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Research Article

Structure Theorem for Functionals in the Space $\mathfrak{S}'_{\omega_1,\omega_2}$

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We introduce the space $\mathfrak{S}_{\omega_1,\omega_2}$ of all C^∞ functions φ such that $\sup_{|\alpha|\le m}\|e^{k\omega_1}\partial^\alpha\varphi\|_\infty$ and $\sup_{|\alpha|\le m}\|e^{k\omega_2}\partial^\alpha\widehat{\varphi}\|_\infty$ are finite for all $k\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^n$, where ω_1 and ω_2 are two weights satisfying the classical Beurling conditions. Moreover, we give a topological characterization of the space $\mathfrak{S}_{\omega_1,\omega_2}$ without conditions on the derivatives. For functionals in the dual space $\mathfrak{S}'_{\omega_1,\omega_2}$, we prove a structure theorem by using the classical Riesz representation thoerem.

1. Introduction

The theory of ultradistributions introduced by Beurling [1] was to find an appropriate context for his work on almost holomorphic extensions. Beurling proved that ultradistributions are limits of holomorphic functions in the upper and lower half-planes. Björck [2] studied and expanded the theory of Beurling on ultradistributions to extend the work of Hörmander [3] on existence, nonexistence, and regularity of solutions of constant coefficient linear partial differential equations.

The Beurling-Björck space $\mathfrak{S}_w$, as defined in [2], consists of C^∞ functions such that the functions and their Fourier transform jointly with all their derivatives decay ultrarapidly at infinity.

In this paper, we introduce the space $\mathfrak{S}_{w_1,w_2}$ of C^∞ functions such that the functions and their Fourier transform jointly with all their derivatives decay ultrarapidly at infinity. Moreover, we give a characterization of the space $\mathfrak{S}_{w_1,w_2}$ and its dual $\mathfrak{S}'_{w_1,w_2}$.

The main difference between the Beurling-Björck space $\mathfrak{S}_w$ and the space $\mathfrak{S}_{w_1,w_2}$ is that the decay of the functions in $\mathfrak{S}_w$ and their Fourier transform are measured by the same submultiplicative function e^{kw}, $k\ge 0$. Whereas the decay of the functions in $\mathfrak{S}_{w_1,w_2}$ and their Fourier transform are measured by two different submultiplicative functions e^{kw_1} and e^{kw_2}, $k \geq 0$.

This paper is organized in three sections. In Section 2, we give preliminary definitions and results and introduce the space $\mathfrak{S}_{w_1,w_2}$. In Section 3, we give a topological characterization of the space $\mathfrak{S}_{w_1,w_2}$ without conditions on the derivatives. In Section 4, we use the topological characterization of the space $\mathfrak{S}_{w_1,w_2}$ that is given in Section 3 to prove a representation theorem for functionals in the dual space $\mathfrak{S}'_{w_1,w_2}$ of the space $\mathfrak{S}_{w_1,w_2}$.

The symbols C^∞, C_0^∞, L^p, and so forth indicate the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_\infty$ indicates the norm in the space L^∞ . When we do not work on the general Euclidean space $\mathbb{R}^n$, we will write $L^p(\mathbb{R})$, and so forth as appropriate. Partial derivatives will be denotedby ∂^α, where α is a multi-index $(\alpha_1, \ldots, \alpha_n)$. If it is necessary to indicate on which variables we are taking the derivative, we will do so by attaching subindexes. We will use the standard abbreviations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$. With $\alpha \leq \beta$, we mean that $\alpha_j \leq \beta_j$ for every j. The Fourier transform of a function g will be denoted by $\mathcal{F}(g)$ or $\widehat{g}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} g(x)dx$. The inverse Fourier transform is then $\mathcal{F}^{-1}(g) = \int_{\mathbb{R}^n} e^{2\pi i x\xi} g(\xi)d\xi$. The letter C will indicate a positive constant, that may be different at different occurrences. If it is important to indicate that a constant depends on certain parameters, we will do so by attaching subindexes to the constant. We will not indicate the dependence of constants on the dimension n or other fixed parameters.

2. Preliminary definitions and results

In this section, we give definitions and results which we will use later.

Definition 2.1 (see [2]). With $\mathcal{M}_c$, we denote the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where

(1) $\Omega : [0, \infty) \to [0, \infty)$ is increasing, continuous, and concave,

(2) $\Omega(0) = 0$,

(3) $\int_{\mathbb{R}} \Omega((t)/(1 + t^2))dt < \infty$,

(4) $\Omega(t) \geq a + b \ln(1 + t)$ for some $a \in \mathbb{R}$ and some $b > 0$.

Standard classes of functions w in $\mathcal{M}_c$ are given by

$$w(x) = |x|^d \quad \text{for } 0 < d < 1, \qquad w(x) = p \ln(1 + |x|) \quad \text{for } p > 0. \tag{2.1}$$

Remark 2.2. Let us observe for future use that if we take an integer $N > (n/b)$, then

$$C_N = \int_{\mathbb{R}^n} e^{-Nw(x)} dx < \infty, \quad \forall w \in \mathcal{M}_c, \tag{2.2}$$

where b is the constant in condition 4 of Definition 2.1.

The following lemma was observed in [2] without proof. Our proof is an adaptation of [4, Proposition 4.6].

Lemma 2.3. *Conditions 1 and 2 in Definition 2.1 imply that w is subadditive for all $w \in \mathcal{M}_c$.*

Proof. Let $0 < k < 1$. Since Ω is increasing, we obtain

$$\begin{aligned} w(x+y) &\leq \Omega\left(\frac{k}{k}|x| + \frac{1-k}{1-k}|y|\right) \\ &\leq \max\left\{\Omega\left(\frac{|x|}{k}\right), \Omega\left(\frac{|y|}{1-k}\right)\right\}. \end{aligned} \tag{2.3}$$

Since Ω is concave on $[0,\infty)$ and $\Omega(0) = 0$, we have

$$\Omega\left(\frac{k}{k}|x|\right) \geq k\Omega\left(\frac{|x|}{k}\right), \qquad \Omega\left(\frac{|y|}{1-k}\right) \geq \frac{1}{1-k}\Omega\left(|y|\right). \tag{2.4}$$

If we take

$$k = \frac{\Omega(|x|)}{\Omega(|x|) + \Omega(|y|)}, \tag{2.5}$$

then we have

$$\begin{aligned} w(x+y) &\leq \max\left\{\Omega\left(\frac{|x|}{k}\right), \Omega\left(\frac{|y|}{1-k}\right)\right\} \\ &\leq w(x) + w(y). \end{aligned} \tag{2.6}$$

This completes the proof of Lemma 2.3. □

We now recall a topological characterization of the Beurling-Björck space $\mathfrak{S}_w$ of test functions for tempered ultradistributions.

Theorem 2.4 (see [5]). *Given $w \in \mathcal{M}_c$, the space $\mathfrak{S}_w$ can be described both as a set and as a topology by*

$$\mathfrak{S}_w = \{\varphi : \mathbb{R}^n\mathbb{C} : \varphi \text{ is continuous and for all } k = 0,1,2,\ldots,\ p_{k,0}(\varphi) < \infty,\ \ p_{k,0} \circ \mathcal{F}(\varphi) < \infty\}, \tag{2.7}$$

where $p_{k,0}(\varphi) = \|e^{kw}\varphi\|_\infty$ and $p_{k,0} \circ \mathcal{F}(\varphi) = \|e^{kw}\widehat{\varphi}\|_\infty$.

We observe that $\mathfrak{S}_w$ becomes the Schwartz space $\mathfrak{S}$ when

$$w(x) = \ln(1 + |x|). \tag{2.8}$$

For $\alpha, \beta > 0$, the Gelfand-Shilov space S_α^β of type S is characterized in [6] by the space of all C^∞ functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ for which the seminorms

$$\|e^{k|x|^{1/\alpha}}\varphi\|_\infty, \qquad \|e^{m|x|^{1/\beta}}\widehat{\varphi}\|_\infty \tag{2.9}$$

are finite for some $k, m \in \mathbb{N}_0$.

Definition 2.5. Given $w_1, w_2 \in \mathcal{M}_c$, the space $\mathfrak{S}_{w_1,w_2}$ is the space of all C^∞ functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ for which the seminorms

$$p_{k,m}(\varphi) = \sup_{|\beta|\le m} \left\| e^{kw_1} \partial^\beta \varphi \right\|_\infty, \qquad \pi_{k,m}(\varphi) = \sup_{|\beta|\le m} \left\| e^{kw_2} \partial^\beta \widehat{\varphi} \right\|_\infty \tag{2.10}$$

are finite, for $k, m \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$.

We can assign to $\mathfrak{S}_{w_1,w_2}$ a structure to Fréchet space by means of the countable family of seminorms

$$S = \left\{ p_{k,m}, \pi_{k,m} \right\}_{k,m=0}^{\infty}. \tag{2.11}$$

Since $p_{k,m}(\varphi) < \infty$ for all $k = 0, 1, 2, \ldots, \varphi$ is integrable, so $\widehat{\varphi}$ is well defined and the formulation of the condition $\pi_{k,m}(\varphi)$ makes sense for all $k = 0, 1, 2, \ldots$.

The space $\mathfrak{S}_{w_1,w_2}$, equipped with the family of seminorms

$$S = \left\{ p_{k,m}, \pi_{k,m} : k, m \in \mathbb{N}_0 \right\}, \tag{2.12}$$

is a Fréchet space.

We observe that the space $\mathfrak{S}_{w_1,w_2}$ becomes the Beurling-Björck space $\mathfrak{S}_{w_1}$, when $w_1 = w_2$. When $w_2(x) = \ln(1+|x|)$, the space of C^∞ functions with compact support $\mathfrak{D}$ is dense subspace of $\mathfrak{S}_{w_1,w_2}$ for all $w_1 \in \mathcal{M}_c$. The conditions imposed on the function w assure that the space $\mathfrak{S}_{w_1,w_2}$ satisfies the properties expected from a space of testing functions. For instance, the operators of differentiation and multiplication by x^α are continuous from $\mathfrak{S}_{w_1,w_2}$ into themselves, the space $\mathfrak{S}_{w_1,w_2}$ is a topological algebra under pointwise multiplication and convolution. Unfortunately, the Fourier transformation on $\mathfrak{S}_{w_1,w_2}$ is not a topological isomorphism from $\mathfrak{S}_{w_1,w_2}$ into itself for some w_1, $w_2 \in \mathcal{M}_c$.For Example, if we take $w_1(x) = |x|^{1/2}$, $w_2(x) = \ln(1+|x|)$, and $f \in \mathfrak{D} \setminus \mathfrak{D}_{w_1}$, then $f \in \mathfrak{S}_{w_1,w_2}$ but $\widehat{f} \notin \mathfrak{S}_{w1,w2}$; see [1, 2].

Theorem 2.6 (Riesz representation theorem [7]). *Given a functional L in the topological dual of the space $\mathcal{C}_0$, there exists a unique regular complex Borel measure μ such that*

$$L(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu. \tag{2.13}$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ. Conversely, any such measure μ defines a continuous linear functional on $\mathcal{C}_0$.

We conclude this section with Lemma 2.7 [8], the version of which is due to Hadamard [9], see also [10].

Lemma 2.7 (see [8, 10]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with continuous derivatives of order ≤ 2. Assume that there exist $P, Q \ge 0$ such that*

$$\begin{gathered} |f(x)| \le P, \\ |f''(x)| \le Q, \end{gathered} \tag{2.14}$$

for all $x \in \mathbb{R}$. Then

$$|f'(x)| \le \sqrt{2PQ} \tag{2.15}$$

for all $x \in \mathbb{R}$.

3. Topological characterization of the space $\mathfrak{S}_{w_1,w_2}$

In this section, we present the following characterization of the space $\mathfrak{S}_{w_1,w_2}$, which imposes no conditions on the derivative.

Theorem 3.1. *Given $w_1, w_2 \in \mathcal{M}_c$, the space $\mathfrak{S}_{w_1,w_2}$ can be described as a set and as a topology by*

$$\mathfrak{S}_{w_1,w_2} = \{\varphi : \mathbb{R}^n \longrightarrow \mathbb{C} : \varphi \text{ is continuous and for all } k = 0,1,2,\ldots, p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty\}, \tag{3.1}$$

where $p_{k,0}(\varphi) = \|e^{kw_1}\varphi\|_\infty$, $\pi_{k,0}(\varphi) = \|e^{kw_2}\widehat{\varphi}\|_\infty$.

Proof. Let us denote by $\mathfrak{B}_{w_1,w_2}$ the space defined in (3.1). The conditions $p_{k,0}(\varphi)$ and $\pi_{k,0}(\varphi)$ imply the smoothness of φ and $\widehat{\varphi}$. The space $\mathfrak{B}_{w_1,w_2}$ becomes a Fréchet space with respect to the family of norms

$$B = \{p_{k,0}, \pi_{k,0}\}_{k=0}^{\infty}. \tag{3.2}$$

From these definitions, it is clear that $\mathfrak{S}_{w_1,w_2} \subseteq \mathfrak{B}_{w_1,w_2}$ and that the inclusion is continuous. To prove the converse, we use the induction on $|\beta|$ and the general idea of Landau's inequality. Fix $\varphi \in \mathfrak{B}_{w_1,w_2} \setminus \{0\}$. We want to show that $\|e^{kw_1(x)}\partial^\beta\varphi\|_\infty$ and $\|e^{kw_2(\xi)}\partial^\beta\widehat{\varphi}\|_\infty$ are finite, for every $k = 0,1,2,\ldots$ and every multi-index β, which is true for all k, when $\beta = 0$. We assume that it is true for all k, when $|\beta| \le m$, and we want to prove it for all k and for $|\beta| = m+1$. We start with $\|e^{kw_1}\partial^\beta\varphi\|_\infty$. Assume that $\beta = (\beta_1 + 1, \beta_2, \ldots, \beta_n)$ with $\beta_1 + \beta_2 + \cdots + \beta_n = m$, $m = 0,1,2,\ldots$. We also indicate $\beta' = (\beta_1, \beta_2, \ldots, \beta_n)$, $\partial^\beta\varphi = \partial_{x_1}\partial^{\beta'}\varphi$, $f_{x'}(x_1) = \partial^{\beta'}\varphi(x_1, x')$ for $x' = (x_2, \ldots, x_n)$ fixed, $\partial^\beta\varphi(x) = f'_{x'}(x_1)$. Moreover, if $h \neq 0$, we have

$$f_{x'}(x_1 + h) = f_{x'}(x_1) + f'_{x'}(x_1)h + \frac{1}{2}f''_{x'}(y)h^2, \tag{3.3}$$

where y is a number between x_1 and $x_1 + h$. Thus,

$$|f'_{x'}(x_1)| \le \frac{|f_{x'}(x_1+h)| + |f_{x'}(x_1)|}{|h|} + \frac{|h|}{2}|f''_{x'}(y)|. \tag{3.4}$$

We can write

$$\begin{gathered} |e^{kw_1(x_1+h,x')}f_{x'}(x_1+h)| \le |e^{kw_1(x_1+h,x')}\partial^{\beta'}\varphi(x_1,x')| \le q_{k,m}(\varphi), \\ |e^{kw_1(x)}f_{x'}(x)| \le q_{k,m}(\varphi). \end{gathered} \tag{3.5}$$

If we take h with the same sign as x_1, we have

$$w_1(x) \le w_1(x_1 + h, x'). \tag{3.6}$$

That is,

$$|f_{x'}(x_1+h)| + |f_{x'}(x_1)| \le C_m p_{k,m}(\varphi)e^{-kw_1(x)}. \tag{3.7}$$

To estimate $f''_{x'}(y) = \partial_{x_1}\partial^{\beta}\varphi(y)$, we write

$$\begin{aligned}\left|\partial_{x_1}\partial^{\beta}\varphi(y)\right| &= \left|\partial_{x_1}\widehat{\widehat{\partial^{\beta}\varphi}}(y)\right| \\ &\le \int_{\mathbb{R}^n}\left|2\pi i\xi_1(2\pi i\xi)^{\beta}\widehat{\varphi}(\xi)\right|d\xi \\ &\le C_{\beta,m}\int_{\mathbb{R}^n}(1+|\xi|)^{m+2}e^{-rw_2(\xi)}e^{rw_2(\xi)}|\widehat{\varphi}(\xi)|d\xi,\end{aligned} \tag{3.8}$$

where $r > (m+n+2)/b$ is an integer and b is the constant in condition 4 of Definition 2.1:

$$\left|\partial_{x_1}\partial^{\beta}\varphi(y)\right| \le C_m\pi_{r,0}(\varphi). \tag{3.9}$$

Thus, we have

$$\left|\partial_{x_1}\partial^{\beta}\varphi(y)\right| \le C_m\pi_{r,0}(\varphi), \tag{3.10}$$

that is,

$$\left|\partial^{\beta}\varphi(x)\right| \le C_m\left[\frac{1}{t}p_{k,m}(\varphi)e^{-kw_1(x)} + t\pi_{r,0}(\varphi)\right] \tag{3.11}$$

for all $t > 0$. As a function of t, the right side of (3.11) has a global minimum at

$$t = \left(p_{k,m}(\varphi)e^{-kw_1(x)}\right)^{1/2}\left(\pi_{r,0}(\varphi)\right)^{-1/2}. \tag{3.12}$$

Thus, we obtain the inequality

$$\left|\partial^{\beta}\varphi(x)\right| \le C_m\left(p_{k,m}(\varphi)\right)^{1/2}\left(\pi_{r,0}(\varphi)\right)^{1/2}e^{(-k/2)w_1(x)}, \tag{3.13}$$

that is,

$$\left|e^{kw_1(x)}\partial^{\beta}\varphi(x)\right| \le C_m\left(p_{2k,m}(\varphi)\right)^{1/2}\left(\pi_{r,0}(\varphi)\right)^{1/2}. \tag{3.14}$$

An argument, similar to the one leading to (3.14), produces

$$\left|e^{kw_2(\xi)}\partial^{\beta}\widehat{\varphi}(\xi)\right| \le C_m\left(\pi_{2k,m}(\varphi)\right)^{1/2}\left(p_{r,0}(\varphi)\right)^{1/2}. \tag{3.15}$$

Combining (3.14), (3.15), the inductive hypothesis implies that $\varphi \in \mathfrak{S}_w$. The open mapping theorem can provide once again the continuity of the inclusion. However, solving the recursive inequalities (3.14), (3.15) , we obtain

$$\begin{aligned}\left|e^{kw(x)}\partial^{\beta}\varphi(x)\right| &\le C_m\left(p_{2^{m+1}k,0}(\varphi)\right)^{2^{-m-1}}\left(\pi_{r,0}(\varphi)\right)^{1-2^{-m-1}}, \\ \left|e^{kw(\xi)}\partial^{\beta}\widehat{\varphi}(\xi)\right| &\le C_m\left(\pi_{2^{m+1}k,0}\circ\mathcal{F}(\varphi)\right)^{2^{-m-1}}\left(p_{r,0}(\varphi)\right)^{1-2^{-m-1}}.\end{aligned} \tag{3.16}$$

This completes the proof of Theorem 3.1. □

When $w_1(x) = w_2(x)$, the characterization of $\mathfrak{S}_{w_1,w_2}$ given by Theorem 3.1 reduces to the characterization of Beurling-Björck space $\mathfrak{S}_{w_1}$ given by Theorem 2.4. In particular, when $w_1(x) = w_2(x) = \ln(1+|x|)$, the characterization of $\mathfrak{S}_{w_1,w_2}$ reduces to the characterization of Schwartz space $\mathfrak{S}$.

Remark 3.2. The Fourier transform is a topological isomorphism between $\mathfrak{S}_{w_1,w_2}$ and $\mathfrak{S}_{w_2,w_1}$. As a consequence, the Fourier transform is also a topological isomorphism between the dual spaces $\mathfrak{S}'_{w_1,w_2}$ and $\mathfrak{S}'_{w_2,w_1}$.

Note that the dual spaces $\mathfrak{S}'_{w_1,w_2}$ and $\mathfrak{S}'_{w_2,w_1}$ are assigned to the weak topologies. For different pairs of admissible functions, the space $\mathfrak{S}_{w_1,w_2}$ has the following embedding properties.

Lemma 3.3. *For every $w_1 < w_1'$ and $w_2 < w_2'$, one has*

$$\mathfrak{S}_{w_1',w_2'} \hookrightarrow \mathfrak{S}_{w_1,w_2}. \tag{3.17}$$

Lemma 3.4. *For $\alpha, \beta > 1$, one has $\mathfrak{S}_{|x|^{1/\alpha},|x|^{1/\beta}} \subseteq S_\alpha^\beta$. As a consequence, $(S_\alpha^\beta)' \subseteq \mathfrak{S}'_{|x|^{1/\alpha},|x|^{1/\beta}}$.*

4. A representation theorem for functionals in the space $\mathfrak{S}'_{w_1,w_2}$

From Theorem 3.1, we can write

$$\mathfrak{S}_{w_1,w_2} = \{\varphi : \mathbb{R}^n \longrightarrow \mathbb{C} : \varphi \text{ is continuous and for all } k = 0,1,2,\ldots, \mathcal{N}_k, (\varphi) < \infty\}, \tag{4.1}$$

where $\mathcal{N}_k(\varphi) = \|e^{kw_1}\varphi\|_\infty + \|e^{kw_2}\widehat{\varphi}\|_\infty$.

Theorem 4.1. *Given $L : \mathfrak{S}_{w_1,w_2} \to \mathbb{C}$, the following statements are equivalent:*

(i) $L \in \mathfrak{S}'_{w_1,w_2}$;

(ii) *there exist two regular complex Borel measures μ_1 and μ_2 of finite total variation and $k \in \{0,1,2,\ldots\}$ such that*

$$L = e^{kw_1}\mu_1 + \mathcal{F}[e^{kw_2}\mu_2], \tag{4.2}$$

in the sense of $\mathfrak{S}'_{w_1,w_2}$.

Proof. (i)⇒(ii). Given $L \in \mathfrak{S}'_{w_1,w_2}$, according to (4.1) there exist k and C so that

$$L(\varphi) \le C\left(\left\|e^{kw_1}\varphi\right\|_\infty + \left\|e^{kw_2}\widehat{\varphi}\right\|_\infty\right) \tag{4.3}$$

for all $\varphi \in \mathfrak{S}_{w_1,w_2}$. Moreover, the map

$$\begin{gathered} \mathfrak{S}_{w_1,w_2} \longrightarrow C_0 \times C_0, \\ \varphi \longrightarrow \left(e^{kw_1}\varphi, e^{kw_2}\widehat{\varphi}\right) \end{gathered} \tag{4.4}$$

is well defined, linear, continuous, and injective. Let $\mathcal{R}$ be the range of this map, on which we define the map

$$l_1(f,g) = L(\varphi), \tag{4.5}$$

where $f = e^{kw_1}\varphi$, $g = e^{kw_2}\widehat{\varphi}$ for a unique $\varphi \in \mathfrak{S}_{w_1,w_2}$. The map $l_1 : \mathcal{R} \to \mathbb{C}$ is linear and continuous. By the Hahn-Banach theorem, there exists a functional L_1 in the topological dual $(\mathcal{C}_0 \times \mathcal{C}_0)'$ of $\mathcal{C}_0 \times \mathcal{C}_0$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1.

Since the spaces $(\mathcal{C}_0 \times \mathcal{C}_0)'$ and $\mathcal{C}_0' \times \mathcal{C}_0'$ are isomorphic as Banach spaces, we can write $L_1(f,g) = L_1(f,0) + L_1(0,g)$. Using Theorem 2.6, there exist regular complex Borel measures μ_1 and μ_2 of finite total variation such that

$$L_1(f,g) = \int_{\mathbb{R}^n} f d\mu_1 + \int_{\mathbb{R}^n} g d\mu_2 \tag{4.6}$$

for all $(f,g) \in \mathcal{C}_0 \times \mathcal{C}_0$. If $(f,g) \in \mathcal{R}$, then we conclude that

$$L(\varphi) = \int_{\mathbb{R}^n} e^{kw_1}\varphi d\mu_1 + \int_{\mathbb{R}^n} e^{kw_2}\widehat{\varphi} d\mu_2 \tag{4.7}$$

for all $\varphi \in \mathfrak{S}_{w_1,w_2}$. In the sense of $\mathfrak{S}'_{w_1,w_2}$,

$$L = e^{kw_1}\mu_1 + \mathcal{F}\left[e^{kw_2}\mu_2\right]. \tag{4.8}$$

(ii)⇒(i). If μ_1 and μ_2 are two regular complex Borel measures satisfying (ii) and $\varphi \in \mathfrak{S}_{w_1,w_2}$, then

$$L(\varphi) = \int_{\mathbb{R}^n} e^{kw_1}\varphi d\mu_1 + \int_{\mathbb{R}^n} e^{kw_2}\widehat{\varphi} d\mu_2. \tag{4.9}$$

This implies that

$$\begin{aligned} |L(\varphi)| &\le \left|\int_{\mathbb{R}^n} e^{kw_1}\varphi d\mu_1\right| + \left|\int_{\mathbb{R}^n} e^{kw_2}\widehat{\varphi} d\mu_2\right| \\ &\le |\mu_1|(\mathbb{R}^n)\left\|e^{kw_1}\varphi\right\|_\infty + |\mu_2|(\mathbb{R}^n)\left\|e^{kw_2}\widehat{\varphi}\right\|_\infty \\ &\le C\left(\left\|e^{kw_1}\varphi\right\|_\infty + \left\|e^{kw_2}\widehat{\varphi}\right\|_\infty\right). \end{aligned} \tag{4.10}$$

It may be noted that μ_1 and μ_2, employed to obtain the above inequality, are of finite total variations. This completes the proof of Theorem 4.1. □

Remark 4.2. When $w_1(x) = w_2(x) = (1+|x|)^k$, (4.2) becomes

$$L = (1+|x|)^k \mu_1 + \mathcal{F}\left[(1+|\xi|)^k \mu_2\right], \tag{4.11}$$

which gives a representation for the tempered distributions.

As consequence of Lemma 3.4, we can view the functionals in $(S_a^b)'$ as functionals in the space $\mathfrak{S}'_{w_1,w_2}$. Then as a result we can characterize $(S_\alpha^\beta)'$ using Theorem 4.1.

Corollary 4.3. *Let $\alpha, \beta > 1$. Then any $L \in (S_\alpha^\beta)'$ can be written as*

$$L = e^{k|x|^{1/\alpha}}\mu_1 + \mathcal{F}\left[e^{k|\xi|^{1/\beta}}\mu_2\right] \tag{4.12}$$

which characterizes the dual space $(S_\alpha^\beta)'$.

References

[1] A. Beurling, *Quasi-Analyticity and General Distributions*, Lectures 4 and 5, Multigraphed Lecture Notes, American Mathematical Society Summer Institute, Stanford, Calif, USA, 1961.

[2] G. Björck, "Linear partial differential operators and generalized distributions," *Arkiv för Matematik*, vol. 6, pp. 351–407, 1966.

[3] L. Hörmander, *Linear Partial Differential Operators*, vol. 116 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, Berlin, Germany, 1963.

[4] M. Andersson and B. Berndtsson, "Almost holomorphic extensions of ultradifferentiable functions," *Journal d'Analyse Mathématique*, vol. 89, pp. 337–365, 2003.

[5] J. Alvarez and H. M. Obiedat, "Characterizations of the Schwartz space $\mathfrak{S}$ and the Beurling-Björck space $\mathfrak{S}_W$," *Cubo*, vol. 6, no. 4, pp. 167–183, 2004.

[6] S.-Y. Chung, D. Kim, and S. Lee, "Characterization for Beurling-Björck space and Schwartz space," *Proceedings of the American Mathematical Society*, vol. 125, no. 11, pp. 3229–3234, 1997.

[7] W. Rudin, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.

[8] J. J. Duistermaat, *Fourier Integral Operators*, Courant Institute of Mathematical Sciences, New York, NY, USA, 1973.

[9] J. Hadamard, "Sur le module maximum d'une function et de ses dérivés," *Comptes Rendus de l'Académie des Sciences*, vol. 41, pp. 68–72, 1914.

[10] E. Landau, "Einige ungleichungen für zweimal diffentiierbare Funktionen," *Proceedings of the London Mathematical Society*, vol. 13, no. 1, pp. 43–49, 1914.