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Research Article

Matrix Transformations and Disk of Convergence in Interpolation Processes

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Let A_ρ denote the set of functions analytic in $|z| < \rho$ but not on $|z| = \rho$ $(1 < \rho < \infty)$. Walsh proved that the difference of the Lagrange polynomial interpolant of $f(z) \in A_\rho$ and the partial sum of the Taylor polynomial of f converges to zero on a larger set than the domain of definition of f. In 1980, Cavaretta et al. have studied the extension of Lagrange interpolation, Hermite interpolation, and Hermite-Birkhoff interpolation processes in a similar manner. In this paper, we apply a certain matrix transformation on the sequences of operators given in the above-mentioned interpolation processes to prove the convergence in larger disks.

1. Introduction

If $x = (x_k)$ is a complex number sequence and $A = [a_{nk}]$ is an infinite matrix, then Ax is the sequence whose nth term is given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k. \tag{1.1}$$

A matrix A is called $X - Y$ matrix if Ax is in the set Y whenever x is in X. For $0 \leq \alpha < \infty$, let

$$G_\alpha = \{x : \limsup |x_k|^{1/k} \leq \alpha\}. \tag{1.2}$$

For various values of α, this sequence space has been studied extensively by several authors [1–5]. In particular, Jacob Jr. [2, page 186] proved the following result.

Theorem 1.1. *An infinite matrix A is a $G_u - G_v$ matrix if and only if for each number w such that $0 < w < 1/v$, there exist numbers B and s such that $0 < s < 1/u$ and*

$$|a_{nk}|w^n \leq Bs^k \tag{1.3}$$

for all n and k.

Let A_ρ denote the collection of functions analytic in the disk $D_\rho = \{z \in \mathcal{C} : |z| < \rho\}$ for some $1 < \rho < \infty$ and having a singularity on the circle $|z| = \rho$. In Section 2, we state the results proved by Cavaretta Jr. et al. [6] on the Lagrange interpolation, Hermite interpolation, and Hermite-Birkhoff interpolation of $f(z) \in A_\rho$ in the nth roots of unity, which will be required. Main results are given in Section 3 and deal with the application of a certain matrix to the various polynomial interpolants in the above results. Interestingly, we are able to show that under the matrix transformation the difference of the interpolant polynomials and the corresponding Taylor polynomials converges to zero in a larger region.

2. Preliminaries

Throughout the paper, we assume that $f(z) \in A_\rho$ with $1 < \rho < \infty$ and $f(z)$ has the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{2.1}$$

For each integer $n \geq 0$, let $L_n(z; f)$ denote the unique Lagrange interpolation polynomial of degree n which interpolates $f(z)$ in the $(n+1)$st roots of unity, that is,

$$L_n(\omega; f) = f(\omega), \tag{2.2}$$

where ω is any $(n+1)$st root of unity. Setting $S_n(z; f) = \sum_{k=0}^{n+1} a_k z^k$, the well-known Walsh equiconvergence theorem [7] states that

$$\lim_{n \to \infty} [L_n(z; f) - S_n(z; f)] = 0, \quad \forall |z| < \rho^2, \tag{2.3}$$

with the convergence being uniform and geometric on any closed subset of $|z| < \rho^2$. This theorem has been extended in various ways by several authors [6, 8–10]. In all that follows, we state some of the results of [6] which are needed for our main results.

Under Lagrange interpolation, letting

$$S_{n,j}(z; f) = \sum_{k=0}^{n} a_{k+j(n+1)} z^k, \quad j = 0, 1, \ldots, \tag{2.4}$$

the authors [6, Theorem 1, page 156] have proved the following result.

Theorem 2.1. *For each positive integer l,*

$$\lim_{n\to\infty}\left\{L_n(z;f)-\sum_{j=0}^{l-1}S_{n,j}(z;f)\right\}=0,\quad \forall|z|<\rho^{l+1} \tag{2.5}$$

and this result is best possible.

In the proof of Theorem 2.1, it has been shown by the authors that

$$L_n(z;f)-\sum_{j=0}^{l-1}S_{n,j}(z;f)=\frac{1}{2\pi i}\int_\Gamma\frac{f(t)(t^{n+1}-z^{n+1})}{(t-z)(t^{n+1}-1)t^{l(n+1)}}dt, \tag{2.6}$$

where Γ is any circle $|t|=R$ with $1<R<\rho$.

In [6] the authors have studied the Hermite interpolation also in a similar way. For a fixed positive integer $r\geq 2$, let $h_{r(n+1)-1}(z;f)$ be the unique Hermite polynomial interpolant to $f,f',\dots,f^{(r-1)}$ in the $(n+1)$st roots of unity, that is,

$$h^{(j)}_{r(n+1)-1}(\omega;f)=f^{(j)}(\omega),\quad j=0,1,\dots,r-1, \tag{2.7}$$

where ω is any $(n+1)$st root of unity.

Setting

$$\begin{aligned} H_{r(n+1)-1,0}(z;f)&=\sum_{k=0}^{r(n+1)-1}a_kz^k,\\ H_{r(n+1)-1,j}(z;f)&=\beta_j(z)\sum_{k=0}^{n}a_{k+(n+1)(r+j-1)}\cdot z^k,\quad j=1,2,\dots, \end{aligned} \tag{2.8}$$

where

$$\beta_j(z)=\sum_{k=0}^{r-1}\binom{r+j-1}{k}\left(z^{n+1}-1\right)^k,\quad j=1,2,\dots, \tag{2.9}$$

in [6, Theorem 3, page 162] the authors proved the following result.

Theorem 2.2. *For each positive integer l,*

$$\lim_{n\to\infty}\left\{h_{r(n+1)-1}(z;f)-\sum_{j=0}^{l-1}H_{r(n+1)-1,j}(z;f)\right\}=0,\quad \forall|z|<\rho^{1+l/r} \tag{2.10}$$

and this result is best possible.

In the proof of Theorem 2.2, it was shown by the authors in [6, page 165] that

$$h_{r(n+1)-1}(z;f)-\sum_{j=0}^{l-1}H_{r(n+1)-1,j}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{(t-z)}K(t,z)dt, \tag{2.11}$$

where Γ is any circle $|t|=R$ with $1<R<\rho$ and

$$|K(t,z)|\leq M\frac{\left(R^{n+1}+|z|^{n+1}\right)(|z|+1)^{r-1}}{R^{(r+l)(n+1)}}. \tag{2.12}$$

Under Hermite-Birkhoff interpolation, the authors in [6] established several results for different cases. We consider here only the $(0,m)$ case. Let n and m be integers with $n\geq m\geq 0$. Let $b_{2n+1}^{(0,m)}(z;f)$ be the unique Hermite-Birkhoff polynomial of degree $2n+1$ which interpolates f in the $(n+1)$st roots of unity and whose mth derivative interpolates $f^{(m)}$ in the $(n+1)$st roots of unity, that is,

$$b_{2n+1}^{(0,m)}(\omega;f)=f(\omega),\qquad \left(b_{2n+1}^{(0,m)}(\omega;f)\right)^{(m)}=f^{(m)}(\omega), \tag{2.13}$$

where ω is any $(n+1)$st root of unity.

Setting

$$\begin{gathered}B_{2n+1,0}^{(0,m)}(z;f)=\sum_{k=0}^{2n+1}a_kz^k,\\ B_{2n+1,\nu}^{(0,m)}(z;f)=\sum_{j=0}^{n}a_{j+(\nu+1)(n+1)}\cdot z^jq_{j,\nu}(z),\quad \nu=1,2\ldots,\end{gathered} \tag{2.14}$$

where $q_{j,\nu}(z)$ is a polynomial of degree $(n+1)$ given by

$$q_{j,\nu}(z)=z^{n+1}+\frac{((\nu+1)(n+1)+j)_m-(n+1+j)_m}{(n+1+j)_m-(j)_m}\left(z^{n+1}-1\right),\quad j=0,1,\ldots,n, \tag{2.15}$$

with the following conventional notation

$$(j)_m=\begin{cases}j(j-1)\cdots(j-m+1), & \text{if } m\leq j,\\ 0, & \text{if } m>j,\end{cases} \tag{2.16}$$

in [6, Theorem 4, page 170] the authors proved the following result.

Theorem 2.3. *For each positive integer l,*

$$\lim_{n\to\infty}\left\{b_{2n+1}^{(0,m)}(z;f)-\sum_{j=0}^{l-1}B_{2n+1,j}^{(0,m)}(z;f)\right\}=0,\quad \text{for } |z|<\rho^{1+l/2} \tag{2.17}$$

and this result is best possible.

In the proof, it was shown in [6, Theorem 4, page 171] that

$$b_{2n+1}^{(0,m)}(z;f)-\sum_{j=0}^{l-1}B_{2n+1,j}^{(0,m)}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}f(t)K_{n,l}(t,z)dt, \tag{2.18}$$

where Γ is any circle $|t|=R$ with $1<R<\rho$ and $|K_{n,l}(t,z)|$ is bounded on the circle $|t|=R<|z|$ by

$$|K_{n,l}(t,z)|\leq\frac{|z|^{n+1}\left(|t|^{n+1}+|z|^{n+1}\right)}{|t-z||t|^{(l+1)(n+1)}|t^{n+1}-1|}+\frac{\left(|z|^{n+1}+1\right)\left(|z|^{n+1}+R^{n+1}\right)}{(|z|-R)R^{(l+2)(n+1)}}. \tag{2.19}$$

3. Main results

Our aim in this section is to apply a G_u-G_v matrix to the polynomial sequences of operators in each of the above three theorems given in Section 2 and prove that the difference of transformed sequences in each case converges to zero in a lager disk $D_{\widehat{\rho}}$. To simplify, let us denote $L_n(z;f)$ and $\sum_{j=0}^{l-1}S_{n,j}(z;f)$ in Theorem 2.1 by L_n and $S_{n,l}$, respectively.

Theorem 3.1. *Let $f(z)\in A_\rho$ and let Γ be any circle $|t|=R$ with $1<R<\rho$. For any $\widehat{\rho}>\rho$, choose $u>\widehat{\rho}/R$ and $0<v<1$. Let A be a G_u-G_v matrix and define*

$$\lambda_n(z)=\sum_{k=0}^{\infty}a_{nk}L_k,\qquad \sigma_{n,l}(z)=\sum_{k=0}^{\infty}a_{nk}S_{k,l} \tag{3.1}$$

for $n=0,1,\ldots$. Then, for each l,

$$\lim_{n\to\infty}[\lambda_n(z)-\sigma_{n,l}(z)]=0,\quad \forall z\in D_{\widehat{\rho}}. \tag{3.2}$$

Proof. Using the integral representation given in (2.6), for each $l=1,2,\ldots$, we have

$$\begin{aligned}\lambda_n(z)-\sigma_{n,l}(z)&=\sum_{k=0}^{\infty}a_{nk}(L_k-S_{k,l})\\&=\sum_{k=0}^{\infty}a_{nk}\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)(t^{k+1}-z^{k+1})}{(t-z)(t^{k+1}-1)t^{l(k+1)}}dt\\&=\sum_{k=0}^{\infty}a_{nk}\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{(t-z)t^{l(k+1)}}\left[1-\left(\frac{z}{t}\right)^{k+1}\right]\frac{t^{k+1}}{t^{k+1}-1}dt.\end{aligned} \tag{3.3}$$

Since $|t| = R > 1$, we have

$$\lambda_n(z) - \sigma_{n,l}(z) = \sum_{k=0}^{\infty} a_{nk} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{(t-z)t^{l(k+1)}} \left[1 - \left(\frac{z}{t}\right)^{k+1}\right] \sum_{q=0}^{\infty} \left(\frac{1}{t^{k+1}}\right)^{q} dt$$
$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{(t-z)} \sum_{q=0}^{\infty} \frac{1}{t^{l+q}} \sum_{k=0}^{\infty} a_{nk} \left[1 - \left(\frac{z}{t}\right)^{k+1}\right] \left(\frac{1}{t^{l+q}}\right)^{k} dt. \tag{3.4}$$

The interchange of the integral and the summations is justified by showing that the series

$$\sum_{k=0}^{\infty} a_{nk} \left(\frac{1}{t^{l+q}}\right)^{k}, \qquad \sum_{k=0}^{\infty} a_{nk} \left(\frac{z}{t}\right)^{k} \left(\frac{1}{t^{l+q}}\right)^{k} \tag{3.5}$$

converge absolutely for each q as follows. From (1.3), for any $1 < w < 1/v$ we have that $|a_{nk}| w^n \le B s^k$, where $s < 1/u < R/\widehat{\rho} < 1$. Thus, for each q, l, and n, we have

$$\sum_{k=0}^{\infty} |a_{nk}| \left(\frac{1}{|t|^{l+q}}\right)^{k} \le \frac{B}{w^n} \sum_{k=0}^{\infty} \left(\frac{s}{R^{l+q}}\right)^{k} = \frac{B}{w^n} \frac{R^{l+q}}{(R^{l+q} - s)}, \tag{3.6}$$

because $s/R^{l+q} < R/\widehat{\rho} R^{l+q} < 1$ and similarly for $|z| < \widehat{\rho}$

$$\sum_{k=0}^{\infty} |a_{nk}| \left|\frac{z}{t}\right|^{k+1} \left(\frac{1}{|t|^{l+q}}\right)^{k} \le \frac{B\widehat{\rho}}{w^n R} \sum_{k=0}^{\infty} \left(\frac{s\widehat{\rho}}{R^{l+q+1}}\right)^{k} = \frac{B\widehat{\rho}}{w^n R} \frac{R^{l+q+1}}{R^{l+q+1} - s\widehat{\rho}}, \tag{3.7}$$

because $s\widehat{\rho}/R^{l+q+1} < R/R^{l+q+1} < 1$. Therefore, from (3.6) and (3.7), identities (3.4) become

$$|\lambda_n(z) - \sigma_{n,l}(z)| \le \frac{B}{2\pi w^n} \int_{\Gamma} \frac{|f(t)|}{|t-z|} \sum_{q=0}^{\infty} \frac{1}{R^{l+q}} \left[\frac{R^{l+q}}{R^{l+q} - s} + \frac{\widehat{\rho} R^{l+q}}{R^{l+q+1} - s\widehat{\rho}}\right] dt$$
$$= \frac{B}{2\pi w^n} \int_{\Gamma} \frac{|f(t)|}{|t-z|} \sum_{q=0}^{\infty} \left[\frac{1}{R^{l+q} - s} + \frac{\widehat{\rho}}{R^{l+q+1} - s\widehat{\rho}}\right] dt. \tag{3.8}$$

It can be easily proved that the two series on the right side of the above expression converge by using the ratio test. Assuming that $f(t)$ is bounded on Γ, $w > 1$ implies that

$$\lim_{n\to\infty} [\lambda_n(z) - \sigma_{n,l}(z)] = 0, \quad \forall |z| < \widehat{\rho}. \tag{3.9}$$

□

Next, we prove a similar result of convergence on a larger disk in the case of the Hermite interpolation. To simplify, let us denote

$$h_{r(n+1)-1}(z; f), \qquad \sum_{j=0}^{l-1} H_{r(n+1)-1,j}(z; f) \tag{3.10}$$

in Theorem 2.2 by h_n and $H_{n,l}$, respectively.

Theorem 3.2. *Let $f(z) \in A_\rho$ and let Γ be any circle $|t| = R$ with $1 < R < \rho$. For any $\widehat{\rho} > \rho$, choose $u > \widehat{\rho}/R$ and $0 < v < 1$. Let A be a $G_u - G_v$ matrix and define*

$$\lambda_n(z) = \sum_{k=0}^{\infty} a_{nk} h_k, \qquad \sigma_{n,l}(z) = \sum_{k=0}^{\infty} a_{nk} H_{k,l} \tag{3.11}$$

for $n = 0, 1, \ldots$. Then, for each l,

$$\lim_{n \to \infty} [\lambda_n(z) - \sigma_{n,l}(z)] = 0, \quad \forall z \in D_{\widehat{\rho}}. \tag{3.12}$$

Proof. Using the integral representation given in (2.11), for each $l = 1, 2, \ldots$, we have

$$\lambda_n(z) - \sigma_{n,l}(z) = \sum_{k=0}^{\infty} a_{nk}(h_k - H_{k,l}) = \sum_{k=0}^{\infty} a_{nk} \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z} K(t, z) dt. \tag{3.13}$$

From (2.12) we obtain that

$$\begin{aligned} |\lambda_n(z) - \sigma_{n,l}(z)| &\leq \sum_{k=0}^{\infty} |a_{nk}| \frac{1}{2\pi} \int_\Gamma \frac{|f(t)|}{|t - z|} \frac{M\left(R^{k+1} + |z|^{k+1}\right)\left(|z| + 1\right)^{r-1}}{R^{(r+l)(k+1)}} dt \\ &\leq \frac{M(|z| + 1)^{r+1}}{2\pi} \int_\Gamma \frac{|f(t)|}{|t - z|} \sum_{k=0}^{\infty} |a_{nk}| \frac{\left(R^{k+1} + |z|^{k+1}\right)}{R^{(r+l)(k+1)}} dt. \end{aligned} \tag{3.14}$$

The interchange of the integral and the summation is justified by showing that the series

$$\sum_{k=0}^{\infty} \frac{|a_{nk}|}{R^{(r+l-1)(k+1)}}, \qquad \sum_{k=0}^{\infty} |a_{nk}| \left(\frac{|z|}{R^{r+l}}\right)^{k+1} \tag{3.15}$$

converge as follows. From (1.3) we have that for any $1 < w < 1/v$, $|a_{nk}| w^n \leq B s^k$, where $s < 1/u < R/\widehat{\rho} < 1$.

Thus for any fixed positive integer $r \geq 2$ and for each l and n

$$\sum_{k=0}^{\infty} |a_{nk}| \left(\frac{1}{R^{r+l-1}}\right)^{k+1} \leq \frac{B}{w^n R^{r+l}} \sum_{k=0}^{\infty} \left(\frac{s}{R^{r+l-1}}\right)^k = \frac{B}{w^n (R^{r+l-1} - s)}, \tag{3.16}$$

because $s/R^{r+l-1} < 1/\widehat{\rho} R^{r+l-2} < 1$, and similarly for $|z| < \widehat{\rho}$,

$$\sum_{k=0}^{\infty} |a_{nk}| \left(\frac{|z|}{R^{r+l}}\right)^{k+l} \leq \frac{B|z|}{w^n R^{r+l}} \sum_{k=0}^{\infty} \left(\frac{s|z|}{R^{r+l}}\right)^k = \frac{B|z|}{w^n} \frac{1}{(R^{r+l} - s|z|)}, \tag{3.17}$$

because $s|z|/R^{r+l} < 1/R^{r+l-1} < 1$.

Therefore, using (3.16) and (3.17) in (3.14) we get that

$$|\lambda_n(z) - \sigma_{n,l}(z)| \leq \frac{MB(|z|+1)^{r+1}}{2\pi w^n} \int_\Gamma \frac{|f(t)|}{|t-z|} \left[\frac{1}{R^{r+l-1} - s} + \frac{|z|}{R^{r+l} - s|z|} \right] dt. \tag{3.18}$$

Assuming that $f(t)$ is bounded on Γ, $w > 1$ implies that

$$\lim_{n\to\infty} [\lambda_n(z) - \sigma_{n,l}(z)] = 0, \quad \text{for } |z| < \widehat{\rho}. \tag{3.19}$$

□

Thus in the Lagrange case, the G_u–G_v matrix transformation of the sequence operators produces new sequences such that the difference between the transformed sequences of polynomials L_n and $S_{n,l}$ converges to zero in an arbitrarily large disk $D_{\widehat{\rho}}$ by choosing $u > \widehat{\rho}/R$. Similarly, in the Hermite case also, the domain of convergence to zero is arbitrarily large under $G_u - G_v$ matrix transformation by choosing $u > \widehat{\rho}/R$. But as we see in the following theorem, in the case of Hermite-Birkhoff interpolation, the domain of convergence to zero of the difference of the transformed polynomials of Theorem 2.3 is arbitrarily large only if we choose $u > \widehat{\rho}^2/R$.

To simplify, let us denote $b_{2n+1}^{(0,m)}(z;f)$ and $\sum_{j=0}^{l-1} B_{2n+1,j}^{(0,m)}(z;f)$ of Theorem 2.3 by b_n and $B_{n,l}$, respectively.

Theorem 3.3. *Let $f(z) \in A_\rho$ and let Γ be any circle $|t| = R$ with $1 < R < \rho$. For any $\widehat{\rho} > \rho$, choose $u > \widehat{\rho}^2/R$ and $0 < v < 1$. Let A be a $G_u - G_v$ matrix and define*

$$\lambda_n(z) = \sum_{k=0}^{\infty} a_{nk} b_k, \qquad \sigma_{n,l}(z) = \sum_{k=0}^{\infty} a_{nk} B_{k,l} \tag{3.20}$$

for $n = 0, 1, \ldots$. Then, for each l,

$$\lim_{n\to\infty} [\lambda_n(z) - \sigma_{n,l}(z)] = 0, \quad \forall z \in D_{\widehat{\rho}}. \tag{3.21}$$

Proof. Using the integral representation given in (2.18), for each $l = 1, 2, \ldots,$ we have

$$\lambda_n(z) - \sigma_{n,l}(z) = \sum_{k=0}^{\infty} a_{nk} \frac{1}{2\pi i} \int_\Gamma f(t) K_{k,l}(t,z)\, dt. \tag{3.22}$$

From (2.19) we obtain that

$$\begin{aligned} |\lambda_n(z) - \sigma_{n,l}(z)| &\leq \sum_{k=0}^{\infty} |a_{nk}| \frac{1}{2\pi} \int_\Gamma |f(t)| |K_{k,l}(t,z)|\, dt \\ &\leq \frac{1}{2\pi} \int_\Gamma \frac{|f(t)|}{|t-z|} \sum_{k=0}^{\infty} \frac{|a_{nk}| |z|^{k+1} \left(|t|^{k+1} + |z|^{k+1}\right)}{|t|^{(l+1)(k+1)} |t^{k+1} - 1|} dt \\ &\quad + \frac{1}{2\pi} \int_\Gamma \frac{|f(t)|}{|z| - R} \sum_{k=0}^{\infty} \frac{|a_{nk}| \left(|z|^{k+1} + 1\right)\left(|z|^{k+1} + R^{k+1}\right)}{R^{(l+2)(k+1)}} dt. \end{aligned} \tag{3.23}$$

The interchange of the integral and the summations is justified by showing that the two series in (3.23) converge as follows. From (1.3) we have that for any $1 < w < 1/v$, $|a_{nk}|w^n \le Bs^k$, where $s < 1/u < R/\widehat{\rho}^{\,2} < 1$. Since $|t| = R$ and $|z| < \widehat{\rho}$, using the same method used in (3.4) we write the first summation as

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{|a_{nk}||z|^{k+1}}{|t|^{(l+1)(k+1)}}\frac{\left(|t|^{k+1}+|z|^{k+1}\right)}{\left(R^{k+1}-1\right)} &\le \sum_{k=0}^{\infty}\frac{Bs^k|z|^{k+1}}{w^n R^{(l+1)(k+1)}}\left[1+\left(\frac{|z|}{R}\right)^{k+1}\right]\sum_{q=0}^{\infty}\left(\frac{1}{R^{k+1}}\right)^{q} \\
&\le \frac{B}{w^n}\sum_{q=0}^{\infty}\frac{\widehat{\rho}}{R^{l+q+1}}\left[\sum_{k=0}^{\infty}\left(\frac{s\widehat{\rho}}{R^{l+q+1}}\right)^{k}+\sum_{k=0}^{\infty}\left(\frac{s\widehat{\rho}}{R^{l+q+1}}\right)^{k}\left(\frac{\widehat{\rho}}{R}\right)^{k+1}\right] \\
&= \frac{B}{w^n}\sum_{q=0}^{\infty}\frac{\widehat{\rho}}{R^{l+q+1}}\left[\frac{R^{l+q+1}}{\left(R^{l+q+1}-s\widehat{\rho}\right)}+\frac{\widehat{\rho}}{R}\frac{\left(R^{l+q+2}\right)}{\left(R^{l+q+2}-s\widehat{\rho}^2\right)}\right],
\end{aligned} \tag{3.24}$$

because $s\widehat{\rho}/R^{l+q+1} < 1/\widehat{\rho}R^{l+q} < 1$ and $s\widehat{\rho}^2/R^{l+q+2} < 1/R^{l+q+1} < 1$ for each l and q. Now, for the second sum in (3.23) we have

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{|a_{nk}|\left(|z|^{k+1}+1\right)\left(|z|^{k+1}+R^{k+1}\right)}{R^{(l+2)(k+1)}} &\le \frac{B}{w^n}\sum_{k=0}^{\infty}\frac{s^k}{R^{(l+2)(k+1)}}\left[\widehat{\rho}^{2k+2}+\widehat{\rho}^{k+1}+\left(\widehat{\rho}R\right)^{k+1}+R^{k+1}\right] \\
&= \frac{B}{w^n}\left[\frac{\widehat{\rho}^2}{R^{l+2}-s\widehat{\rho}^2}+\frac{\widehat{\rho}}{R^{l+2}-s\widehat{\rho}}+\frac{\widehat{\rho}}{R^{l+1}-s\widehat{\rho}}+\frac{1}{R^{l+1}-s}\right],
\end{aligned} \tag{3.25}$$

because for each l,

$$\begin{gathered}
\frac{s\widehat{\rho}^2}{R^{l+2}} < \frac{1}{R^{l+1}} < 1, \\
\frac{s\widehat{\rho}}{R^{l+2}} < \frac{1}{\widehat{\rho}R^{l+1}} < 1, \\
\frac{s\widehat{\rho}}{R^{l+1}} < \frac{1}{\widehat{\rho}R^{l}} < 1, \\
\frac{s}{R^{l+1}} < \frac{1}{\widehat{\rho}^2R^{l}} < 1.
\end{gathered} \tag{3.26}$$

Using (3.24) and (3.25), the expression (3.23) becomes

$$
\begin{aligned}
|\lambda_n(z) - \sigma_{n,l}(z)| \leq{} & \frac{B}{w^n 2\pi} \int_\Gamma \frac{|f(t)|}{|t-z|} \sum_{q=0}^{\infty} \left[\frac{\widehat{\rho}}{R^{l+q+1} - s\widehat{\rho}} + \frac{\widehat{\rho}^2}{R^{l+q+2} - s\widehat{\rho}^2} \right] dt \\
& + \frac{B}{2\pi w^n} \int_\Gamma \frac{|f(t)|}{|z| - R} \left[\frac{\widehat{\rho}^2}{R^{l+2} - s\widehat{\rho}^2} + \frac{\widehat{\rho}}{R^{l+2} - s\widehat{\rho}} + \frac{\widehat{\rho}}{R^{l+1} - s\widehat{\rho}} + \frac{1}{R^{l+1} - s} \right] dt.
\end{aligned}
\tag{3.27}
$$

It can be easily proved that the two series on the right side of the above expression converge by using the ratio test. Assuming that $f(t)$ is bounded on Γ, $w > 1$ implies that

$$
\lim_{n \to \infty} [\lambda_n(z) - \sigma_{n,l}(z)] = 0 \tag{3.28}
$$

for each $|z| < \widehat{\rho}$. □

In two of the above three theorems, for any R and $\widehat{\rho}$ satisfying the stated conditions, we have chosen $u > \widehat{\rho}/R > 1$ and $0 < v < 1$. In the case of the last theorem, we have chosen $u > \widehat{\rho}^2/R > 1$ and $0 < v < 1$. Now, to see if there exists such a $G_u - G_v$ matrix, we give below an obvious example.

Define a matrix A by

$$
a_{nk} = \frac{v^n}{p^k}, \quad p > u. \tag{3.29}
$$

Then for any w such that $0 < w < 1/v$, we have that

$$
|a_{nk}| w^n = \frac{(vw)^n}{p^k} < \left(\frac{1}{p} \right)^k, \tag{3.30}
$$

where $1/p < 1/u$. Hence by Theorem 1.1, A is a $G_u - G_v$ matrix.

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