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Research Article

Strong Convergence Theorem of Implicit Iteration Process for Generalized Asymptotically Nonexpansive Mappings in Hilbert Space

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Let C be a nonempty closed and convex subset of a Hilbert space H, let T and $S : C \to C$ be two commutative generalized asymptotically nonexpansive mappings. We introduce an implicit iteration process of S and T defined by $x_n = \alpha_n x_0 + (1-\alpha_n)(2/((n+1)(n+2)))\sum_{k=0}^{n}\sum_{i+j=k} S^i T^j x_n$, and then prove that $\{x_n\}$ converges strongly to a common fixed point of S and T. The results generalize and unify the corresponding results.

1. Introduction

Let H be a Hilbert space, C a nonempty closed and convex subset of H. A mapping S is said to be generalized asymptotically nonexpansive if

$$\left\|S^n x - S^n y\right\| \le k_n \|x-y\| + r_n \tag{1.1}$$

with $k_n \ge 0$, $r_n \ge 0$, $k_n \to 1$, $r_n \to 0$ $(n \to \infty)$, for each $x, y \in C$, $n = 0, 1, 2, \ldots$. If $k_n = 1$ and $r_n = 0$, (1.1) reduces to nonexpansive mapping; if $r_n = 0$, (1.1) reduces to asymptotically nonexpansive mapping; if $k_n = 1$, (1.1) reduces to asymptotically nonexpansive-type mapping. So, a generalized asymptotically nonexpansive mapping is much more general than many other mappings.

Browder [1] introduced the following implicit iteration process of a nonexpansive self-mapping for arbitrary $x_0 \in C$:

$$x_n = \alpha_n x_0 + (1-\alpha_n) T x_n, \tag{1.2}$$

and then proved that $\{x_n\}$ converges strongly to the fixed point of T which is nearest to x_0.

After that, Baillon [2] proved the first nonlinear ergodic theorem: let T be a nonexpansive self-mapping defined on a nonempty bounded closed and convex set. Then for arbitrary $x \in C$, $\{(1/(n+1))\sum_{i=0}^{n} T^i x\}$ converges weakly to a fixed point of T. Those results have been extended by several authors (see, e.g., [3–7]).

Recently, Shimizu and Takahashi [8] studied the following implicit iteration process of asymptotically nonexpansive mappings for arbitrary $x_0 \in C$:

$$x_n = \alpha_n x_0 + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n, \tag{1.3}$$

where $\alpha_n = (b_n - 1)/(b_n - 1 + a)$, $b_n = (1/(n+1))\sum_{i=0}^{n}(1 + |1-k_i| + e^{-i})$, $\{k_n\}$ is the asymptotical coefficient of T. And then proved that $\{x_n\}$ converges strongly to the fixed point of T which is nearest to x_0.

They also studied the following explicit iteration process of two commutative nonexpansive self-mappings in [9]:

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S^i T^j x_n, \tag{1.4}$$

where $\{\alpha_n\} \subseteq [0,1]$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$. And then proved that $\{x_n\}$ converges strongly to the common fixed point of S and T which is nearest to x_0.

In this paper, we prove the strong convergence theorem of implicit iteration process for generalized asymptotically nonexpansive mappings which is defined by

$$x_n = \alpha_n x_0 + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S^i T^j x_n, \tag{1.5}$$

where $\{\alpha_n\} \subseteq [0,1]$, $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}(1-\alpha_n)(2/((n+1)(n+2)))\sum_{k=0}^{n}\sum_{i+j=k} k_i t_j < 1$.

It is well known that a Hilbert space H satisfies Opial's condition [5], that is, if a sequence $\{x_n\}$ converges weakly to an element $y \in H$ and $y \neq z$, then

$$\liminf_{n\to\infty}\|x_n - y\| < \liminf_{n\to\infty}\|x_n - z\|. \tag{1.6}$$

Throughout this paper, S and T are two commutative generalized asymptotically nonexpansive mappings. We denote by $F(S)$ and $F(T)$ the set of fixed points of S and T, respectively; and suppose that $F(S) \bigcap F(T) \neq \varnothing$. S and T satisfy the following conditions:

$$\begin{aligned} \|S^n x - S^n y\| &\le k_n\|x-y\| + r_n, \\ \|T^n x - T^n y\| &\le t_n\|x-y\| + b_n, \end{aligned} \tag{1.7}$$

where $k_n \ge 0$, $r_n \ge 0$, $t_n \ge 0$, $b_n \ge 0$, and $k_n \to 1$, $r_n \to 0$, $t_n \to 1$, $b_n \to 0$ $(n \to \infty)$.

2. Auxiliary lemmas

This section collects some lemmas which will be used to prove the main results in the next section.

Lemma 2.1 (see [9]). *Letting $L_n = ((n+1)(n+2))/2$, there holds the identity in a Hilbert space H:*

$$\|y_n - v\|^2 = \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|x_{i,j} - v\|^2 - \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|x_{i,j} - y_n\|^2 \tag{2.1}$$

for $\{x_{i,j}\}_{i,j=0}^{\infty} \subseteq H$, $y_n = (1/L_n)\sum_{k=0}^{n}\sum_{i+j=k}x_{i,j} \in H$, and $v \in H$.

Lemma 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, S and T two generalized asymptotically nonexpansive mappings of C into itself such that $ST = TS$. For any $x \in C$, put $F_n(x) = (2/((n+1)(n+2)))\sum_{k=0}^{n}\sum_{i+j=k}S^iT^jx$. Then*

$$\begin{aligned}&\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{x\in C}\|F_n(x) - S^lF_n(x)\| = 0,\\ &\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{x\in C}\|F_n(x) - T^lF_n(x)\| = 0.\end{aligned} \tag{2.2}$$

Proof. Put $x_{i,j} = S^iT^jx$, $v = S^lF_n(x)$ and $L_n = ((n+1)(n+2))/2$. It follows from Lemma 2.1 that

$$\begin{aligned}
&\|F_n(x) - S^lF_n(x)\|^2\\
&\quad= \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|S^iT^jx - S^lF_n(x)\|^2 - \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|S^iT^jx - F_n(x)\|^2\\
&\quad= \frac{1}{L_n}\sum_{k=0}^{l-1}\sum_{i+j=k}\|S^iT^jx - S^lF_n(x)\|^2 + \frac{1}{L_n}\sum_{k=l}^{n}\sum_{i+j=k,\, i\le l-1}\|S^iT^jx - S^lF_n(x)\|^2\\
&\qquad+ \frac{1}{L_n}\sum_{k=l}^{n}\sum_{i+j=k,\, i\ge l}\|S^iT^jx - S^lF_n(x)\|^2 - \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|S^iT^jx - F_n(x)\|^2\\
&\quad\le \frac{1}{L_n}\sum_{k=0}^{l-1}\sum_{i+j=k}\|S^iT^jx - S^lF_n(x)\|^2 + \frac{1}{L_n}\sum_{k=l}^{n}\sum_{i+j=k,\, i\le l-1}\|S^iT^jx - S^lF_n(x)\|^2\\
&\qquad+ \frac{1}{L_n}\sum_{k=l}^{n}\sum_{i+j=k,\, i\ge l}\left(k_l\|S^{i-l}T^jx - F_n(x)\| + r_l\right)^2 - \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|S^iT^jx - F_n(x)\|^2\\
&\quad= \frac{1}{L_n}\sum_{k=0}^{l-1}\sum_{i+j=k}\|S^iT^jx - S^lF_n(x)\|^2 + \frac{1}{L_n}\sum_{k=l}^{n}\sum_{i+j=k,\, i\le l-1}\|S^iT^jx - S^lF_n(x)\|^2\\
&\qquad+ \frac{1}{L_n}\sum_{k=0}^{n-l}\sum_{i+j=k}\left(k_l^2\|S^iT^jx - F_n(x)\|^2 + 2k_lr_l\|S^iT^jx - F_n(x)\| + r_l^2\right)\\
&\qquad- \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\|S^iT^jx - F_n(x)\|^2
\end{aligned}$$

$$\leq \frac{1}{L_n}\sum_{k=0}^{l-1}\sum_{i+j=k}\left\|S^iT^jx - S^lF_n(x)\right\|^2 + \frac{1}{L_n}\sum_{k=l}^{n}\sum_{i+j=k,\, i\leq l-1}\left\|S^iT^jx - S^lF_n(x)\right\|^2$$
$$+ \frac{1}{L_n}\sum_{k=0}^{n-l}\sum_{i+j=k}\left(k_l^2-1\right)\left\|S^iT^jx - F_n(x)\right\|^2 + \frac{2k_lr_l}{L_n}\sum_{k=0}^{n-l}\sum_{i+j=k}\left\|S^iT^jx - F_n(x)\right\|$$
$$+ \frac{(n+2-l)(n+1-l)}{(n+2)(n+1)}r_l^2. \tag{2.3}$$

Choose $p \in F(S)\bigcap F(T)$, then there exists a constant $M > 0$ such that

$$\begin{gathered}\left\|S^iT^jx - p\right\| \leq k_i\left\|T^jx - p\right\| + r_i \leq k_it_j\|x-p\| + k_ib_j + r_i \leq \frac{M}{2}, \\ \left\|F_n(x) - p\right\| \leq \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k}\left\|S^iT^jx - p\right\| \leq \frac{M}{2}, \\ \left\|S^lF_n(x) - p\right\| \leq k_l\left\|F_n(x) - p\right\| + r_l \leq \frac{M}{2}\end{gathered} \tag{2.4}$$

for all nonnegative integer i, j, l, and n. Hence, $\|S^iT^jx - S^lF_n(x)\| \leq M$, $\|S^iT^jx - F_n(x)\| \leq M$ for all nonnegative integer i, j, l, and n. So

$$\begin{aligned}&\sup_{x\in C}\left\|F_n(x) - S^lF_n(x)\right\|^2 \\ &\quad\leq \frac{(l+1)l}{(n+2)(n+1)}M^2 + \frac{2(n+1-l)l}{(n+2)(n+1)}M^2 + \frac{\left(k_l^2-1\right)(n+2-l)(n+1-l)}{(n+2)(n+1)}M^2 \\ &\qquad + \frac{2k_lr_l(n+2-l)(n+1-l)}{(n+2)(n+1)}M + \frac{(n+2-l)(n+1-l)}{(n+2)(n+1)}r_l^2 \\ &\quad\longrightarrow 0 \ (n\longrightarrow\infty, l\longrightarrow\infty).\end{aligned} \tag{2.5}$$

Similarly, we can prove that

$$\lim_{l\to\infty}\lim_{n\to\infty}\sup_{x\in C}\left\|F_n(x) - T^lF_n(x)\right\| = 0. \tag{2.6}$$

□

Remark 2.3. Lemma 2.2 extends [8, Lemma 3] and [9, Lemma 1].

Lemma 2.4. *Let S and T be two continuous generalized asymptotically nonexpansive mappings defined on a nonempty bounded closed convex subset C of a Hilbert space H with $ST = TS$. Let $L_n = ((n+1)(n+2))/2$. If $\{x_n\}$ is a sequence in C such that $\{x_n\}$ converges weakly to some $x \in C$ and $\{x_n - (1/L_n)\sum_{k=0}^{n}\sum_{i+j=k}S^iT^jx_n\}$ converges strongly to 0, then $x \in F(S)\bigcap F(T)$.*

Proof. We claim that $\{S^l x\}$ converges strongly to x as $l \to \infty$. If not, there exist a positive number ε_0 and a subsequence $\{l_m\}$ of $\{l\}$ such that $\|S^{l_m}x - x\| \geq \varepsilon_0$ for all m. However, we have

$$\begin{aligned}
\|x_{n_i} - S^{l_m}x\| &\leq \left\| x_{n_i} - \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} \right\| \\
&\quad + \left\| \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} - S^{l_m}\left(\frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i}\right) \right\| \\
&\quad + \left\| S^{l_m}\left(\frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i}\right) - S^{l_m}x \right\| \\
&\leq \left\| x_{n_i} - \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} \right\| \\
&\quad + \left\| \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} - S^{l_m}\left(\frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i}\right) \right\| \\
&\quad + k_{l_m}\left\| \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} - x \right\| + r_{l_m} \\
&\leq \left\| x_{n_i} - \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} \right\| \\
&\quad + \left\| \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} - S^{l_m}\left(\frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i}\right) \right\| \\
&\quad + k_{l_m}\left\| \frac{1}{L_{n_i}}\sum_{k=0}^{n_i}\sum_{i+j=k} S^iT^jx_{n_i} - x_{n_i} \right\| + k_{l_m}\|x_{n_i} - x\| + r_{l_m}.
\end{aligned} \tag{2.7}$$

By Opial's condition, for any $y \in C$ with $y \neq x$, we have

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|. \tag{2.8}$$

Let $r = \liminf_{n\to\infty}\|x_n - x\|$ and choose a positive number ρ such that

$$\rho < \sqrt{r^2 + \frac{\varepsilon_0^2}{4}} - r. \tag{2.9}$$

Then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i} - x\| = r$ and $\|x_{n_i} - x\| < r + \rho/5$ for all i. By definition of $\{k_{l_m}\}$ and $\{r_{l_m}\}$, there exists a positive integer m_0 such that

$$k_{l_m}\|x_{n_i} - x\| < r + \frac{\rho}{5}, \quad r_{l_m} < \frac{\rho}{5} \tag{2.10}$$

for all $m > m_0$. Since

$$\lim_{n\to\infty}\left\| x_n - \frac{1}{L_n}\sum_{k=0}^{n}\sum_{i+j=k} S^iT^jx_n \right\| = 0 \tag{2.11}$$

and $\{k_{l_m}\}$ is bounded , there exists a positive integer i_0 such that

$$\left\| x_{n_i} - \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right\| < \frac{\rho}{5},$$
$$k_{l_m} \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - x_{n_i} \right\| < \frac{\rho}{5} \tag{2.12}$$

for all m and $i > i_0$. As $\{x_{n_i}\} \subset C$ is bounded and by Lemma 2.2, there exist $m_1 > m_0$ and $i_1 > 0$ such that

$$\left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - S^{l_{m_1}} \left(\frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right) \right\| < \frac{\rho}{5} \tag{2.13}$$

for all $i > i_1$. By (2.7), (2.10), (2.12), and (2.13), we have

$$\left\| x_{n_i} - S^{l_{m_1}} x \right\| < \frac{\rho}{5} + \frac{\rho}{5} + \frac{\rho}{5} + r + \frac{\rho}{5} + \frac{\rho}{5} = r + \rho \tag{2.14}$$

for all $i > \max\{i_0, i_1\}$. However,

$$\begin{aligned} \left\| x_{n_i} - \frac{S^{l_{m_1}} x + x}{2} \right\|^2 &= \frac{1}{2} \left\| x_{n_i} - S^{l_{m_1}} x \right\|^2 + \frac{1}{2} \left\| x_{n_i} - x \right\|^2 - \frac{1}{4} \left\| S^{l_{m_1}} x - x \right\|^2 \\ &< \frac{(r+\rho)^2}{2} + \frac{(r+\rho/5)^2}{2} - \frac{\varepsilon_0^2}{4} \\ &< (r+\rho)^2 - \frac{\varepsilon_0^2}{4} < r^2 \end{aligned} \tag{2.15}$$

for all $i > \max\{i_0, i_1\}$. This contradicts with (2.8). So $\{S^l x\}$ converges strongly to x and then $x \in F(S)$. Similarly, we can get $x \in F(T)$. Hence, x is a common fixed point of S and T. □

3. Main results

In this section, we prove our main theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let S, T be two continuous generalized asymptotically nonexpansive mappings of C into itself such that*

(i) $\|S^n x - S^n y\| \le k_n \|x - y\| + r_n$, $k_n \ge 0$, $r_n \ge 0$, $k_n \to 1$, $r_n \to 0$ $(n \to \infty)$,

(ii) $\|T^n x - T^n y\| \le t_n \|x - y\| + b_n$, $t_n \ge 0$, $b_n \ge 0$, $t_n \to 1$, $b_n \to 0$ $(n \to \infty)$,

(iii) $ST = TS$ *and* $F(S) \bigcap F(T) \ne \emptyset$.

For arbitrary $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} S^i T^j x_n, \quad n \ge 0, \tag{3.1}$$

with $\alpha_n \in [0,1]$ *and* $\lim_{n\to\infty}\alpha_n = 0$. *If* $\limsup_{n\to\infty}(1-\alpha_n)(2/((n+1)(n+2)))\sum_{k=0}^{n}\sum_{i+j=k}k_it_j < 1$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the common fixed point* Px_0 *of* S *and* T, *where* P *is the metric projection of* H *onto* $F(S)\bigcap F(T)$.

Proof. Firstly, we show that $\{x_n\}_{n=0}^{\infty}$ is bounded. By $\limsup_{n\to\infty}(1-\alpha_n)(2/((n+1)(n+2)))\sum_{k=0}^{n}\sum_{i+j=k}k_it_j < 1$, there exist $N > 0$ and $0 < a < 1$ such that $(1-\alpha_n)(2/((n+1)(n+2)))\sum_{k=0}^{n}\sum_{i+j=k}k_it_j \le a$ for all $n > N$. Choose $p \in F(S)\bigcap F(T)$, then we have

$$\begin{aligned}\|x_n - p\| &= \left\|\alpha_n(x_0-p) + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}\left(S^iT^jx_n - S^iT^jp\right)\right\| \\ &\le \alpha_n\|x_0-p\| + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}\left(k_it_j\|x_n-p\| + k_ib_j + r_i\right) \\ &= \alpha_n\|x_0-p\| + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}k_it_j\|x_n-p\| \\ &\quad + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}k_ib_j + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}r_i.\end{aligned} \tag{3.2}$$

Hence

$$\begin{aligned}(1-a)\|x_n-p\| \le \alpha_n\|x_0-p\| &+ (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}k_ib_j \\ &+ (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}r_i\end{aligned} \tag{3.3}$$

for all $n > N$. So $\{x_n\}$ is bounded.

Letting $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$, there exists a subsequence $\{x_{m_t}\}$ of $\{x_{n_i}\}$ such that $\{x_{m_t}\}$ converges weakly to some $u \in C$. For $\{x_{m_t}\}$ is bounded, so is $\{(2/((m_t+1)(m_t+2)))\sum_{k=0}^{m_t}\sum_{i+j=k}S^iT^jx_{m_t}\}$. Thus

$$\begin{aligned}x_{m_t} - \frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}S^iT^jx_{m_t} &= \alpha_{m_t}\left(x_0 - \frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}S^iT^jx_{m_t}\right) \\ &\longrightarrow 0(t\longrightarrow\infty).\end{aligned} \tag{3.4}$$

It follows from Lemma 2.4 that $u \in F(S)\bigcap F(T)$. For

$$\alpha_{m_t}Px_0 = Px_0 - (1-\alpha_{m_t})\frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}S^iT^jPx_0, \tag{3.5}$$

we have

$$\begin{aligned}
&\alpha_{m_t}\langle x_0 - Px_0, x_{m_t} - Px_0\rangle \\
&\quad = \left\langle x_{m_t} - Px_0 - (1-\alpha_{m_t})\frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}\left(S^iT^jx_{m_t} - S^iT^jPx_0\right), x_{m_t} - Px_0\right\rangle \\
&\quad = \|x_{m_t} - Px_0\|^2 - (1-\alpha_{m_t})\left\langle \frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}\left(S^iT^jx_{m_t} - S^iT^jPx_0\right), x_{m_t} - Px_0\right\rangle \\
&\quad \geq \|x_{m_t} - Px_0\|^2 - (1-\alpha_{m_t})\frac{2}{(m_t+1)(m_t+2)} \\
&\qquad \times \sum_{k=0}^{m_t}\sum_{i+j=k}\left(k_it_j\|x_{m_t} - Px_0\| + k_ib_j + r_i\right)\|x_{m_t} - Px_0\| \\
&\quad = \left(1 - (1-\alpha_{m_t})\frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}k_it_j\right)\|x_{m_t} - Px_0\|^2 \\
&\qquad - (1-\alpha_{m_t})\frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}\left(k_ib_j + r_i\right)\|x_{m_t} - Px_0\|.
\end{aligned} \tag{3.6}$$

By hypothesis, there exist $M > 0$ and $0 < a < 1$ such that $(1-\alpha_{m_t})(2/((m_t+1)(m_t+2)))\sum_{k=0}^{m_t}\sum_{i+j=k}k_it_j \leq a$ for all $t > M$. So

$$\begin{aligned}
&(1-a)\left\|x_{m_t} - Px_0\right\|^2 \\
&\quad \leq \alpha_{m_t}\langle x_0 - Px_0, x_{m_t} - Px_0\rangle + (1-\alpha_{m_t})\frac{2}{(m_t+1)(m_t+2)}\sum_{k=0}^{m_t}\sum_{i+j=k}\left(k_ib_j + r_i\right)\left\|x_{m_t} - Px_0\right\|
\end{aligned} \tag{3.7}$$

for all $t > M$. We can easily prove that $\lim_{n\to\infty}(1-\alpha_{m_t})(2/((m_t+1)(m_t+2)))\sum_{k=0}^{m_t}\sum_{i+j=k}(k_ib_j + r_i)\|x_{m_t} - Px_0\| = 0$. Since P is metric projection, $\langle x_0 - Px_0, x - Px_0\rangle \leq 0$ for all $x \in F(S)\bigcap F(T)$. Hence

$$\begin{aligned}
\langle x_0 - Px_0, x_{m_t} - Px_0\rangle &= \langle x_0 - Px_0, x_{m_t} - u\rangle + \langle x_0 - Px_0, u - Px_0\rangle \\
&\leq \langle x_0 - Px_0, x_{m_t} - u\rangle.
\end{aligned} \tag{3.8}$$

Since $\{x_{m_t}\}$ converges weakly to u, $\limsup_{t\to\infty}\langle x_0 - Px_0, x_{m_t} - Px_0\rangle \leq 0$. It follows from (3.7) that $\{x_{m_t}\}$ converges strongly to Px_0. Hence, $\{x_n\}$ converges strongly to Px_0. This completes the proof. □

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