatorics of the diagram Γ. This is especially evident in the study of $\mathrm{Aut}(W)$. For example, the groupoid $\mathfrak{F}(\Gamma)$ consisting of the vertex sets of complete subgraphs of Γ plays an important role in [5] where Tits exhibits a split exact sequence

$$1 \longrightarrow \mathrm{Aut}^{\circ}(W) \longrightarrow \mathrm{Aut}(W) \longrightarrow \mathrm{Aut}(\mathfrak{F}(\Gamma)) \longrightarrow 1. \tag{1.1}$$

Tits also defines W to have propriété I if the complementary graph Γ^C has no triangles. He goes on to show that if W has propriété I, then $\mathrm{Aut}^{\circ}(W)$ is isomorphic to $\mathrm{Inn}(W)$.

In the present article, we focus on the set $\mathfrak{M}(\Gamma)$ whose members are the vertex sets of *maximal* complete subgraphs. We say that Γ has condition C if there exist $\mathbb{Z} \in \mathfrak{M}(\Gamma)$ and a collection $T_1, \ldots, T_n \in \mathfrak{M}(\Gamma) - \{Z\}$ such that

(C1) $T_i \cap Z \neq \varnothing$ for $1 \leq i \leq n$,

(C2) for each $v \in Z$, the cardinality of the set $\{i \mid v \notin T_i\}$ is odd.

When Γ has condition C, the subgraph spanned by $Z \cup T_1 \cup \cdots \cup T_n$ has propriété I. Thus, our condition C is a "local version" of Tits' propriété I.

Motivated by the results of Tits regarding sequence (1.1), we consider the sequence

$$1 \longrightarrow \mathrm{Inn}(W) \longrightarrow \mathrm{Aut}(W) \longrightarrow \mathrm{Out}(W) \longrightarrow 1. \tag{1.2}$$

A splitting of this sequence implies that all extensions of the form

$$1 \longrightarrow W \longrightarrow G \longrightarrow Q \longrightarrow 1 \tag{1.3}$$

are trivial (cf. [6]). Clearly (1.2) does not always split. For example, $\mathbb{Z}_2$ is a right-angled Coxeter group and

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \tag{1.4}$$

is a nontrivial extension. On the other hand, if W has no center, then finding nontrivial extensions of W is surprisingly difficult. Whether (1.2) splits for all W with trivial center is currently an open question.

The group $\mathrm{Aut}^\circ(W)$ has been studied extensively in [5, 7] and is called the group of symmetric automorphisms of W. We approach the problem of whether (1.2) splits by considering the following.

(a) Does the sequence

$$1 \longrightarrow \mathrm{Inn}(W) \longrightarrow \mathrm{Aut}^\circ(W) \longrightarrow \mathrm{Aut}^\circ(W)/\mathrm{Inn}(W) \longrightarrow 1 \tag{1.5}$$

split?

(b) Are the splittings of (1.1) and (1.5) compatible?

A positive answer to both (a) and (b) implies that (1.2) is a split extension. We show in Theorem 5.4 that if Γ has condition C, then (1.5) splits. To obtain a splitting of (1.2), we show that the action of $\mathrm{Aut}(\mathfrak{F}(\Gamma))$ on $\mathrm{Aut}^\circ(W)$ is compatible if Γ is not "too symmetrical." More precisely, our main result is the following.

Theorem 1.1. *If Γ has condition C and each $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$ leaves all vertices of Z invariant, then* (1.2) *is a split exact sequence.*

It was recently shown [8] that (1.5) always splits. However, this result does not lead to a generalization of Theorem 1.1 as the splitting given there is not, in general, compatible with (1.1).

2. Right-angled Coxeter groups

Coxeter groups are typically defined by presentations, and there are various conventions for representing such presentations diagramatically. In this section, we review some definitions and important properties, focusing exclusively on the right-angled case. See [9] or [10] for a comprehensive treatment.

If X is any set, let $P_2(X)$ denote the set of subsets of X with cardinality 2.

Definition 2.1. Given a finite set S and $E \subseteq P_2(S)$, let $\Gamma = (S, E)$ denote the undirected graph with vertex set S and edge set E (note that Γ does not have loops or parallel edges). As such graphs are often used to represent right-angled Coxeter groups, Γ is called a *Coxeter diagram*.

Definition 2.2. Given $T \subseteq S$, set $E_T = E \cap P_2(T)$. The graph $\Gamma_T = (T, E_T)$ is the subgraph of Γ *spanned* by T. A complete subgraph is *maximal* if it is not properly contained in any complete subgraph of Γ.

Definition 2.3. The presentation

$$\wp(\Gamma) = \langle S \mid s^2, (tu)^2; s \in S, \{t, u\} \in E \rangle \tag{2.1}$$

is the *Coxeter presentation defined by* Γ.

Definition 2.4. A group W is a *right-angled Coxeter group* if it has a presentation $\wp(\Gamma)$ defined by some Coxeter diagram Γ. In this case, one writes $W = W(\Gamma)$ and calls W the right-angled Coxeter group *defined* by Γ. The pair (W, S) is a *right-angled Coxeter system*.

Remark 2.5. The Coxeter diagram is not the same as the traditional Dynkin diagram. Indeed, as graphs, the Coxeter and Dynkin diagrams are complementary.

Clearly, each Coxeter diagram defines a unique right-angled Coxeter group. On the other hand, to recover the diagram from a group one must first choose a particular Coxeter presentation. It is natural to wonder whether nonisomorphic diagrams might define isomorphic groups. The relationship between right-angled Coxeter groups and their diagrams is clarified by the following result.

Theorem 2.6 (Radcliffe [3]). *If (W, S) and (W, S') are right-angled Coxeter systems for W, then there is an automorphism $\rho : W \to W$ such that $\rho(S) = S'$.*

A similar result was also obtained by Castella in [4].

Definition 2.7. A subgroup of W generated by a subset of S is called a *special subgroup*. If $T \subseteq S$, it is customary to write W_T for the subgroup generated by T. Finite special subgroups are called *spherical subgroups*. A subgroup of W is *parabolic* if it is conjugate to a special subgroup.

We conclude this section with statements of some of the remarkable properties enjoyed by special subgroups. For proofs, see [9] or [10].

Theorem 2.8. *If $A, B \subset S$, then $W_A \cap W_B = W_{A \cap B}$.*

Theorem 2.9. *If $T \subseteq S$, then W_T is the right-angled Coxeter group defined by Γ_T.*

Corollary 2.10. *The following are equivalent:*

(a) W_T *is spherical;*

(b) W_T *is an elementary abelian 2-group of rank* $|T|$*;*

(c) Γ_T *is complete.*

3. Automorphisms of right-angled Coxeter groups

For the remainder of this article, let $W = W(\Gamma)$ be the right-angled Coxeter group defined by the connected Coxeter diagram $\Gamma = (S, E)$. It is easily verified that, under the operation of symmetric difference, the set

$$\mathfrak{F}(\Gamma) = \{T \subseteq S \mid W_T \text{ is finite}\} \tag{3.1}$$

is a commutative groupoid with identity $\varnothing$.

In [5], Tits uses the group of automorphisms of $\mathfrak{F}(\Gamma)$ to exhibit $\mathrm{Aut}(W)$ as a semidirect product. We sketch the construction. Let $\sigma \in \mathrm{Aut}(W)$. It is well known that every maximal finite subgroup of W is parabolic (see, e.g., [11, Lemma 3.2.1]). Consequently, every finite subgroup of W is conjugate into a spherical subgroup. It follows that, for each $T \in \mathfrak{F}(\Gamma)$, there is a unique minimal $\overline{T} \in \mathfrak{F}(\Gamma)$ such that $\sigma(W_T)$ is conjugate into $W_{\overline{T}}$. The map

$$q : \mathrm{Aut}(W) \longrightarrow \mathrm{Aut}(\mathfrak{F}(\Gamma)) \tag{3.2}$$

given by $q(\sigma)(T) = \overline{T}$ is an epimorphism.

Definition 3.1. Let $\mathrm{Aut}^\circ(W)$ be the kernel of q. Elements of $\mathrm{Aut}^\circ(W)$ are called *symmetric automorphisms* of W.

Given $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$, consider $\widehat{\alpha} \in \mathrm{Aut}(W)$ defined by

$$\widehat{\alpha}(s) = \prod_{t \in \alpha(s)} t \tag{3.3}$$

for all $s \in S$. A main result of [5] is the following theorem.

Theorem 3.2. *The mapping* $\mathrm{Aut}(\mathfrak{F}(\Gamma)) \to \mathrm{Aut}(W)$ *given by* $\alpha \mapsto \widehat{\alpha}$ *is a splitting of the sequence*

$$1 \longrightarrow \mathrm{Aut}^\circ(W) \longrightarrow \mathrm{Aut}(W) \longrightarrow \mathrm{Aut}(\mathfrak{F}(\Gamma)) \longrightarrow 1. \tag{3.4}$$

Let d_Γ be the standard path metric on Γ that assigns length one to each edge. For each $s \in S$, define

$$\begin{aligned} s^* &= \{t \in S \mid d_\Gamma(s,t) \le 1\}, \\ s^\perp &= \{t \in S \mid d_\Gamma(s,t) \ge 2\}. \end{aligned} \tag{3.5}$$

The subgraph $\Gamma_{s^\perp}$ spanned by the elements of $s^\perp$ gives rise to certain generators of $\mathrm{Aut}^\circ(W)$ as follows. If K is the vertex set of a connected component of $\Gamma_{s^\perp}$, then the map $\sigma : S \to W$ given by

$$\sigma(t) = \begin{cases} sts, & t \in K, \\ t, & t \notin K, \end{cases} \tag{3.6}$$

extends to a unique involution $\sigma_{sK} \in \mathrm{Aut}^\circ(W)$. The following is easily deduced from [7].

Theorem 3.3. *For each $s \in S$, let $K_1^s, \dots, K_{m_s}^s$ be the vertex sets of the components of $\Gamma_{s^\perp}$. Then* $\mathrm{Aut}^\circ(W)$ *is generated by the set* $\{\sigma_{sK_i^s} \mid s \in S, 1 \le i \le m_s\}$.

Remark 3.4. In [7], Mühlherr gives a complete presentation for $\mathrm{Aut}^\circ(W)$ based on a slightly different set of generators.

4. Symmetric automorphisms

The subsets of S that generate maximal spherical subgroups of W play a key role in the subsequent development. As such, we define

$$\mathfrak{M}(\Gamma) = \{T \subseteq S \mid W_T \text{ is a maximal finite subgroup}\}. \tag{4.1}$$

Note that $\mathfrak{M}(\Gamma)$ is in one-to-one correspondence with the family of maximal complete subgraphs of Γ. The global behavior of a symmetric automorphism is governed by the following observation: if $\phi \in \mathrm{Aut}^\circ(W)$ and $T \in \mathfrak{M}(\Gamma)$, then there exists an element $a_T = a_T(\phi) \in W$ such that $\phi(x) = a_T x a_T^{-1}$ for all $x \in W_T$.

Remark 4.1. When $T \in \mathfrak{M}(\Gamma)$, W_T is its own centralizer. Thus, the element a_T above is determined up to right multiplication by any member of W_T (i.e., up to choice of representative for the coset $a_T W_T$).

If $T, U \in \mathfrak{M}(\Gamma)$ and $T \cap U \ne \varnothing$, then $a_T x a_T^{-1} = a_U x a_U^{-1}$ for all $x \in W_T \cap W_U = W_{T\cap U}$. Consequently, $a_T^{-1} a_U$ lies in the centralizer of $W_{T\cap U}$.

Definition 4.2. With ϕ, T, U as above, let $\gamma_{TU} = \gamma_{TU}(\phi) = a_T^{-1} a_U$. A representative of the double coset $W_T \gamma_{TU} W_U$ is called a *ϕ-transition* from W_T to W_U.

Remark 4.3. If $T, U, V \in \mathfrak{M}(\Gamma)$ have pairwise nonempty intersection, then $\gamma_{TT} = 1$, $\gamma_{VU} = (\gamma_{UV})^{-1}$, and $\gamma_{TU}\gamma_{UV} = \gamma_{TV}$. As the terminology suggests, the transitions are in some sense a group-theoretic analogue of the change-of-coordinate maps on a manifold.

For the remainder of this article we fix an element $Z \in \mathfrak{M}(\Gamma)$. For much of what follows, Z may be chosen arbitrarily; in Section 5 we show that a preferred choice may exist.

Given $x \in W$, let φ_x be the inner automorphism of W given by $w \mapsto xwx^{-1}$ for all $w \in W$. Restricting our attention to Z we define

$$\begin{aligned} \mathrm{Inn}(Z) &= \{\varphi_s \in \mathrm{Inn}(W) \mid s \in W_Z\}, \\ \mathrm{Fix}^\circ(Z) &= \{\phi \in \mathrm{Aut}^\circ(W) \mid \phi(x) = x \,\forall x \in Z\}. \end{aligned} \tag{4.2}$$

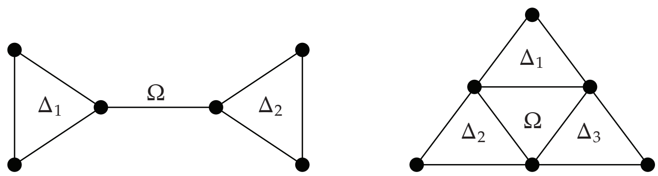
In other words, $\mathrm{Fix}^\circ(Z)$ is the pointwise stabilizer of W_Z under the action of $\mathrm{Aut}^\circ(W)$ on W. Observe that, since the subgraph Γ_Z spanned by Z is complete, $\mathrm{Inn}(Z)$ is Abelian.

Clearly, $\mathrm{Inn}(Z)$ is a subgroup of $\mathrm{Fix}^\circ(Z)$. Let π be the restriction to $\mathrm{Fix}^\circ(Z)$ of the natural map $\mathrm{Aut}^\circ(W) \to \mathrm{Out}^\circ(W)$ and choose a class $[\phi] \in \mathrm{Out}^\circ(W)$ with representative $\phi \in \mathrm{Aut}^\circ(W)$. If f is the restriction of ϕ to W_Z, then $f(x) = a_Z x a_Z^{-1}$ for all $x \in W_Z$. Since $[f^{-1}\phi] = [\phi]$ and $f^{-1}\phi \in \mathrm{Fix}^\circ(Z)$, it follows that π is onto. We have established the following.

Theorem 4.4. *The sequence*

$$0 \longrightarrow \mathrm{Inn}(Z) \xrightarrow{\ i\ } \mathrm{Fix}^\circ(Z) \xrightarrow{\ \pi\ } \mathrm{Out}^\circ(W) \longrightarrow 1 \tag{4.3}$$

is a central extension.



(a) (b)

Figure 1

In Section 5, we construct a retraction of the mapping $\mathrm{Inn}(Z) \xrightarrow{i} \mathrm{Fix}^{\circ}(Z)$. We now describe the key ingredient used in this construction. For each $T \in \mathfrak{F}(\Gamma)$, let

$$C(T) = \bigcap_{v \in T} v^{*}. \tag{4.4}$$

It follows at once that $W_{C(T\cap Z)}$ is the centralizer of $W_{T\cap Z}$ in W (cf. [5, page 350]). The function

$$\pi_T : C(T \cap Z) \longrightarrow Z - T \tag{4.5}$$

given by

$$\pi_T(s) = \begin{cases} 1, & s \notin Z - T, \\ s, & s \in Z - T, \end{cases} \tag{4.6}$$

extends uniquely to a retraction $W_{C(T\cap Z)} \to W_{Z-T}$, which we also denote by π_T.

5. Splittings

As in Section 4, we assume that Z is a fixed element of $\mathfrak{M}(\Gamma)$.

Definition 5.1. One says that Z satisfies *condition C* if there exist elements $T_1, T_2, \ldots, T_n \in \mathfrak{M}(\Gamma) - \{Z\}$ such that the following conditions hold.

(C1) $T_i \cap Z \neq \varnothing$ for each $1 \leq i \leq n$.

(C2) For each $v \in Z$, the cardinality of the set $\{i \mid v \notin T_i\}$ is odd.

If Γ contains a maximal complete subgraph Ω whose vertex set Z satisfies condition C, then one says that Γ has *condition C*. When Γ has condition C, then for each $1 \leq i \leq n$ above let Δ_i denote the maximal complete subgraph of Γ spanned by T_i.

Note that, since Γ is connected, when $n = 0$ our hypotheses imply that Γ is complete. In this case, W is Abelian and the results below are trivial. Thus, we may assume that n is a positive integer.

Example 5.2. We illustrate condition C with some examples.

(a) If Γ contains a maximal complete subgraph Ω with an even number of vertices each of which meets exactly one other complete subgraph of Γ, then Γ has condition C. For example, *any* Coxeter diagram Γ that contains Figure 1(a) as a subgraph (where Ω, Δ_1, and Δ_2 are maximal complete subgraphs of Γ) has condition C.

(b) Suppose n is odd and Γ has maximal complete subgraphs $\Omega, \Delta_1, \ldots, \Delta_n$. If the vertex set of Ω is $Z = \{v_1, \ldots, v_n\}$ and the vertex set of Δ_i contains every element of Z but v_i, then Γ has condition C. For example, *any* Coxeter diagram Γ that contains Figure 1(b) as a subgraph (where Ω, Δ_1, Δ_2, and Δ_3 are maximal complete subgraphs of Γ) has condition C.

Remark 5.3. In [5], Tits says that a right-angled Coxeter group with Coxeter diagram Γ has "propriété I" if the complementary graph Γ^C has no triangles. In this case, the inclusion $\mathrm{Inn}(W) \to \mathrm{Aut}^\circ(W)$ is an isomorphism. If Γ has condition C, then the union $\Delta_1 \cup \cdots \cup \Delta_n \cup \Omega$ defines a right-angled Coxeter group that has propriété I. Thus, condition C is in some sense a "local" version of Tits' propriété I.

Assume Γ satisfies condition C and let $\phi \in \mathrm{Fix}^\circ(Z)$. In this case, we can choose $a_Z(\phi) = 1$. Then, for each $T \in \mathfrak{M}(\Gamma)$ with $T \cap Z \neq \varnothing$, the transition $\gamma_{ZT}(\phi) = a_T(\phi)$ is an element of the coset $a_T W_T$. It must be emphasized that no left multiplication by elements of W_Z is permitted.

Theorem 5.4. *If Γ has condition C, then the sequence*

$$1 \longrightarrow \mathrm{Inn}(W) \longrightarrow \mathrm{Aut}^\circ(W) \longrightarrow \mathrm{Out}^\circ(W) \longrightarrow 1 \tag{5.1}$$

splits.

Proof. Let $Z, T_1, \ldots, T_n \in \mathfrak{M}(\Gamma)$ satisfying (C1) and (C2) of Definition 5.1. For each $\phi \in \mathrm{Fix}^\circ(Z)$, define an inner automorphism $r(\phi) = \psi_x$ where

$$x = \prod_{i=1}^{n} \pi_{T_i}\left(\gamma_{ZT_i}(\phi)\right) = \prod_{i=1}^{n} \pi_{T_i}\left(a_{T_i}(\phi)\right) \tag{5.2}$$

(recall that ψ_x is the inner automorphism that conjugates each element of W by x). By definition, $\pi_{T_i}(\gamma_{ZT_i}(\phi)) = \pi_{T_i}(a_{T_i}(\phi))$ lies in $W_{Z-T_i} \subseteq W_Z$. Consequently, the terms in the product (5.2) commute and so the mapping

$$r : \mathrm{Fix}^\circ(Z) \longrightarrow \mathrm{Inn}(Z) \tag{5.3}$$

is well defined. To see that r is a homomorphism, choose $\phi, \kappa \in \mathrm{Fix}^\circ(Z)$. Then, for each T_i we have

$$a_{T_i}(\phi\kappa) = \phi\left(a_{T_i}(\kappa)\right) \cdot a_{T_i}(\phi). \tag{5.4}$$

Since $a_{T_i}(\kappa)$ lies in the centralizer of $W_{T_i \cap Z}$, any reduced expression for $a_{T_i}(\kappa)$ is of the form $a_{T_i}(\kappa) = v_1 \cdots v_m$, where each $v_j \in C(T_i \cap Z)$. Observe that

$$\pi_{T_i}\left(\phi(v_j)\right) = \begin{cases} v_j, & \text{if } v_j \in Z - T_i, \\ 1, & \text{if } v_j \notin Z - T_i, \end{cases} \qquad = \pi_{T_i}\left(v_j\right) \tag{5.5}$$

and so $\pi_{T_i}(\phi(a_{T_i}(\kappa))) = \pi_{T_i}(a_{T_i}(\kappa))$. It follows from (5.4) that

$$\pi_{T_i}\left(a_{T_i}(\phi\kappa)\right) = \pi_{T_i}\left(a_{T_i}(\kappa)\right) \cdot \pi_{T_i}\left(a_{T_i}(\phi)\right). \tag{5.6}$$

Now, using the fact that the image of each π_{T_i} lies in the Abelian subgroup W_Z, we have that $r(\phi\kappa) = r(\phi)r(\kappa)$.

To see that r is a retraction, let $v \in Z$ and let k be a number of T_i's that do not contain v. If ψ_v is the inner automorphism of W that conjugates by v, then $a_{T_i}(\psi_v) = v$ for every $1 \le i \le n$. From (5.2) we have that $r(\psi_v)$ conjugates every element by

$$\prod_{i=1}^{n} \pi_{T_i}(a_{T_i}(\psi_v)) = \prod_{i=1}^{n} \pi_{T_i}(v) = v^k. \tag{5.7}$$

Since k is odd, $v^k = v$ and so $r(\psi_v) = \psi_v$.

Since r is a retraction and the sequence (4.3) is central, the mapping

$$\mathrm{Fix}^{\circ}(Z) \longrightarrow \mathrm{Inn}(Z) \times \ker(r) \tag{5.8}$$

defined by

$$\phi \longmapsto \left(r(\phi), r(\phi)^{-1}\phi\right) \tag{5.9}$$

is an isomorphism. Consequently, $j = \pi|_{\ker(r)}$ is an isomorphism from $\ker(r)$ onto $\mathrm{Out}^{\circ}(W)$. A section $\mathrm{Out}^{\circ}(W) \to \mathrm{Aut}^{\circ}(W)$ is given by the composition $h \circ j^{-1}$, where $h : \ker(r) \to \mathrm{Aut}^{\circ}(W)$ is the inclusion

$$\ker(r) \subseteq \mathrm{Fix}^{\circ}(Z) \subseteq \mathrm{Aut}^{\circ}(W). \tag{5.10}$$

□

As noted in the introduction, to obtain a splitting of sequence (1.2), Γ must satisfy an "asymmetry" condition. This is obtained by imposing a restriction on the action of $\mathrm{Aut}(\mathfrak{F}(\Gamma))$, and motivates the following definition.

Definition 5.5. Let $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$ and $T \in \mathfrak{M}(\Gamma)$. One says that α *fixes* T if $\alpha(\{t\}) = \{t\}$ for every $t \in T$.

Note that if α fixes T, then $\alpha(T') = T'$ for every $T' \subseteq T$.

Theorem 5.6. *If* Γ *has condition C and each* $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$ *fixes* Z*, then*

$$1 \longrightarrow \mathrm{Inn}(W) \longrightarrow \mathrm{Aut}(W) \longrightarrow \mathrm{Out}(W) \longrightarrow 1 \tag{5.11}$$

splits.

Proof. Let $\mathrm{Fix}(Z) = \{\beta \in \mathrm{Aut}(W) \mid \beta(z) = z \text{ for all } z \in Z\}$. Since each $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$ fixes Z, it follows that $\hat{\alpha}$ (as defined in (3.3) above) lies in $\mathrm{Fix}(Z)$ and the mapping $\alpha \mapsto \hat{\alpha}$ is a splitting of the sequence

$$1 \longrightarrow \mathrm{Fix}^{\circ}(Z) \longrightarrow \mathrm{Fix}(Z) \xrightarrow{\overline{q}} \mathrm{Aut}(\mathfrak{F}(\Gamma)) \longrightarrow 1 \tag{5.12}$$

(where $\overline{q}$ is the restriction of the mapping q given in (3.2) above). As in the proof of Theorem 4.4, the projection $\mathrm{Fix}(Z) \to \mathrm{Out}(W)$ is onto and its kernel is

$$\mathrm{Inn}(Z) \cap \mathrm{Fix}(Z) = \mathrm{Inn}\big(C(Z)\big). \tag{5.13}$$

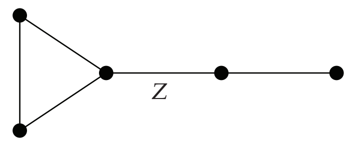


Figure 2

But $Z \in \mathfrak{M}(\Gamma)$ and so $C(Z) = Z$. Thus, we have an extension

$$0 \longrightarrow \mathrm{Inn}(Z) \longrightarrow \mathrm{Fix}(Z) \longrightarrow \mathrm{Out}(W) \longrightarrow 1 \tag{5.14}$$

which is easily seen to be central.

If $\phi \in \mathrm{Fix}^{\circ}(Z)$ and $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$, then $\widehat{\alpha}\phi\widehat{\alpha}^{-1} \in \mathrm{Fix}^{\circ}(Z)$ and, because $\mathrm{Inn}(Z)$ is Abelian,

$$r\left(\widehat{\alpha}\phi\widehat{\alpha}^{-1}\right) = r(\phi). \tag{5.15}$$

Since (5.12) splits, every element $f \in \mathrm{Fix}(Z)$ has a unique expression $f = \phi \cdot \widehat{\alpha}$, where $\phi \in \mathrm{Fix}^{\circ}(Z)$ and $\alpha \in \mathrm{Aut}(\mathfrak{F}(\Gamma))$. Consequently, we may define

$$r' : \mathrm{Fix}(Z) \longrightarrow \mathrm{Inn}(Z) \tag{5.16}$$

by $r'(f) = r(\phi)$ for all $f \in \mathrm{Fix}(Z)$. If $g \in \mathrm{Fix}(Z)$ is written as $\psi \cdot \widehat{\beta}$, then, by (5.15),

$$\begin{aligned} r'(fg) &= r'\left(\phi\widehat{\alpha}\psi\widehat{\alpha}^{-1}\widehat{\alpha\beta}\right) \\ &= r\left(\phi\widehat{\alpha}\psi\widehat{\alpha}^{-1}\right) \\ &= r(\phi)r\left(\widehat{\alpha}\psi\widehat{\alpha}^{-1}\right) \\ &= r(\phi)r(\psi) \\ &= r'(f)r'(g). \end{aligned} \tag{5.17}$$

Thus, r' is a homomorphism and is easily verified to be a retraction. The proof is completed in a manner similar to the conclusion of the proof of Theorem 5.4, replacing $\mathrm{Out}^{\circ}(W)$ with $\mathrm{Out}(W)$, and so on. □

As an immediate consequence we obtain the following corollary (cf. [6]).

Corollary 5.7. *If W is a right-angled Coxeter group satisfying the hypotheses of Theorem 5.6, then every extension*

$$1 \longrightarrow W \longrightarrow G \longrightarrow Q \longrightarrow 1 \tag{5.18}$$

splits.

Example 5.8. The graph in Figure 2 clearly satisfies the hypothesis of Theorem 5.6. On the other hand, Theorem 5.6 does not apply to either of the graphs given in Figure 1.

Remark 5.9. As noted in the introduction, it is currently unknown whether, for W with trivial center, the sequence

$$1 \longrightarrow \mathrm{Inn}(W) \longrightarrow \mathrm{Aut}(W) \longrightarrow \mathrm{Out}(W) \longrightarrow 1 \tag{5.19}$$

always splits. However, it was recently shown in [8] that sequence (1.5) always splits, but the splitting found there is not, in general, compatible with (1.1). In particular, one cannot obtain generalizations of Theorem 5.6 and Corollary 5.7 from [8].

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