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Research Article

The Generalizations of Hilbert's Inequality

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By introducing some parameters and estimating the weight functions, we establish a new Hilbert-type inequalities with best constant factors. The equivalent inequalities are also considered.

1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \pi\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is the well-known Hilbert's inequality. Inequality (1.1) had been generalized by Hardy-Riesz (see [2]) in 1925 as if f, g are real functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(x)dx\right\}^{1/q}, \tag{1.2}$$

where the constant factor $c = \pi/\sin(\pi/p)$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1). Inequality (1.2) is named after Hardy-Hilbert's integral inequality, which is important in the analysis and its applications (see [3]), it has been studied and generalized in many directions by a number of mathematicians (see [4–8]).

Under the same condition of (1.2), we have Hardy-Hilbert's type inequality (see [1, Theorems 341 and 342]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dx\,dy < 4\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2},$$
$$\int_0^\infty\int_0^\infty \frac{\ln x-\ln y}{x-y}f(x)g(y)dx\,dy < \pi^2\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.3}$$

where the constant factors 4 and π^2 are both the best possible.

Recently, Li et al. [9], by introducing the function $|\ln x-\ln y|/(x+y+|x-y|)$, establish new inequalities similar to Hilbert-type inequality for integrals.

Theorem 1.1. *If $f(x),g(x)\geq 0$, $0<\int_0^\infty f^2(x)dx<\infty$, $0<\int_0^\infty g^2(x)dx<\infty$. Then, one has*

$$\int_0^\infty\int_0^\infty \frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)g(y)dx\,dy < 4\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(x)dx\right)^{1/2}, \tag{1.4}$$

where the constant factor 4 is the best possible.

In this paper, we give further analogs of Hilbert-type inequality and its applications. The main result unifies and generalizes the classical results as follows.

Assume that $r>0$, $s>-\min\{1,r\}$, $t\geqslant 0$, $p,q>1$, and $1/p+1/q=1$. If $f,g\geqslant 0$, such that $0<\int_0^\infty f^p(x)dx<\infty$, $0<\int_0^\infty g^q(x)dx<\infty$, then

$$\int_0^\infty\int_0^\infty \frac{|\ln x-\ln y|^t}{x+ry+s|x-y|}f(x)g(y)dx\,dy < K_1\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(x)dx\right)^{1/q}, \tag{1.5}$$

where the constant factor

$$K_1=\int_0^\infty \frac{|\ln u|^t}{1+ru+s|1-u|}u^{-1/q}du \tag{1.6}$$

is the best possible.

2. Some lemmas

Our results will be based on the following results. In the following lemmas, assume that $r>0$, $s>-\min\{1,r\}$, $t\geqslant 0$, $p,q>1$, and $1/p+1/q=1$.

Lemma 2.1. *Define the following weight functions:*

$$\omega_1(x) = \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|}\left(\frac{x}{y}\right)^{1/q} dy, \quad x > 0,$$
$$\omega_2(y) = \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|}\left(\frac{y}{x}\right)^{1/p} dx, \quad y > 0. \tag{2.1}$$

Then, $\omega_1 = \omega_2 = K_1$, where K_1 is defined by (1.1).

Proof. For $\omega_1(x)$, let $u = y/x$, and we have

$$\omega_1 = \int_0^\infty \frac{|\ln u|^t}{1 + ru + s|1-u|} u^{-1/q} du = K_1. \tag{2.2}$$

For $\omega_2(y)$, first let $v = x/y$, and then let $u = 1/v$. Thus, we have

$$\omega_2 = \int_0^\infty \frac{|\ln v|^t}{1 + rv + s|v-1|} v^{-1/p} dv = \int_0^\infty \frac{|\ln u|^t}{1 + ru + s|1-u|} u^{-1/q} du = K_1. \tag{2.3}$$

Hence, the lemma is proved. □

Lemma 2.2. *Assume that $\varepsilon > 0$, then*

$$\int_0^\infty \frac{|\ln u|^t}{1 + ru + s|1-u|} u^{-(1+\varepsilon)/q} du = K_1 + o(1) \quad (\varepsilon \to 0^+). \tag{2.4}$$

Proof. Since

$$\lim_{u\to 0} u^{1/2p}|\ln u|^t = 0, \qquad \lim_{u\to +\infty} \frac{u^{1/2p}}{|\ln u|^t} = +\infty, \tag{2.5}$$

it follows that there exists $\delta_1 \in (0,1)$, such that if $u \in (0, \delta_1)$, then $u^{-1/2p} > |\ln u|^t$. Moreover, there exists $\delta_2 \in (1, +\infty)$, such that if $u \in (\delta_2, +\infty)$, then $u^{1/2q} > |\ln u|^t$. Using the expression of K_1, if $s \leqslant 0$, then

$$\begin{aligned}
&\left|\int_0^\infty \frac{|\ln u|^t}{1+ru+s|1-u|}u^{-(1+\varepsilon)/q}du - K_1\right| \\
&\quad= \left|\int_0^\infty \frac{|\ln u|^t}{1+ru+s|1-u|}\left(u^{-(1+\varepsilon)/q}-u^{-1/q}\right)du\right| \\
&\quad\leqslant \left(\int_0^{\delta_1}+\int_{\delta_1}^{\delta_2}+\int_{\delta_2}^{\infty}\right)\frac{|\ln u|^t}{1+ru+s|1-u|}\left|u^{-(1+\varepsilon)/q}-u^{-1/q}\right|du \\
&\quad< \int_0^{\delta_1}\frac{|\ln u|^t}{1+s}\left(u^{-(1+\varepsilon)/q}-u^{-1/q}\right)du + \int_{\delta_1}^{\delta_2}\frac{|\ln u|^t}{1+ru+s|1-u|}\left|u^{-(1+\varepsilon)/q}-u^{-1/q}\right|du \\
&\qquad+ \int_{\delta_2}^{\infty}\frac{|\ln u|^t}{(r+s)u}\left(u^{-1/q}-u^{-(1+\varepsilon)/q}\right)du \\
&\quad< \int_0^{\delta_1}\frac{u^{-1/2p}}{1+s}\left(u^{-(1+\varepsilon)/q}-u^{-1/q}\right)du + \int_{\delta_1}^{\delta_2}\frac{|\ln u|^t}{1+ru+s|1-u|}\left|u^{-(1+\varepsilon)/q}-u^{-1/q}\right|du \\
&\qquad+ \int_{\delta_2}^{\infty}\frac{u^{1/2q}}{(r+s)u}\left(u^{-1/q}-u^{-(1+\varepsilon)/q}\right)du \\
&\quad= \frac{1}{1+s}\left(\frac{1}{1-1/2p-(1+\varepsilon)/q}\delta_1^{1-1/2p-(1+\varepsilon)/q} - \frac{1}{1-1/2p-1/q}\delta_1^{1-1/2p-1/q}\right) \\
&\qquad+ \int_{\delta_1}^{\delta_2}\frac{|\ln u|^t}{1+ru+s|1-u|}\left|u^{-(1+\varepsilon)/q}-u^{-1/q}\right|du + \frac{1}{r+s}\left(2q\delta_2^{-1/2q} - \frac{2q}{1+2\varepsilon}\delta_2^{-(1+2\varepsilon)/2q}\right) \\
&\quad= o(1) \quad (\varepsilon \to 0^+).
\end{aligned} \tag{2.6}$$

Therefore, Lemma 2.2 is proved for $s \leqslant 0$. If $s > 0$, then we can replace the right-hand side of the first strict inequality above with

$$\begin{aligned}
&\int_0^{\delta_1}\frac{|\ln u|^t}{1}\left(u^{-(1+\varepsilon)/q}-u^{-1/q}\right)du + \int_{\delta_1}^{\delta_2}\frac{|\ln u|^t}{1+ru+s|1-u|}\left|u^{-(1+\varepsilon)/q}-u^{-1/q}\right|du \\
&\qquad+ \int_{\delta_2}^{\infty}\frac{|\ln u|^t}{ru}\left(u^{-1/q}-u^{-(1+\varepsilon)/q}\right)du.
\end{aligned} \tag{2.7}$$

By the same way, we can show that the lemma is valid for $s > 0$. Hence, the lemma is proved. □

Lemma 2.3. *Assume that $\varepsilon > 0$, then*

$$\int_1^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}}\frac{|\ln u|^t}{1+ru+s|1-u|}u^{-(1+\varepsilon)/q}du = O(1) \quad (\varepsilon \to 0^+). \tag{2.8}$$

Proof. Using δ_1 introduced in the proof of Lemma 2.2, if $s \leqslant 0$, then

$$\begin{aligned}
0 &< \int_1^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+ru+s|1-u|}u^{-(1+\varepsilon)/q}du \\
&= \int_1^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+ru+s(1-u)}u^{-(1+\varepsilon)/q}du \\
&< \int_1^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&= \left(\int_1^{1/\delta_1} + \int_{1/\delta_1}^\infty\right)x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&= \int_1^{1/\delta_1} x^{-1-\varepsilon}dx\left(\int_0^{\delta_1} + \int_{\delta_1}^{x^{-1}}\right)\frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du + \int_{1/\delta_1}^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&= \int_1^{1/\delta_1} x^{-1-\varepsilon}dx\int_0^{\delta_1}\frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du + \int_1^{1/\delta_1} x^{-1-\varepsilon}dx\int_{\delta_1}^{x^{-1}}\frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&\quad + \int_{1/\delta_1}^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&< \int_1^{1/\delta_1} x^{-1-\varepsilon}dx\int_0^{\delta_1}\frac{u^{-1/2p}}{1+s}u^{-(1+\varepsilon)/q}du + \int_1^{1/\delta_1} x^{-1-\varepsilon}dx\int_{\delta_1}^{x^{-1}}\frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&\quad + \int_{1/\delta_1}^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{u^{-1/2p}}{1+s}u^{-(1+\varepsilon)/q}du \\
&= \frac{(1-\delta_1)\delta_1^{1-1/2p-(1+\varepsilon)/q}}{\varepsilon(1+s)(1-1/2p-(1+\varepsilon)/q)} + \int_1^{1/\delta_1} x^{-1-\varepsilon}dx\int_{\delta_1}^{x^{-1}}\frac{|\ln u|^t}{1+s}u^{-(1+\varepsilon)/q}du \\
&\quad + \frac{2p\delta_1^{(1+2\varepsilon)/2p}}{(1+2\varepsilon)(1+s)(1-1/2p-(1+\varepsilon)/q)} \\
&= O(1) \quad (\varepsilon \to 0^+).
\end{aligned} \tag{2.9}$$

Therefore, Lemma 2.3 is proved for $s \leqslant 0$. If $s > 0$, then we can replace the right-hand side of the second strict inequality mentioned above with

$$\int_1^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1}u^{-(1+\varepsilon)/q}du. \tag{2.10}$$

By the same way, we can show that the lemma is valid for $s > 0$. Therefore, the lemma is proved. □

3. Integral case

In this section, we will state our main results.

Proposition 3.1. *Assume that* $r > 0$, $s > -\min\{1, r\}$, $t \geqslant 0$, $p, q > 1$, *and* $1/p + 1/q = 1$. *If* $f, g \geqslant 0$, *such that* $0 < \int_0^\infty f^p(x)dx < \infty$, $0 < \int_0^\infty g^q(x)dx < \infty$, *then*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)g(y)dx\,dy < K_1 \left(\int_0^\infty f^p(x)dx\right)^{1/p} \left(\int_0^\infty g^q(x)dx\right)^{1/q}, \tag{3.1}$$

where the constant factor

$$K_1 = \int_0^\infty \frac{|\ln u|^t}{1 + ru + s|1-u|} u^{-1/q} du \tag{3.2}$$

is the best possible.

Proof. Using Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)g(y)dx\,dy \\
&\quad = \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} \left[\left(\frac{x}{y}\right)^{1/pq} f(x)\right] \left[\left(\frac{y}{x}\right)^{1/pq} g(y)\right] dx\,dy \\
&\quad \leqslant \left(\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} \left(\frac{x}{y}\right)^{1/q} f^p(x)dx\,dy\right)^{1/p} \\
&\qquad \times \left(\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} \left(\frac{y}{x}\right)^{1/p} g^q(y)dx\,dy\right)^{1/q} \\
&\quad = \left(\int_0^\infty \omega_1(x) f^p(x)dx\right)^{1/p} \left(\int_0^\infty \omega_2(y) g^q(y)dy\right)^{1/q} \\
&\quad = K_1 \left(\int_0^\infty f^p(x)dx\right)^{1/p} \left(\int_0^\infty g^q(x)dx\right)^{1/q}.
\end{aligned} \tag{3.3}$$

If (3.3) takes the form of equality, then there are constants a and b, such that they are not all zero, and

$$a \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} \left(\frac{x}{y}\right)^{1/q} f^p(x) = b \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} \left(\frac{y}{x}\right)^{1/p} g^q(y), \quad \text{a.e. in } (0,\infty)\times(0,\infty). \tag{3.4}$$

Hence,

$$axf^p(x) = byg^q(y), \quad \text{a.e. in } (0,\infty)\times(0,\infty). \tag{3.5}$$

Therefore, there is a constant c, such that

$$axf^p(x) = byg^q(y) = c, \quad \text{a.e. in } (0,\infty)\times(0,\infty). \tag{3.6}$$

We claim that $a = 0$. In fact, if $a \neq 0$, then $f^p(x) = c/ax$, a.e., in $(0,\infty)$, which contradicts the fact that $0 < \int_0^\infty f^p(x)dx < \infty$. In the same way, we claim that $b = 0$. This is a contradiction. Hence, by (3.3), we have (3.1).

If the constant factor K_1 in (3.1) is not the best possible factor, then there exists a positive constant H (with $H < K_1$), such that (3.1) is still valid if we replace K_1 with H. For $\varepsilon > 0$ sufficiently small, construct the following functions:

$$\begin{aligned} f_\varepsilon(x) &= \begin{cases} 0, & x \in (0,1), \\ x^{-(1+\varepsilon)/p}, & x \in (0,+\infty), \end{cases} \\ g_\varepsilon(x) &= \begin{cases} 0, & x \in (0,1), \\ x^{-(1+\varepsilon)/q}, & x \in (0,+\infty), \end{cases} \end{aligned} \tag{3.7}$$

Thus, we obtain

$$\begin{aligned} &H\left(\int_0^\infty f_\varepsilon^p(x)dx\right)^{1/p}\left(\int_0^\infty g_\varepsilon^q(x)dx\right)^{1/q} = \frac{H}{\varepsilon}, \\ I &:= \int_0^\infty\int_0^\infty \frac{|\ln x - \ln y|^t}{x+ry+s|x-y|} f_\varepsilon(x)g_\varepsilon(y)dx\,dy \\ &= \int_1^\infty\int_1^\infty \frac{|\ln x - \ln y|^t}{x+ry+s|x-y|} x^{-(1+\varepsilon)/p}y^{-(1+\varepsilon)/q}dx\,dy \\ &= \int_1^\infty x^{-(1+\varepsilon)/p}dx\int_1^\infty \frac{|\ln x - \ln y|^t}{x+ry+s|x-y|} y^{-(1+\varepsilon)/q}dy. \end{aligned} \tag{3.8}$$

Setting $u = y/x$, we have

$$\begin{aligned} I &= \int_1^\infty x^{-1-\varepsilon}dx\int_{x^{-1}}^\infty \frac{|\ln u|^t}{1+ru+s|1-u|} u^{-(1+\varepsilon)/q}du \\ &= \int_1^\infty x^{-1-\varepsilon}dx\int_0^\infty \frac{|\ln u|^t}{1+ru+s|1-u|} u^{-(1+\varepsilon)/q}du - \int_1^\infty x^{-1-\varepsilon}dx\int_0^{x^{-1}} \frac{|\ln u|^t}{1+ru+s|1-u|} u^{-(1+\varepsilon)/q}du \\ &= \frac{1}{\varepsilon}(K_1 + o(1)) + O(1) \quad (\varepsilon \to 0^+). \end{aligned} \tag{3.9}$$

Since

$$I < \frac{H}{\varepsilon}, \tag{3.10}$$

it follows that

$$K_1 + o(1) < H \quad (\varepsilon \to 0^+). \tag{3.11}$$

Hence,

$$K_1 \leqslant H. \tag{3.12}$$

This contradicts the fact that $H < K_1$. So the constant factor K_1 in (3.1) is the best possible. Then Proposition 3.1 is proved. □

Proposition 3.2. *Under the same assumptions of Proposition 3.1, one has*

$$\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)dx\right]^p dy < K_1^p \int_0^\infty f^p(x)dx, \tag{3.13}$$

where the constant factor K_1^p is the best possible. Inequalities (3.1) *and* (3.13) *are equivalent.*

Proof. Setting

$$g(y) = \left[\int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)dx\right]^{p-1}, \quad y \in (0,\infty), \tag{3.14}$$

by (3.1), we have

$$\begin{aligned}
\int_0^\infty g^q(y)dy &= \int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)dx\right]^p dy \\
&= \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)g(y)dx\,dy \\
&< K_1\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(x)dx\right)^{1/q}.
\end{aligned} \tag{3.15}$$

As a result,

$$\left(\int_0^\infty g^q(x)dx\right)^{1/p} < K_1\left(\int_0^\infty f^p(x)dx\right)^{1/p}. \tag{3.16}$$

Hence,

$$\int_0^\infty g^q(x)dx < K_1^p \int_0^\infty f^p(x)dx. \tag{3.17}$$

Then, we have (3.13).

Conversely, by Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)g(y)dx\,dy \\
&\quad = \int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)dx\right] g(y)dy \\
&\quad \leqslant \left(\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|^t}{x + ry + s|x-y|} f(x)dx\right]^p dy\right)^{1/p} \left(\int_0^\infty g^q(x)dx\right)^{1/q} \\
&\quad < K_1 \left(\int_0^\infty f^p(x)dx\right)^{1/p} \left(\int_0^\infty g^q(x)dx\right)^{1/q}.
\end{aligned} \tag{3.18}$$

Then, by (3.13), we have (3.1). Hence inequalities (3.1) and (3.13) are equivalent.

If the constant factor K_1^p in (3.13) is not the best possible, then by (3.18) we can get a contradiction that the constant factor K_1 in (3.1) is not the best possible. Hence Proposition 3.2 is proved. □

Remark 3.3. In (3.1), let $r = 1$, $s = 0$, $t = 0$, $p = q = 2$, and we have Hilbert's integral inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\,dy < \pi \left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}. \tag{3.19}$$

Let $r = 1$, $s = 1$, $t = 0$, $p = q = 2$; we have Hardy-Hilbert's classical inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{2\max\{x,y\}} dx\,dy < 2 \left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}. \tag{3.20}$$

Let $r = 1$, $s = 1/3$, $t = 0$, $p = q = 2$, and we can combine the above two inequalities as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(2/3)(x + y + \max\{x,y\})} dx\,dy < K_1 \left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}. \tag{3.21}$$

Let $r = 1$, $s = -1/3$, $t = 0$, and we can get the following inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(2/3)(x + y + \min\{x, y\})} dx\, dy < K_1 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}. \tag{3.22}$$

4. Discrete case

We also give results for the discrete case.

Proposition 4.1. *Assume that* $r > 0$, $s \in (-\min\{1, r\}, \min\{1, r\}]$, $p, q > 1$ *and* $1/p + 1/q = 1$. *If* $a_m, b_n \geqslant 0$, *such that*

$$0 < \sum_{m=1}^\infty a_m^p < \infty, \qquad 0 < \sum_{n=1}^\infty b_n^q < \infty, \tag{4.1}$$

then one has

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m + rn + s|m - n|} < K_1 \left(\sum_{m=1}^\infty a_m^p \right)^{1/p} \left(\sum_{n=1}^\infty b_n^q \right)^{1/q}, \tag{4.2}$$

$$\sum_{n=1}^\infty \left[\sum_{m=1}^\infty \frac{a_m}{m + rn + s|m - n|} \right]^p < K_1^p \sum_{m=1}^\infty a_m^p, \tag{4.3}$$

where the constant factor

$$K_1 = \int_0^\infty \frac{1}{1 + ru + s|1 - u|} u^{-1/q} du \tag{4.4}$$

is the best possible, and inequalities (4.2) *and* (4.3) *are equivalent.*

Proof. Define the following weight functions:

$$\begin{aligned} \varpi_1(m) &= \sum_{n=1}^\infty \frac{1}{m + rn + s|m - n|} \left(\frac{m}{n} \right)^{1/q}, \quad m = 1, 2, \ldots, \\ \varpi_2(n) &= \sum_{m=1}^\infty \frac{1}{m + rn + s|m - n|} \left(\frac{n}{m} \right)^{1/p}, \quad n = 1, 2, \ldots, \end{aligned} \tag{4.5}$$

then,

$$\varpi_1(m) < \omega_1(m) = K_1, \qquad \varpi_2(n) < \omega_2(n) = K_1, \tag{4.6}$$

where ω_1, ω_2 are defined in Lemma 2.1. By Hölder's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+rn+s|m-n|} \\
&\quad= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m+rn+s|m-n|}\left[\left(\frac{m}{n}\right)^{1/pq}a_m\right]\left[\left(\frac{n}{m}\right)^{1/pq}b_n\right] \\
&\quad\le \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m+rn+s|m-n|}\left(\frac{m}{n}\right)^{1/q}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m+rn+s|m-n|}\left(\frac{n}{m}\right)^{1/p}b_n^q\right)^{1/q} \\
&\quad= \left(\sum_{m=1}^{\infty}\varpi_1(m)a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\varpi_2(n)b_n^q\right)^{1/q} \\
&\quad< K_1\left(\sum_{m=1}^{\infty}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}.
\end{aligned} \tag{4.7}$$

The last strict inequality holds because both series $\{a_m\}$ and $\{b_n\}$ have positive terms. Thus, we have (4.2).

If the constant factor K_1 in (4.2) is not the best possible, then there exists a positive constant H (with $H < K_1$), such that (4.2) is still valid if we replace K_1 by H. For $\varepsilon > 0$ small enough, construct series

$$\tilde{a}_m = m^{-(1+\varepsilon)/p}, \qquad \tilde{b}_n = n^{-(1+\varepsilon)/q}. \tag{4.8}$$

Then, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{m+rn+s|m-n|} > \int_0^{\infty}\int_0^{\infty}\frac{f_\varepsilon(x)g_\varepsilon(y)\,dx\,dy}{x+ry+s|x-y|} = \frac{1}{\varepsilon}\left(K_1+o(1)\right)+O(1), \tag{4.9}$$

$$\left(\sum_{m=1}^{\infty}\tilde{a}_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\tilde{b}_n^q\right)^{1/q} = \sum_{n=1}^{\infty}n^{-1-\varepsilon} < 1+\int_1^{\infty}x^{-1-\varepsilon}dx = 1+\frac{1}{\varepsilon}, \tag{4.10}$$

where $f_\varepsilon(x)$ and $g_\varepsilon(x)$ are defined in the proof of Proposition 3.1. Since

$$\frac{1}{\varepsilon}\left(K_1+o(1)\right)+O(1) < H\left(1+\frac{1}{\varepsilon}\right), \tag{4.11}$$

it follows that

$$K_1+o(1) < H. \tag{4.12}$$

Hence,

$$K_1 \leqslant H. \tag{4.13}$$

This contradicts with the fact that $H < K_1$. So the constant factor K_1 in (4.2) is the best possible one.

Setting

$$b_n = \left[\sum_{m=1}^{\infty} \frac{a_m}{m + rn + s|m-n|}\right]^{p-1}, \quad n = 1, 2, \ldots, \tag{4.14}$$

we find

$$\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{a_m}{m + rn + s|m-n|}\right]^{p} = \sum_{n=1}^{\infty} b_n^q = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m + rn + s|m-n|}. \tag{4.15}$$

By the same argument used in the proof of Proposition 3.2, we can show that (4.3) is valid, the constant factor K_1^p in (4.3) is the best possible one and inequalities (4.2) and (4.3) are equivalent. □

Remark 4.2. Proposition 4.1 is the corresponding series form of Propositions 3.1 and 3.2 for $s \in (-\min\{1, r\}, \min\{1, r\}]$, $t = 0$, and it is also a generalization of Hilbert's inequality. Here, we restrict the constants s, t so that we can use the monotony of functions to obtain (4.6) and (4.9).

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