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Research Article

Generalized Newman Phenomena and Digit Conjectures on Primes

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We prove that the ratio of the Newman sum over numbers multiple of a fixed integer, which is not a multiple of 3, and the Newman sum over numbers multiple of a fixed integer divisible by 3 is o(1) when the upper limit of summing tends to infinity. We also discuss a connection of our results with a digit conjecture on primes.

1. Introduction

Denote for $x, m \in \mathbb{N}$,

$$S_m(x) = \sum_{0 \le n < x,\, n \equiv 0 \pmod{m}} (-1)^{s(n)}, \tag{1.1}$$

where $s(n)$ is the number of 1's in the binary expansion of n. Sum (1.1) is a *Newman digit sum.* From the fundamental paper of Gelfond [1], it follows that

$$S_m(x) = O(x^{\lambda}), \quad \lambda = \frac{\ln 3}{\ln 4}. \tag{1.2}$$

The case $m = 3$ was studied in detail in [2–4].

So, from Coquet's theorem [3, 5] it follows that

$$-\frac{1}{3} + \frac{2}{\sqrt{3}} x^{\lambda} \le S_3(3x) \le \frac{1}{3} + \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda} \tag{1.3}$$

with a microscopic improvement [4]:

$$\frac{2}{\sqrt{3}}x^{\lambda} \le S_3(3x) \le \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda}, \quad x \ge 2, \tag{1.4}$$

and, moreover,

$$\left\lfloor 2\left(\frac{x}{6}\right)^{\lambda} \right\rfloor \le S_3(x) \le \left\lceil \frac{55}{3}\left(\frac{x}{65}\right)^{\lambda} \right\rceil. \tag{1.5}$$

These estimates give the most exact modern limits of the so-called *Newman phenomena*. Note that Drmota and Skałba [6], using a close function $(S_m^{(m)}(x))$, proved that if m is a multiple of 3, then for sufficiently large x,

$$S_m(x) > 0, \quad x \ge x_0(m). \tag{1.6}$$

In this paper, we study a general case for $m \ge 5$ (in the cases of $m = 2$ and $m = 4$, we have $|S_m(n)| \le 1$).

To formulate our results, put for $m \ge 5$,

$$\lambda_m = 1 + \log_2 b_m, \tag{1.7}$$

$$\mu_m = \frac{2}{2b_m - 1}, \tag{1.8}$$

where

$$b_m^2 = \begin{cases} \sin\left(\frac{\pi}{3}\left(1 + \frac{3}{m}\right)\right)\left(\sqrt{3} - \sin\left(\frac{\pi}{3}\left(1 + \frac{3}{m}\right)\right)\right), & \text{if } m \equiv 0 \pmod 3, \\ \sin\left(\frac{\pi}{3}\left(1 - \frac{1}{m}\right)\right)\left(\sqrt{3} - \sin\left(\frac{\pi}{3}\left(1 - \frac{1}{m}\right)\right)\right), & \text{if } m \equiv 1 \pmod 3, \\ \sin\left(\frac{\pi}{3}\left(1 + \frac{1}{m}\right)\right)\left(\sqrt{3} - \sin\left(\frac{\pi}{3}\left(1 + \frac{1}{m}\right)\right)\right), & \text{if } m \equiv 2 \pmod 3. \end{cases} \tag{1.9}$$

Directly, one can see that

$$\frac{\sqrt{3}}{2} > b_m \ge \begin{cases} 0.86184088\cdots, & \text{if } (m,3) = 1, \\ 0.85559967\cdots, & \text{if } (m,3) = 3, \end{cases} \tag{1.10}$$

and thus,

$$\lambda_m < \lambda,$$

$$2.73205080\cdots < \mu_m \leq \begin{cases} 2.76364572\cdots, & \text{if } (m,3) = 1, \\ 2.81215109\cdots, & \text{if } (m,3) = 3. \end{cases} \tag{1.11}$$

Below, we prove the following results.

Theorem 1.1. *If* $(m,3) = 1$, *then*

$$\left|S_m(x)\right| \leq 1 + \mu_m x^{\lambda_m}. \tag{1.12}$$

Theorem 1.2 (Generalized Newman phenomena). *If* $m > 3$ *is a multiple of 3, then*

$$\left|S_m(x) - \frac{3}{m}S_3(x)\right| \leq 1 + \mu_m x^{\lambda_m}. \tag{1.13}$$

Using Theorem 1.2 and (1.5), one can estimate $x_0(m)$ in (1.6). For example, one can prove that $x_0(21) < e^{909}$.

2. Explicit formula for $S_m(N)$

We have

$$\begin{aligned} S_m(N) &= \sum_{n=0,\, m|n}^{N-1} (-1)^{s(n)} \\ &= \frac{1}{m}\sum_{t=0}^{m-1}\sum_{n=0}^{N-1}(-1)^{s(n)} e^{2\pi i(nt/m)} \\ &= \frac{1}{m}\sum_{t=0}^{m-1}\sum_{n=0}^{N-1} e^{2\pi i((t/m)n+(1/2)s(n))}. \end{aligned} \tag{2.1}$$

Note that the interior sum has the form

$$F_\alpha(N) = \sum_{n=0}^{N-1} e^{2\pi i(\alpha n+(1/2)s(n))}, \quad 0 \leq \alpha < 1. \tag{2.2}$$

Lemma 2.1. *If* $N = 2^{\nu_0} + 2^{\nu_1} + \cdots + 2^{\nu_r}$, $\nu_0 > \nu_1 > \cdots > \nu_r \geq 0$, *then*

$$F_\alpha(N) = \sum_{h=0}^{r} e^{2\pi i(\alpha\sum_{j=0}^{h-1}2^{\nu_j}+h/2)} \prod_{k=0}^{\nu_h-1}\left(1 + e^{2\pi i(\alpha 2^k+1/2)}\right), \tag{2.3}$$

where as usual $\sum_{j=0}^{-1} = 0$, $\prod_{k=0}^{-1} = 1$.

Proof. Let $r = 0$, then by (2.2),

$$\begin{aligned} F_\alpha(N) &= \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i\alpha n} \\ &= 1 - \sum_{j=0}^{\nu_0-1} e^{2\pi i\alpha 2^j} + \sum_{0\le j_1<j_2\le \nu_0-1} e^{2\pi i\alpha(2^{j_1}+2^{j_2})} - \cdots \\ &= \prod_{k=0}^{\nu_0-1}\left(1 - e^{2\pi i\alpha 2^k}\right), \end{aligned} \tag{2.4}$$

which corresponds to (2.3) for $r = 0$.

Assuming that (2.3) is valid for every N with $s(N) = r + 1$, let us consider $N_1 = 2^{\nu_r} a + 2^{\nu_{r+1}}$ where a is odd, $s(a) = r + 1$, and $\nu_{r+1} < \nu_r$. Let

$$\begin{aligned} N &= 2^{\nu_r} a = 2^{\nu_0} + \cdots + 2^{\nu_r}; \\ N_1 &= 2^{\nu_0} + \cdots + 2^{\nu_r} + 2^{\nu_{r+1}}. \end{aligned} \tag{2.5}$$

Notice that for $n \in [0, 2^{\nu_{r+1}})$, we have

$$s(N + n) = s(N) + s(n). \tag{2.6}$$

Therefore,

$$\begin{aligned} F_\alpha(N_1) &= F_\alpha(N) + \sum_{n=N}^{N_1-1} e^{2\pi i(\alpha n + (1/2)s(n))} \\ &= F_\alpha(N) + \sum_{n=0}^{2^{\nu_{r+1}}-1} e^{2\pi i(\alpha n + \alpha N + (1/2)(s(N)+s(n)))} \\ &= F_\alpha(N) + e^{2\pi i(\alpha N + (1/2)s(N))} \sum_{n=0}^{2^{\nu_{r+1}}-1} e^{2\pi i(\alpha n + (1/2)s(n))}. \end{aligned} \tag{2.7}$$

Thus, by (2.3) and (2.4),

$$\begin{aligned} F_\alpha(N_1) &= \sum_{h=0}^{r} e^{2\pi i(\alpha \sum_{j=0}^{h-1} 2^{\nu_j} + h/2)} \prod_{k=0}^{\nu_h-1}\Big(1 + e^{2\pi i(\alpha 2^k + 1/2)} \\ &\quad + e^{2\pi i(\alpha \sum_{j=0}^{r} 2^{\nu_j} + (r+1)/2)} \prod_{k=0}^{\nu_{r+1}-1}\left(1 + e^{2\pi i(\alpha 2^k + 1/2)}\right) \\ &= \sum_{h=0}^{r+1} e^{2\pi i(\alpha \sum_{j=0}^{h-1} 2^{\nu_j} + h/2)} \prod_{k-0}^{\nu_h-1}\Big(1 + e^{2\pi i(\alpha 2^k + 1/2)}\Big). \end{aligned} \tag{2.8}$$

□

Formulas (2.1)–(2.3) give an explicit expression for $S_m(N)$ as a linear combination of the products of the form

$$\prod_{k=0}^{\nu_h-1}\left(1+e^{2\pi i(\alpha 2^k+1/2)}\right), \quad \alpha=\frac{t}{m},\ 0\le t\le m-1. \tag{2.9}$$

Remark 2.2. On can extract (2.3) from a very complicated general Gelfond formula [1], however, we prefer to give an independent proof.

3. Proof of Theorem 1.1

Note that in (2.3)

$$r\le \nu_0=\left\lfloor\frac{\ln N}{\ln 2}\right\rfloor. \tag{3.1}$$

By Lemma 2.1, we have

$$\begin{aligned}\left|F_\alpha(N)\right| &\le \sum_{\nu_h=\nu_0,\nu_1,\dots,\nu_r}\left|\prod_{k=1}^{\nu_h}\left(1+e^{2\pi i(\alpha 2^{k-1}+1/2)}\right)\right| \\ &\le \sum_{h=0}^{\nu_0}\left|\prod_{k=1}^{h}\left(1+e^{2\pi i(\alpha 2^{k-1}+1/2)}\right)\right|.\end{aligned} \tag{3.2}$$

Furthermore,

$$1+e^{2\pi i(2^{k-1}\alpha+1/2)}=2\sin\left(2^{k-1}\alpha\pi\right)\left(\sin\left(2^{k-1}\alpha\pi\right)-i\cos\left(2^{k-1}\alpha\pi\right)\right) \tag{3.3}$$

and, therefore,

$$\left|1+e^{2\pi i(2^{k-1}\alpha+1/2)}\right|\le 2\left|\sin\left(2^{k-1}\alpha\pi\right)\right|. \tag{3.4}$$

According to (3.2), let us estimate the product

$$\prod_{k=1}^{h}\left(2\left|\sin\left(2^{k-1}\alpha\pi\right)\right|\right)=2^h\prod_{k=1}^{h}\left|\sin\left(2^{k-1}\alpha\pi\right)\right|, \tag{3.5}$$

where by (2.1),

$$\alpha=\frac{t}{m}, \quad 0\le t\le m-1. \tag{3.6}$$

Repeating arguments of [1], put

$$\left|\sin\left(2^{k-1}\alpha\pi\right)\right| = t_k. \tag{3.7}$$

Considering the function

$$\rho(x) = 2x\sqrt{1-x^2}, \quad 0 \le x \le 1, \tag{3.8}$$

we have

$$t_k = 2t_{k-1}\sqrt{1-t_{k-1}^2} = \rho(t_{k-1}). \tag{3.9}$$

Note that

$$\rho'(x) = 2\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right) \le -1 \tag{3.10}$$

for $x_0 \le x \le 1$, where

$$x_0 = \frac{\sqrt{3}}{2} \tag{3.11}$$

is the only positive root of the equation $\rho(x) = x$.

Show that either

$$t_k \le \sin\left(\frac{\pi}{m}\left\lfloor\frac{m}{3}\right\rfloor\right) = \sin\left(\frac{\pi}{m}\left\lceil\frac{2m}{3}\right\rceil\right) = g_m < \frac{\sqrt{3}}{2} \tag{3.12}$$

or, simultaneously, $t_k > g_m$, and

$$\begin{aligned} t_k t_{k+1} &\le \max_{0\le l\le m-1}\left(\left|\sin\frac{l\pi}{m}\right|\left(\sqrt{3} - \left|\sin\frac{l\pi}{m}\right|\right)\right) \\ &= \begin{cases} \left(\sin\left(\dfrac{\pi}{m}\left\lfloor\dfrac{m}{3}\right\rfloor\right)\right)\left(\sqrt{3} - \sin\left(\dfrac{\pi}{m}\left\lfloor\dfrac{m}{3}\right\rfloor\right)\right), & \text{if } m \equiv 1 \pmod 3 \\ \left(\sin\left(\dfrac{\pi}{m}\left\lceil\dfrac{m}{3}\right\rceil\right)\right)\left(\sqrt{3} - \sin\left(\dfrac{\pi}{m}\left\lceil\dfrac{m}{3}\right\rceil\right)\right), & \text{if } m \equiv 2 \pmod 3 \end{cases} = h_m < \frac{3}{4}. \end{aligned} \tag{3.13}$$

Indeed, let for a fixed values of $t \in [0, m-1]$ and $k \in [1, n]$,

$$t2^{k-1} \equiv l (\operatorname{mod} m), \quad 0 \le l \le m-1. \tag{3.14}$$

Then,

$$t_k = \left| \sin \frac{l\pi}{m} \right|. \tag{3.15}$$

Now, distinguish two cases: (1) $t_k \le \sqrt{3}/2$, (2) $t_k > \sqrt{3}/2$.

In case (1),

$$t_k = \frac{\sqrt{3}}{2} \leftrightarrows \frac{l\pi}{m} = \frac{r\pi}{3}, \quad (r,3) = 1, \tag{3.16}$$

and since $0 \le l \le m-1$, then

$$m = \frac{3l}{r}, \quad r = 1,2. \tag{3.17}$$

Because of the condition $(m,3) = 1$, we have $t_k < \sqrt{3}/2$.

Thus, in (3.15),

$$l \in \left[0, \left\lfloor \frac{m}{3} \right\rfloor\right] \cup \left[\left\lceil \frac{2m}{3} \right\rceil, m\right], \tag{3.18}$$

and (3.12) follows.

In case (2), let $t_k > \sqrt{3}/2 = x_0$. For $\varepsilon > 0$, put

$$1 + \varepsilon = \frac{t_k}{x_0} = \frac{2}{\sqrt{3}} \left| \sin\left(\pi 2^{k-1}\alpha\right) \right| \tag{3.19}$$

such that

$$1 - \varepsilon = 2 - \frac{2}{\sqrt{3}} \left| \sin\left(\pi 2^{k-1}\alpha\right) \right|, \tag{3.20}$$

$$1 - \varepsilon^2 = \frac{4}{3} \left| \sin\left(\pi 2^{k-1}\alpha\right) \right| \left(\sqrt{3} - \left| \sin\left(\pi 2^{k-1}\alpha\right) \right|\right). \tag{3.21}$$

By (3.9) and (3.19), we have

$$t_{k+1} = \rho(t_k) = \rho((1+\varepsilon)x_0) = \rho(x_0) + \varepsilon x_0 \rho'(c), \tag{3.22}$$

where $c \in (x_0, (1+\varepsilon)x_0)$.

Thus, according to (3.10) and taking into account that $\rho(x_0) = x_0$, we find

$$t_{k+1} \le x_0(1+\varepsilon), \tag{3.23}$$

while by (3.19)

$$t_k = x_0(1+\varepsilon). \tag{3.24}$$

Now, in view of (3.21) and (3.11),

$$t_k t_{k+1} \le \left|\sin \pi 2^{k-1}\alpha\right| \left(\sqrt{3} - \left|\sin\left(\pi 2^{k-1}\alpha\right)\right|\right), \tag{3.25}$$

and according to (3.14), (3.15), we obtain that

$$t_k t_{k+1} \le h_m, \tag{3.26}$$

where h_m is defined by (3.13).

Notice that from simple arguments and according to (1.9),

$$g_m \le \sqrt{h_m} = b_m. \tag{3.27}$$

Therefore,

$$\prod_{k=1}^{h} \left|\sin\left(\pi 2^{k-1}\alpha\right)\right| \le \left(b_m^{\lfloor h/2 \rfloor}\right)^2 \le b_m^{h-1}. \tag{3.28}$$

Now, by (3.2)-(3.4), for $\alpha = t/m,\ t = 0, 1, \ldots, m-1$, we have

$$\begin{aligned} \left|F_{t/m}(N)\right| &\le \sum_{h=0}^{\nu_0} \left|\prod_{k=1}^{h} \left(1 + e^{2\pi i(\alpha 2^{k-1} + 1/2)}\right)\right| \\ &\le \sum_{h=0}^{\nu_0} 2^h \prod_{k=1}^{h} \left|\sin\left(2^{k-1}\alpha\pi\right)\right| \\ &\le 1 + 2\sum_{h=1}^{\nu_0} (2b_m)^{h-1} \\ &\le 1 + 2\frac{(2b_m)^{\nu_0}}{2b_m - 1}. \end{aligned} \tag{3.29}$$

Note that, according to (1.7) and (3.1),

$$(2b_m)^{\nu_0} = 2^{\lambda_m \nu_0} \le 2^{\lambda_m \log_2 N} = N^{\lambda_m}. \tag{3.30}$$

Thus, by (1.8)

$$\left|F_{t/m}(N)\right| \le 1 + \frac{2}{2b_m - 1} N^{\lambda_m} = 1 + \mu_m N^{\lambda_m}. \tag{3.31}$$

Thus, the theorem follows from (2.1).

4. Proof of Theorem 1.2

Select in (2.1) the summands which correspond to $t = 0,\ m/3,\ 2m/3$.

We have

$$mS_m(N) = \sum_{n=0}^{N-1}\left(e^{\pi i s(n)} + e^{2\pi i(n/3+(1/2)s(n))} + e^{2\pi i(2n/3+(1/2)s(n))}\right) + \sum_{t=1,\, t\,\neq\, m/3,2m/3}^{m-1}\ \sum_{n=0}^{N-1} e^{2\pi i((t/m)n+(1/2)s(n))}. \tag{4.1}$$

Since the chosen summands do not depend on m and, for $m = 3$, the latter sum is empty, then we find

$$mS_m(N) = 3S_3(N) + \sum_{t=1,\, t\,\neq\, m/3,2m/3}^{m-1}\ \sum_{n=0}^{N-1} e^{2\pi i((t/m)n+(1/2)s(n))}. \tag{4.2}$$

Further, the last double sum is estimated by the same way as in Section 3 such that

$$\left|S_m(N) - \frac{3}{m}S_3(N)\right| \leq \mu_m N^{\lambda_m}. \tag{4.3}$$

Remark 4.1. Notice that from elementary arguments it follows that if $m \geq 5$ is a multiple of 3, then

$$\left(\sin\frac{\pi}{m}\left\lfloor\frac{m-1}{3}\right\rfloor\right)\left(\sqrt{3} - \sin\frac{\pi}{m}\left\lfloor\frac{m-1}{3}\right\rfloor\right) \leq \left(\sin\frac{\pi}{m}\left\lceil\frac{m+1}{3}\right\rceil\right)\left(\sqrt{3} - \sin\frac{\pi}{m}\left\lceil\frac{m+1}{3}\right\rceil\right). \tag{4.4}$$

The latter expression is the value of b_m^2 in this case (see (1.9)).

Example 4.2. Let us find some x_0 such that $S_{21}(x) > 0$ for $x \geq x_0$.

Supposing that x is multiple of 3 and using (1.4), we obtain that

$$S_3(x) \geq \frac{2}{3^{\lambda+1/2}}x^{\lambda}. \tag{4.5}$$

Therefore, putting $m = 21$ in Theorem 1.2, we have

$$S_{21}(x) \geq \frac{1}{7}S_3(x) - \mu_{21}x^{\lambda_{21}} - 1 \geq \frac{2}{7\cdot 3^{\lambda+1/2}}x^{\lambda} - \mu_{21}x^{\lambda_{21}} - 1. \tag{4.6}$$

Now, calculating λ and λ_m by (1.2) and (1.8), we find a required x_0:

$$x_0 = \left(3.5\cdot 3^{\lambda+1/2}\mu_{21}\right)^{1/(\lambda-\lambda_{21})} = e^{908.379\ldots}. \tag{4.7}$$

Corollary 4.3. *For m which is not a multiple of 3, denote $U_m(x)$ the set of the positive integers not exceeding x which are multiples of m and not multiples of 3. Then,*

$$\sum_{n\in U_m(x)} (-1)^{s(n)} = -\frac{1}{m} S_3(x) + O\left(x^{\lambda_m}\right). \tag{4.8}$$

In particular, for sufficiently large x, we have

$$\sum_{n\in U_m(x)} (-1)^{s(n)} < 0. \tag{4.9}$$

Proof. Since

$$\left|U_m(x)\right| = S_m(x) - S_{3m}(x), \tag{4.10}$$

then the corollary immediately follows from Theorems 1.1, 1.2. □

5. On Newman sum over primes

In [7], we put the following binary digit conjectures on primes.

Conjecture 5.1. For all $n \in \mathbb{N}$, $n \neq 5, 6$,

$$\sum_{p\le n} (-1)^{s(p)} \le 0, \tag{5.1}$$

where the summing is over all primes not exceeding n.

More precisely, by the observations, $\sum_{p\le n}(-1)^{s(p)} < 0$ beginning with $n = 31$. Moreover, the following conjecture holds.

Conjecture 5.2.

$$\lim_{n\to\infty} \frac{\ln\left(-\sum_{p\le n}(-1)^{s(p)}\right)}{\ln n} = \frac{\ln 3}{\ln 4}. \tag{5.2}$$

A heuristic proof of Conjecture 5.2 was given in [8]. For a prime p, denote $V_p(x)$ the set of positive integers not exceeding x for which p is the least prime divisor. Show that the correctness of Conjectures 5.1 (for $n \ge n_0$) follows from the following very plausible statement, especially in view of the above estimates.

Conjecture 5.3. For sufficiently large n, we have

$$\begin{aligned} \left|\sum_{5\le p\le\sqrt{n}} \sum_{j\in V_p(n),\, j>p} (-1)^{s(j)}\right| &< \sum_{j\in V_3(n)} (-1)^{s(j)} \\ &= S_3(n) - S_6(n). \end{aligned} \tag{5.3}$$

Indeed, in the "worst case" (really is not satisfied), in which for all $n \geq p^2$

$$\sum_{j\in V_p(n),\, j>p} (-1)^{s(j)} < 0, \quad p \geq 5, \tag{5.4}$$

we have a decreasing but *positive* sequence of sums:

$$\begin{gathered}\sum_{j\in V_3(n),\, j>3} (-1)^{s(j)}, \sum_{j\in V_3(n),\, j>3} (-1)^{s(j)} + \sum_{j\in V_5(n),\, j>5} (-1)^{s(j)}, \dots, \\ \sum_{j\in V_3(n),\, j>3} (-1)^{s(j)} + \sum_{5\leq p<\sqrt{n}} \sum_{j\in V_p(n),\, j>p} (-1)^{s(j)} > 0.\end{gathered} \tag{5.5}$$

Hence, the "balance condition" for odd numbers [8]

$$\left| \sum_{j\leq n,\ j \text{ is odd}} (-1)^{s(j)} \right| \leq 1 \tag{5.6}$$

must be ensured permanently by the excess of the odious primes. This explains Conjecture 5.1.

It is very interesting that for some primes p the inequality (5.4), indeed, is satisfied for all $n \geq p^2$. Such primes we call "resonance primes." Our numerous observations show that all resonance primes not exceeding 1000 are

$$\begin{gathered}11, 19, 41, 67, 107, 173, 179, 181, 307, 313, 421, 431, 433, 587, \\ 601, 631, 641, 647, 727, 787.\end{gathered} \tag{5.7}$$

In conclusion, note that for $p \geq 3$, we have

$$\lim_{n\to\infty} \frac{|V_p(n)|}{n} = \frac{1}{p} \prod_{2\leq q<p} \left(1 - \frac{1}{q}\right) \tag{5.8}$$

such that

$$\lim_{n\to\infty} \left(\sum_{p\geq 3} \frac{|V_p(n)|}{n} \right) = \frac{1}{2}. \tag{5.9}$$

Thus, using Theorems 1.1, 1.2 in the form

$$S_m(n) = \begin{cases} o(S_3(n)), & (m,3) = 1, \\ \dfrac{3}{m} S_3(n)\bigl(1 + o(1)\bigr), & 3 \mid m, \end{cases} \tag{5.10}$$

and inclusion-exclusion for $p \geq 5$, we find

$$\sum_{j \in V_p(n)} (-1)^{\sigma(j)} = -\frac{3}{3p} \prod_{2 \leq q < p,\, q \neq 3} \left(1 - \frac{1}{q}\right) S_3(n)(1 + o(1)) = -\frac{3}{2p} \prod_{2 \leq q < p} \left(1 - \frac{1}{q}\right) S_3(n)\big(1 + o(1)\big). \tag{5.11}$$

Now, in view of (1.5), we obtain the following absolute result as an approximation of Conjectures 5.1, 5.2.

Theorem 5.4. *For every prime number $p \geq 5$ and sufficiently large $n \geq n_p$, we have*

$$\sum_{j \in V_p(n)} (-1)^{s(j)} < 0 \tag{5.12}$$

and, moreover,

$$\lim_{n \to \infty} \frac{\ln\left(-\sum_{j \in V_p(n)} (-1)^{s(j)}\right)}{\ln n} = \frac{\ln 3}{\ln 4}. \tag{5.13}$$

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