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Research Article

Skew Polynomial Extensions over Zip Rings

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In this article, we study the relationship between left (right) zip property of R and skew polynomial extension over R, using the skew versions of Armendariz rings.

1. Introduction

Throughout this paper R denotes an associative ring with identity and $\sigma : R \to R$ an automorphism of R, otherwise unless stated. We denote $R[[x;\sigma]]$ $(R[[x,x^{-1};\sigma]])$ the skew series rings (skew Laurent series rings) whose elements are the series $\sum_{i\geq 0} a_i x^i$ $(\sum_{j=p}^{\infty} b_j x^j)$, where the addition is defined as usual and the multiplication is defined by the rule, $xa = \sigma(a)x$ $(xa = \sigma(a)x$ and $x^{-1}a = \sigma^{-1}(a)x)$, for any $a \in R$. Note that the skew polynomial rings of automorphism type $R[x;\sigma]$ (skew Laurent of polynomial $R[x,x^{-1};\sigma]$) are subrings of $R[[x;\sigma]]$ $(R[[x,x^{-1};\sigma]])$ whose elements are $\sum_{i=0}^{n} a_i x^i$ $(\sum_{j=q}^{m} b_j x^j)$ where the sum and multiplication are defined as before.

Rege and Chhawchharia in [1] introduced the notion of an Armendariz ring. A ring R is called Armendariz if whenever polynomials $\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $0 \leq i \leq n$ and $0 \leq j \leq m$. The name Armendariz ring was chosen because Armendariz [2] had shown that a reduced ring (i.e., ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings have been studied by Rege and Chhawchharia [1], Armendariz [2], Anderson and Camillo [3], and Kim and Lee [4].

Faith in [5] called a ring R right zip if the right annihilator $r_R(X)$ of a subset X of R is zero, then $r_R(Y) = 0$ for a finite subset $Y \subseteq X$; equivalently, for a left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. R is zip if it is right and left zip. The concept of zip rings was initiated by Zelmanowitz [6] and appeared in various papers [5, 7–12], and references therein. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring (although not so-called at that time), but the converse does not hold. Extensions of zip rings were studied by several authors. Beachy and Blair [7] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is zip. The authors in [13] proved that R is a right (left) zip ring if and only if $R[x]$ is a right (left) zip ring when R is an Armendariz ring.

In this paper, we study skew polynomial extensions over zip rings by using skew versions of Armendariz rings and we generalized the results of [13]. Our skew versions of Armendariz rings follow the ideas of [14, Definition]. Moreover, we provide some examples to display some of the phenomenas of Section 2.

2. Skew polynomial extensions over zip rings

Throughout this paper σ is an automorphism of R unless otherwise stated and S will denote one of the following rings: $R[x;\sigma]$, $R[[x;\sigma]]$, $R[x,x^{-1}\sigma]$, and $R[[x,x^{-1};\sigma]]$. A left (right) annihilator of a subset U of R is defined by $l_R(U)=\{a\in R : aU=0\}$ ($r_R(U)=\{a\in R : Ua=0\}$). For a ring R, put $r\mathrm{Ann}_R(2^R)=\{r_R(U) : U\subseteq R\}$ and $l\mathrm{Ann}_R(2^R)=\{l_R(U) : U\subseteq R\}$.

We begin with the following lemma and use it without further mention.

Lemma 2.1. *Let S be one of the rings above and U a subset of R. The following statements hold:*

(i) $l_S(U)=Sl_R(U)$,

(ii) $r_S(U)=r_R(U)S$.

Proof. (i) We only prove for the case $S=R[x;\sigma]$ because the other cases are similar. Let $f(x)=\sum_{i=0}^{n}a_ix^i\in R[x;\sigma]$ such that $f(x)U=0$. Then $\sigma^{-i}(a_i)U=0$ for all $0\le i\le n$ and it follows that $\sigma^{-i}(a_i)\in l_R(U)$ for all $0\le i\le n$. Hence $f(x)=\sum_{i=0}^{n}x^i\sigma^{-i}(a_i)\in R[x;\sigma]l_R(U)$. So $l_{R[x;\sigma]}(U)\subseteq R[x;\sigma]l_R(U)$. We clearly have that $R[x;\sigma]l_R(U)\subseteq l_{R[x;\sigma]}(U)$. Therefore, we have $l_{R[x;\sigma]}(U)=R[x;\sigma]l_R(U)$.

(ii) We only prove for the case $S=R[x;\sigma]$ because the other cases are similar. Let $f(x)=\sum_{i=0}^{n}a_ix^i\in R[x;\sigma]$ such that $Uf(x)=0$. Then $Ua_i=0$ for all $0\le i\le n$ and it follows that $a_i\in r_R(U)$ for all $0\le i\le n$. Hence $f(x)=\sum_{i=0}^{n}a_ix^i\in r_R(U)R[x;\sigma]$. So $r_{R[x;\sigma]}(U)\subseteq r_R(U)R[x;\sigma]$. We clearly have that $r_R(U)R[x;\sigma]\subseteq r_{R[x;\sigma]}(U)$. Therefore, we have $r_{R[x;\sigma]}(U)=r_R(U)R[x;\sigma]$. □

With the above lemma, we have maps $\phi : r\mathrm{Ann}_R(2^R)\to r\mathrm{Ann}_S(2^S)$ defined by $\phi(I)=IS$ for every $I\in r\mathrm{Ann}_R(2^R)$ and

$$\Psi : l\mathrm{Ann}_R(2^R)\longrightarrow l\mathrm{Ann}_S(2^S) \tag{2.1}$$

defined by $\Psi(I)=SI$ for every $I\in l\mathrm{Ann}_R(2^R)$. Moreover, we have maps $\Phi : r\mathrm{Ann}_S(2^S)\to r\mathrm{Ann}_R(2^R)$ defined by $\Phi(J)=J\cap R$ for every $J\in r\mathrm{Ann}_S(2^S)$ and $\Gamma : l\mathrm{Ann}_S(2^S)\to l\mathrm{Ann}_R(2^R)$ defined by $\Gamma(J)=J\cap R$ for every $J\in l\mathrm{Ann}_S(2^S)$. Obviously, ϕ is injective and Φ is surjective. Clearly, ϕ is surjective if and only if Φ is injective, and in this case ϕ and Φ are the inverses of each other. Note that Ψ and Γ satisfy the same relations as above. The first item of the definition below appears in [14, Definition].

Definition 2.2. (i) Suppose that σ is an endomorphism of R. A ring R satisfies SA1′ if for $f(x)=\sum_{i=0}^{n}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j$ in $R[x;\sigma]$, $f(x)g(x)=0$ implies that $a_i\sigma^i(b_j)=0$ for all $0\le i\le n$ and $0\le j\le m$.

(ii) Suppose that σ is an endomorphism of R. A ring R satisfies SA2$'$ if for $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\sigma]]$, $f(x)g(x) = 0$ implies that $a_i\sigma^i(b_j) = 0$ for all $i \geq 0$, $j \geq 0$.

(iii) Suppose that σ is an automorphism of R. A ring R satisfies SA3$'$ if for $f(x) = \sum_{i=s}^{q} a_i x^i$ and $g(x) = \sum_{j=t}^{n} b_j x^j \in R[x, x^{-1};\sigma]$, $f(x)g(x) = 0$ implies that $a_i\sigma^i(b_j) = 0$ for all $s \leq i \leq q$ and $t \leq j \leq n$.

(iv) Suppose that σ is an automorphism of R. A ring R satisfies SA4$'$ if for $f(x) = \sum_{i=s}^{\infty} a_i x^i$ and $g(x) = \sum_{j=t}^{\infty} b_j x^j \in R[[x, x^{-1};\sigma]]$, $f(x)g(x) = 0$ implies that $a_i\sigma^i(b_j) = 0$ for all $i \geq s$ and $j \geq t$.

Note that if R satisfies one of the conditions above, then all subrings S of R such that $\sigma(S) \subseteq S$ satisfies the same property. The following implications are easy to verify: SA4$'$ $\Rightarrow$ SA3$'$ and SA2$'$ $\Rightarrow$ SA1$'$. Following [15, Example 2.1] when $\sigma = id_R$, the last implication is not reversible.

Lemma 2.3. *Let σ be an automorphism of R. Then*

(i) *R satisfies* SA1$'$ *if and only if R satisfies* SA3$'$;

(ii) *R satisfies* SA2$'$ *if and only if R satisfies* SA4$'$.

Proof. Let $f(x), g(x) \in R[x, x^{-1};\sigma]$ such that $f(x)g(x) = 0$, where $f(x) = \sum_{i=-p}^{q} a_i x^i$ and $g(x) = \sum_{j=-t}^{s} b_j x^j$. We clearly have $x^p f(x) \in R[x;\sigma]$ and $g(x)x^t \in R[x;\sigma]$, then $x^p f(x)g(x)x^t = 0$. By assumption, $\sigma^p(a_i)\sigma^{i+p}(b_j) = 0$ for all $-p \leq i \leq q$ and $-t \leq j \leq s$. Hence $a_i\sigma^i(b_j) = 0$ for all $-p \leq i \leq q$ and $-t \leq j \leq s$. Since $R[x;\sigma] \subseteq R[x, x^{-1};\sigma]$, the converse follows.

The proof of the other statement is similar. □

The following definition appears in [16, Definition 2.1].

Definition 2.4. Let R be a ring and σ an endomorphism of R. Then R is said σ-compatible like right R-module, if $ar = 0$ if and only if $a\sigma(r) = 0$ for any $a \in R$ and $r \in R$.

Let R be a ring and α an endomorphism of R. Following [17], the endomorphism α is said α-rigid if $r\alpha(r) = 0$, then $r = 0$. A ring R is said a rigid ring if it exists a rigid endomorphism α of R.

Proposition 2.5. *Let σ be an endomorphism of R. If R is a reduced ring and σ-compatible like right R-module, then R is a σ-rigid ring and hence satisfies* SA1$'$ *and* SA2$'$.

Proof. We only prove the case of SA2$'$ because the other are similar. We claim that $R[[x;\sigma]]$ is a reduced ring. In fact, let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ such that $(f(x))^2 = 0$. We have that $a_0^2 = 0$. Since R is reduced, then $a_0 = 0$. Next, we have $a_1\sigma(a_1) = 0$, since R is σ-compatible and reduced, then $a_1 = 0$. By induction, we get $f(x) = 0$. Hence $R[[x;\sigma]]$ is reduced. Using the same ideas of [14, Proposition 3], we have that R is σ-rigid and using similar ideas of [14, Corollary 4], we obtain that R satisfies SA2$'$. □

Without the assumption that R is σ-compatible, Proposition 2.5 is not true. In fact, let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\sigma : R \rightarrow R$, defined by $\sigma((a,b)) = (b,a)$. By [14, Example 2], R does not satisfy SA2$'$ because R does not satisfy SA1$'$. Observe that $(1,0)(0,1) = (0,0)$ but $(1,0)\sigma(0,1) \neq (0,0)$ and so R is not σ-compatible. We have the following natural questions.

Questions

(i) Let σ be an endomorphism of R. Suppose that R satisfies SA2′. Is R σ-compatible like right R-module?

(ii) Let σ be an endomorphism of R. Suppose that R is σ-compatible like right R-module. Does R satisfy SA2′?

The question (i) is false. Let R_0 be any domain and $R = R_0[x]$. Let $\sigma : R \to R$ be defined by $\sigma(t) = 0$ and $\sigma|_{R_0} = id_{R_0}$. By [16, Example 4.1], R is not σ-compatible and using the similar ideas of the proof of [14, Proposition 10], we have that R satisfies SA2′ and consequently R satisfies SA1′.

The question (ii) is false. Let $R = K[x, y]/(x^2, y^2)$, where K is a field of characteristic 2, and consider $T = M_2(R)$. In this case, take $\sigma = id_T$. By [18, Example 3.6], S does not satisfy SA2′ because T does not satisfy SA1′. Moreover, T is σ-compatible like right T-module.

In [19] the authors introduced the following version of skew Armendariz rings.

(i) Suppose that σ is an endomorphism of R. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\sigma]$ such that $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all $0 \le i \le n$ and $0 \le j \le m$.

(ii) Suppose that σ is an endomorphism of R. Let $f(x) = \sum_{i\ge 0} a_i x^i$, $g(x) = \sum_{j\ge 0} b_j x^j \in R[[x;\sigma]]$ such that $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all $i \ge 0$ and $j \ge 0$.

Note that the item (i) above in [20, Definition 1.1] the authors called it by σ-Armendariz, the item (ii) above is similar with [20, Definition 1.1] and we call it here by σ-power Armendariz.

In the next proposition, we give a relationship between the definition above and the skew versions of Armendariz rings used in this paper. Using [21, Lemma 2.1] and [20, Theorem 1.8], the proof of next proposition is easy to verify.

Proposition 2.6. *Let σ be an endomorphism of R and suppose that R is σ-compatible like right R-module. Then*

(i) *R satisfies* SA1′ *if and only if R is σ-Armendariz;*

(ii) *R satisfies* SA2′ *if and only if R is σ-power Armendariz.*

The proposition above without the compatibility assumption is not true according to [20, Example 1.9] and the authors in [22, Theorem 2.2] obtained an approach of the result above without the compatibility assumption.

The following proposition is a generalization of [18, Proposition 3.4] and partially generalizes [15, Proposition 2.6].

Lemma 2.7. *Let S be any of the rings $R[x;\sigma]$ and $R[[x;\sigma]]$. The following conditions are equivalent:*

(i) *R satisfies* SA2′ (SA1′)*;*

(ii) *$\phi : r\mathrm{Ann}_R(2^R) \to r\mathrm{Ann}_S(2^S)$ defined by $\phi(J) = JS$ is bijective;*

(iii) *$\Psi : l\mathrm{Ann}_R(2^R) \to l\mathrm{Ann}_S(2^S)$ defined by $\Psi(J) = SJ$ is bijective.*

Proof. We only prove the proposition in the case of SA2′ because the equivalence of (i) and (ii) when R satisfies SA1′ was proved in [23, Proposition 3.2]. The equivalence between (i) and (iii) when R satisfies SA1′ has similar proof.

(i)→(ii). It is only necessary to show that ϕ is surjective. For an element $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\sigma]]$, define $C_{f(x)} = \{\sigma^{-i}(a_i),\ i \ge 0\}$, and for a subset T of $R[[x;\sigma]]$, we denote the set

$\cup_{f(x)\in T} C_{f(x)}$ by C_T. We show that $r_{R[[x;\sigma]]}(f(x)) = r_{R[[x;\sigma]]}(C_{f(x)})$. In fact, given $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $r_{R[[x;\sigma]]}(f(x))$, we have $f(x)g(x) = 0$. Since R satisfies SA2′, then $a_i\sigma^i(b_j) = 0$ for all $i \geq 0$ and $j \geq 0$. In particular, $\sigma^{-i}(a_i)b_j = 0$ for all $i \geq 0$ and $j \geq 0$. Hence $g(x) \in r_{R[[x;\sigma]]}(C_{f(x)})$.

On the other hand, let $h(x) = \sum_{k=0}^{\infty} c_k x^k$ be an element in $R[[x;\sigma]]$ such that $C_{f(x)}h(x) = 0$. It is clear that $a_i\sigma^i(c_k) = 0$ for all $i \geq 0$ and $k \geq 0$. So $f(x)h(x) = (0)$. Since R satisfies SA2′ then $r_{R[[x;\sigma]]}(T) = r_{R[[x;\sigma]]}(\cup_{f(x)\in T} C_{f(x)})$. Thus

$$\begin{aligned} r_{R[[x;\sigma]]}(T) &= \bigcap_{f(x)\in T} r_{R[[x;\sigma]]}(f(x)) = \bigcap_{f(x)\in T} r_{R[[x;\sigma]]}(C_{f(x)}) \\ &= \left(\bigcap_{f(x)\in T} r_R(C_{f(x)})\right) R[[x;\sigma]] = r_R(C_T)R[[x;\sigma]]. \end{aligned} \tag{2.2}$$

Therefore, ϕ is surjective.

(ii)→(i). Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be elements in $R[[x;\sigma]]$ such that $f(x)g(x) = 0$. By assumption, $r_{R[[x,\sigma]]}(f(x)) = BR[[x;\sigma]]$, for some right ideal B of R. Hence $g(x) \in BR[[x;\sigma]]$ and we have that $b_j \in B \subset r_{R[[x;\sigma]]}(f(x))$ for all $j \geq 0$. So $a_i\sigma^i(b_j) = 0$ for all $i \geq 0$ and $j \geq 0$.

(iii)→(i). Let $f(x) = \sum_{i\geq 0} a_i x^i$ and $g(x) = \sum_{j\geq 0} b_j x^j$ be elements in $R[[x;\sigma]]$ such that $f(x)g(x) = 0$. By assumption, $l_{R[[x;\sigma]]}(g(x)) = R[[x;\sigma]]B$ for some left ideal B of R. We can write $f(x) = \sum_{i\geq 0} x^i\sigma^{-i}(a_i) \in R[[x;\sigma]]B$. By the equality of the polynomials with the coefficients on the right side, we have that $\sigma^{-i}(a_i) \in B \subseteq l_{R[[x;\sigma]]}(g(x))$ for all $i \geq 0$. So $a_i\sigma^i(b_j) = 0$ for all $i \geq 0$ and $j \geq 0$.

(i)→(iii). It is only necessary to show that Ψ is surjective. Let $f(x) = \sum_{i\geq 0} a_i x^i \in R[[x;\sigma]]$. Define $C_{f(x)} = \{a_i,\ i \geq 0\}$, and for a subset T of $R[[x;\sigma]]$, we denote the set $\cup_{f(x)\in T} C_{f(x)}$ by C_T. We show that

$$l_{R[[x;\sigma]]}(f(x)) = l_{R[[x;\sigma]]}(C_{f(x)}). \tag{2.3}$$

In fact, given $g(x) = \sum_{j\geq 0} b_j x^j \in l_{R[[x;\sigma]]}(f(x))$, we have $g(x)f(x) = 0$. Since R satisfies SA2′, then $b_j\sigma^j(a_i) = 0$ for all $i \geq 0$ and $j \geq 0$. Hence $g(x) = \sum_{j\geq 0} x^j\sigma^{-j}(b_j) \in l_{R[[x;\sigma]]}(C_{f(x)})$.

On the other hand, let $g(x) \in R[[x;\sigma]]$ such that $g(x)C_{f(x)} = 0$. Thus $g(x)a_i = 0$ for all $i \geq 0$. So $g(x)\sum_{i\geq 0} a_i x^i = g(x)f(x) = 0$, and we have that $g(x) \in l_{R[[x;\sigma]]}(f(x))$.

We easily have that for each subset T of $R[[x;\sigma]]$,

$$l_{R[[x;\sigma]]}(T) = l_{R[[x;\sigma]]}\left(\bigcup_{f(x)\in T} C_{f(x)}\right). \tag{2.4}$$

We claim that $l_{R[[x;\sigma]]}(C_{f(x)}) = R[[x;\sigma]]l_R(C_{f(x)})$. In fact, let $g(x) = \sum_{j\geq 0} b_j x^j$ such that $g(x)C_{f(x)} = 0$. Then we have that $0 = g(x)a_i = \sum_{j\geq 0} b_j x^j a_i = \sum_{j\geq 0} x^j\sigma^{-j}(b_j)a_i$. Thus $\sigma^{-j}(b_j) \in l_R(C_{f(x)})$, and it follows that

$$\sum_{j\geq 0} x^j\sigma^{-j}(b_j) \in R[[x;\sigma]]l_R(C_{f(x)}). \tag{2.5}$$

The other inclusion is trivial. So

$$\begin{aligned} l_{R[[x;\sigma]]}(T) &= \bigcap_{f(x)\in T} l_{R[[x;\sigma]]}(C_{f(x)}) = \bigcap_{f(x)\in T} l_{R[[x;\sigma]]}(C_{f(x)}) \\ &= R[[x;\sigma]]\left(\bigcap_{f(x)\in T} l_R(C_{f(x)})\right) = R[[x;\sigma]]l_R(C_T). \end{aligned} \tag{2.6}$$

Therefore, Ψ is surjective. □

Now we are able to prove the main results of this paper.

Theorem 2.8. *Let σ be an automorphism of R.*

(i) *Suppose that R satisfies* SA1′. *The following conditions are equivalent:*

(a) *R is a right (left) zip ring;*
(b) *$R[x;\sigma]$ is a right (left) zip ring;*
(c) *$R[x,x^{-1},\sigma]$ is a right (left) zip ring.*

(ii) *Suppose that R satisfies* SA2′. *The following conditions are equivalent:*

(a) *R is right (left) zip ring;*
(b) *$R[[x;\sigma]]$ is right (left) zip ring;*
(c) *$R[[x,x^{-1};\sigma]]$ is right (left) zip ring.*

Proof. (i) We will show the right case because the left case is similar.

Suppose that $R[x;\sigma]$ is right zip. Let X be a subset of R such that $r_R(X)=0$, and $f(x)=\sum_{i=0}^{n}a_ix^i \in R[x;\sigma]$ such that $Xf(x)=0$. Thus $a_i\in r_R(X)=0$ and it follows that $f(x)=0$. By assumption, there exists $X_1=\{x_0,\dots,x_n\}$ such that $r_{R[x;\sigma]}(X_1)=0$. Hence $r_R(X_1)=r_{R[x;\sigma]}(X_1)\cap R=(0)$.

Conversely, let $Y\subseteq R[x;\sigma]$ such that $r_{R[x;\sigma]}(Y)=0$. By Lemma 2.7, $r_{R[x;\sigma]}(Y)=r_R(T)R[x;\sigma]$, where $T=C_Y=\cup_{f(x)\in Y}C_{f(x)}$ such that $C_{f(x)}=\{\sigma^{-i}(a_i):0\le i\le n\}$ with $f(x)=\sum_{i=0}^{n}a_ix^i\in Y$. We have that $r_R(T)=0$ and, by assumption, there exists $T_1=\{\sigma^{-i_1}(a_{i_1}),\dots,\sigma^{-i_n}(a_{i_n})\}$ such that $r_R(T_1)=0$. For each $\sigma^{-i_j}(a_{i_j})\in T_1$, there exists $g_{a_{i_j}}(x)\in Y$ such that some of the coefficients of $g_{a_{i_j}}(x)$ are a_{i_j} for each $1\le j\le n$. Let Y_0 be a minimal subset of Y such that $g_{a_{i_j}}(x)\in Y_0$ for each $1\le j\le n$. Then Y_0 is nonempty finite subset of Y. Set $T_0=\cup_{f(x)\in Y_0}(C_{f(x)})$ and we have that $T_1\subseteq T_0$. Hence $r_R(T_0)\subseteq r_R(T_1)=0$. By Lemma 2.7, $r_{R[x;\sigma]}(Y_0)=r_R(T_0)R[x;\sigma]$ and it follows that $r_{R[x;\sigma]}(Y_0)=0$.

The proofs of (a)⇔(c) and of item (ii) follow similarly. □

Let σ be an endomorphism of R and $\delta: R\to R$ an additive map of R. The application δ is said to be a σ-derivation if $\delta(ab)=\delta(a)b+\sigma(a)\delta(b)$. The Ore extension $R[x;\sigma,\delta]$ is the set of polynomials $\sum_{i=0}^{n}a_ix^i$ with the usual sum, and the multiplication rule is $xa=\sigma(a)x+\delta(a)$.

Following [16], R is said to be (σ,δ)-compatible, where σ is an endomorphism of R and δ is a σ-derivation of R if $ab=0\Leftrightarrow a\sigma(b)=0$ and $ab=0$ implies that $a\delta(b)=0$.

In the next result we obtain a necessary and sufficient condition for $R[x;\sigma,\delta]$ to be left zip, when σ is an endomorphism of R using the skew version of Armendariz rings of [19].

Theorem 2.9. *Let σ be an endomorphism of R and δ a σ-derivation of R. Suppose that if $f(x)g(x)=0$ for $f(x)=\sum_{i=0}^{n}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j\in R[x;\sigma,\delta]$, then $a_ib_j=0$ for all $0\le i\le n$ and $0\le j\le m$. Then R is left zip if and only if $R[x;\sigma,\delta]$ is left zip.*

Proof. Let X be any subset of $R[x;\sigma,\delta]$ and $C_X=\cup_{f(x)\in X}C_{f(x)}$, where $C_{f(x)}=\{a_i,\ 0\le i\le n\}$ with $f(x)=\sum_{i=0}^{n}a_ix^i$. Suppose that $l_{R[x;\sigma,\delta]}(X)=0$. We clearly have $l_R(C_X)=0$. By assumption, there exists $\{b_0,\dots,b_t\}\subseteq C_X$ such that $l_R(Y)=0$. Let $f_{b_i}(x)\in X$ be an element of X with some of its coefficients are equal to b_i for all $1\le i\le t$. Take X_0 be a minimal subset of X with this property. We clearly have that X_0 is a finite set. We claim that $l_{R[x;\sigma,\delta]}(X_0)=0$. In fact, we easily have $l_R(C_{X_0}) = 0$, where $C_{X_0} = \cup_{f(x)\in X_0} C_{f(x)}$ with $C_{f(x)}$ being defined as before. Next, let $g(x) = \sum_{j=0}^{m} b_j x^j$ such that $g(x)X_0 = 0$. Hence for any $f(x) = \sum_{i=0}^{n} a_i x^i \in X_0$, $g(x)f(x) = 0$, and we have, by assumption, $b_j a_i = 0$ for all $0 \leq j \leq m$ and $0 \leq i \leq n$. Thus $b_j C_{X_0} = 0$ for all $0 \leq j \leq m$ and it follows that $g(x) = 0$. So $l_{R[x;\sigma,\delta]}(X_0) = 0$.

Using the methods of Theorem 2.8, the converse follows. □

Remark 2.10. Let R be a ring and σ an endomorphism of R. Suppose that R is σ-power Armendariz and left zip. Using similar methods of [20, Theorem 1.8], R satisfies SA2′ and with similar ideas of Theorem 2.9, we have that R is a left zip ring if and only if $R[[x;\sigma]]$ is a left zip ring.

3. Examples

In this section, we present some examples of rings that satisfy SA1′ and SA2′, and they are zip rings. Moreover, an example of a σ-rigid ring that is a zip ring is given.

Example 3.1. Let F be any field and $\sigma : F \to F$ any automorphism of F. Following [14, page 113], we consider the ring $T(F,F)$ with automorphism $\overline{\sigma}(a,b) = (\sigma(a),\sigma(b))$ and we denote it by σ. Note that

$$T(F,F) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a,b \in F \right\}. \tag{3.1}$$

By [14, Proposition 15], $T(F,F)$ satisfies SA1′, and using similar methods, we can prove that $T(F,F)$ satisfies SA2′. We claim that $T(F,F)$ is a zip ring. In fact, the unique one-sided ideals of $T(F,F)$ are $\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$,

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F \right\}, \tag{3.2}$$

and $T(F,F)$. Note that $r_{T(F,F)}(I) \neq \{0\}$ and $l_{T(F,F)}(I) \neq 0$. So we easily have that $T(F,F)$ is a zip ring.

Example 3.2. Let F be any field and σ a monomorphism of F, and let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a,b,c \in F \right\} \tag{3.3}$$

with usual addition and multiplication of matrix. Note that the monomorphism σ is naturally extended to R, and R has the following one-sided ideals:

$$I_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} : a \in F \right\}, \qquad I_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \in F \right\}, \tag{3.4}$$

R and the zero ideal. We easily have $r_R(I_2) \neq 0$, $l_R(I_2) \neq 0$, $r_R(I_1) \neq 0$, and $l_R(I_1) \neq 0$. Now we clearly have that R is a zip ring and by [14, Proposition 17], R satisfies SA1′, and with similar methods of [14, Proposition 17], we can prove that R satisfies SA2′.

Example 3.3. Let D be any domain with identity, $R = D[x]$, σ an endomorphism of R defined by $\sigma(f(x)) = f(0)$. Since R is a domain, then R is right and left zip. Moreover, using similar methods of [14, Example 5], we have that R satisfies SA1′ and SA2′.

Example 3.4. Let D and D_1 be any domains, σ an monomorphism of D, and τ an monomorphism of D_1. Set $R = D \times D_1$ with usual addition and multiplication, and we define an endomorphism γ of R by $\gamma(a,b) = (\sigma(a), \tau(b))$. We easily have that γ is a monomorphism of R. Since D is σ-rigid and D_1 is τ-rigid, we easily obtain that R is γ-rigid. We claim that R is left and right zip. In fact, let I be any left ideal of R. It is well known that $I = A \times B$, where A is a left ideal of D and B is a left ideal of D_1. Suppose that $r_R(I) = 0$. Then $A \neq 0$ and $B \neq 0$. It is not difficult to show that $r_D(A) = 0$ and $r_{D_1}(B) = 0$. Since D and D_1 are left zip, then there exists a left finitely generated ideal L of D contained in A such that $r_D(L) = 0$ and a left finitely generated ideal L_1 of D_1 contained in B such that $r_{D_1}(L_1) = 0$. Thus $r_R(L \times L_1) = 0$ and $L \times L_1$ is a left finitely generated ideal of R contained in $A \times B$. Hence R is left zip. Using similar methods, we have that R is right zip.

Example 3.5. Let F be a field, σ an automorphism of F,

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c \in F \right\}, \tag{3.5}$$

and D a domain with automorphism τ. Set $T = R \times D$ and we define an endomorphism γ of T by $\gamma(a,b) = (\sigma(a), \tau(t))$. It is clear that γ is an automorphism of T and it is not difficult to show that T satisfies SA1′ and SA2′ because R and D satisfy SA1′ by [14, Proposition 17] and [14, Proposition 10], respectively, and using similar methods of [14, Proposition 17] and [14, Proposition 10], R and D satisfy SA2′, respectively.

Using similar methods of Example 3.4, we have that T is right and left zip and note that T is not γ-rigid, since T is not a reduced ring.

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