Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 381650, 8 pages
doi:10.1155/2008/381650

Research Article

On a Generalization of Hilbert-Type Integral Inequality

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Received 1 March 2008; Accepted 9 May 2008

Recommended by Feng Qi

By introducing some parameters, we establish generalizations of the Hilbert-type inequality. As applications, the reverse and its equivalent form are considered.

1. Introduction

Considerable attention has been given to Hilbert inequalities and Hilbert-type inequalities by several authors including Gao and Yang [1], Yang [2–4], Jichang and Debnath [5], Pachpatte [6], Zhao [7], Brnetić and Pečarić [8]. In 2007, Li et al. [9] gave a new inequality similar to Hilbert inequality for integrals:

If $f(x),\ g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty,\ 0 < \int_0^\infty g^2(x)dx < \infty$, then one has

$$\iint_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y)dx\,dy < 4\left[\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right]^{1/2}, \tag{1.1}$$

where constant factor 4 is the best possible.

An equivalent inequality is

$$\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)dx\right]^2 dy < 16\int_0^\infty f^2(x)dx. \tag{1.2}$$

In this paper, by introducing some parameters we generalize (1.1), (1.2), and we obtain the reverse form for each of them. The equivalent forms are also considered.

2. Main results

Lemma 2.1. *Suppose that $\lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, define weight functions $w_\lambda(x,q)$, $w_\lambda(y,p)$, respectively as*

$$w_\lambda(x,q) = \int_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\left(\frac{x}{y}\right)^{\lambda/q}\frac{1}{y^{1-\lambda}}dy,$$
$$w_\lambda(y,p) = \int_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\left(\frac{y}{x}\right)^{\lambda/p}\frac{1}{x^{1-\lambda}}dx. \tag{2.1}$$

One has $w_\lambda(x,q) = w_\lambda(y,p) = (1/2\lambda^2)(p^2+q^2)$.

Proof. Letting $t = y^\lambda/x^\lambda$, we have

$$w_\lambda(x,q) = \frac{1}{\lambda^2}\int_0^\infty \frac{|\ln t|}{1+t+|1-t|}\cdot t^{-1/q}dt = \frac{1}{\lambda^2}\left[\int_0^1 \frac{-\ln t}{2}\cdot t^{-1/q}dt + \int_1^\infty \frac{\ln t}{2t}\cdot t^{-1/q}dt\right] = \frac{1}{2\lambda^2}(p^2+q^2). \tag{2.2}$$

By symmetry we have

$$w_\lambda(y,p) = \frac{1}{2\lambda^2}(p^2+q^2). \tag{2.3}$$

The lemma is proved. □

Lemma 2.2. *Let $p > 1$ (or $0 < p < 1$), $1/p + 1/q = 1$, $\lambda > 0$, and $0 < \varepsilon < q\lambda/2p$, setting*

$$J(\varepsilon) = \iint_1^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p}y^{-[1+\varepsilon+(q-1)(1-\lambda)]/q}dx\,dy. \tag{2.4}$$

Then for $\varepsilon \to 0^+$, one gets

$$\frac{p^2+q^2}{2\lambda^2\varepsilon}[1+o(1)] - O(1) < J(\varepsilon) < \frac{p^2+q^2}{2\lambda^2\varepsilon}[1+o(1)]. \tag{2.5}$$

Proof. Letting $t = y^\lambda/x^\lambda$, we have

$$\begin{aligned} J(\varepsilon) &= \frac{1}{\lambda^2}\int_1^\infty \frac{1}{x^{1+\varepsilon}}\left[\int_{1/x^\lambda}^\infty \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt\right]dx \\ &= \frac{1}{\lambda^2}\int_1^\infty \frac{1}{x^{1+\varepsilon}}\left[\int_0^\infty \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt\right]dx \\ &\quad - \frac{1}{\lambda^2}\int_1^\infty \frac{1}{x^{1+\varepsilon}}\left[\int_0^{1/x^\lambda} \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt\right]dx \\ &= \frac{1}{\lambda^2\varepsilon}\int_1^\infty \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt - \frac{1}{\lambda^2}\int_1^\infty \frac{1}{x^{1+\varepsilon}}\left[\int_0^{1/x^\lambda} \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt\right]dx \\ &:= I_1 - I_2. \end{aligned} \tag{2.6}$$

Now, observe that

$$\int_0^\infty \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt = \frac{p^2+q^2}{2}\left[1+o(1)\right], \quad (\varepsilon \longrightarrow 0^+), 0 < \int_0^{1/x^\lambda} \frac{|\ln t|}{1+t+|1-t|}\cdot t^{1/p-1-\varepsilon/\lambda q}dt$$
$$= \int_0^{1/x^\lambda} \frac{-\ln t}{2}\cdot t^{1/p-1-\varepsilon/\lambda q}dt$$
$$< \int_0^{1/x^\lambda} \frac{-\ln t}{2}\cdot t^{1/2p-1}dt$$
$$= p\lambda x^{-\lambda/2p}\ln x + 2p^2 x^{-\lambda/2p}, \tag{2.7}$$

then

$$I_1 = \frac{p^2+q^2}{2\lambda^2\varepsilon}\left[1+o(1)\right],$$
$$0 < I_2 < \frac{1}{\lambda^2}\left[p\lambda\int_1^\infty x^{-1-\lambda/2p}\ln x\,dx + 2p^2\int_1^\infty x^{-1-\lambda/2p}dx\right] = \frac{8p^3}{\lambda^3}. \tag{2.8}$$

We get

$$J(\varepsilon) > \frac{p^2+q^2}{2\lambda^2\varepsilon}\left[1+o(1)\right] - O(1). \tag{2.9}$$

On the other hand,

$$J(\varepsilon) = \iint_1^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + \left|x^\lambda - y^\lambda\right|} x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p} y^{-[1+\varepsilon+(q-1)(1-\lambda)]/q} dx\,dy$$
$$< \int_1^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + \left|x^\lambda - y^\lambda\right|} x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p} y^{-[1+\varepsilon+(q-1)(1-\lambda)]/q} dy\right] dx \tag{2.10}$$
$$= \frac{p^2+q^2}{2\lambda^2\varepsilon}\left[1+o(1)\right].$$

Hence, (2.5) is valid. The lemma is proved. □

Theorem 2.3. *Let $p > 1$, $1/p + 1/q = 1$, $\lambda > 0$, $f(x) \ge 0$, $g(y) \ge 0$. If $0 < \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx < \infty$, $0 < \int_0^\infty y^{(q-1)(1-\lambda)} g^q(y)dy < \infty$, then one has*

$$\iint_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + \left|x^\lambda - y^\lambda\right|} f(x)g(y)dx\,dy$$
$$< \frac{p^2+q^2}{2\lambda^2}\left(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{(q-1)(1-\lambda)} g^q(y)dy\right)^{1/q}, \tag{2.11}$$

where constant factor $(p^2+q^2)/2\lambda^2$ *is the best possible. In particular, for* $\lambda = 1$*, inequality* (2.11) *reduces to*

$$\iint_0^\infty \frac{|\ln x - \ln y|}{x+y+|x-y|} f(x)g(y)dx\,dy < \frac{p^2+q^2}{2}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}. \tag{2.12}$$

Proof. Applying Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
&\iint_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|} f(x)g(y)dx\,dy \\
&= \iint_0^\infty \left[\left(\frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\right)^{1/p}\left(\frac{x}{y}\right)^{\lambda/pq}\frac{x^{(1-\lambda)/q}}{y^{(1-\lambda)/p}} f(x)\right] \\
&\quad\times\left[\left(\frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\right)^{1/q}\left(\frac{y}{x}\right)^{\lambda/pq}\frac{y^{(1-\lambda)/p}}{x^{(1-\lambda)/q}} f(y)\right]dx\,dy \\
&\le \left\{\iint_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\left(\frac{x}{y}\right)^{\lambda/q}\frac{x^{(p/q)(1-\lambda)}}{y^{1-\lambda}} f^p(x)dx\,dy\right\}^{1/p} \\
&\quad\times\left\{\iint_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\left(\frac{y}{x}\right)^{\lambda/p}\frac{y^{(q/p)(1-\lambda)}}{x^{1-\lambda}} g^q(y)dx\,dy\right\}^{1/q} \\
&\le \frac{p^2+q^2}{2\lambda^2}\left(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{(q-1)(1-\lambda)} g^q(y)dy\right)^{1/q}.
\end{aligned} \tag{2.13}$$

If (2.13) takes the form of equality, then there exist constants A and B, such that they are not all zero, and

$$A\frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\left(\frac{x}{y}\right)^{\lambda/q}\frac{x^{(p/q)(1-\lambda)}}{y^{1-\lambda}} f^p(x) = B\frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|}\left(\frac{y}{x}\right)^{\lambda/p}\frac{y^{(q/p)(1-\lambda)}}{x^{1-\lambda}} g^q(y), \tag{2.14}$$

a.e. (x,y) in $(0,\infty)\times(0,\infty)$. It follows that there exists a constant C, such that

$$Ax\cdot x^{(p-1)(1-\lambda)} f^p(x) = By\cdot y^{(q-1)(1-\lambda)} g^q(y) = C, \quad \text{a.e. } (x,y) \text{ in } (0,\infty)\times(0,\infty). \tag{2.15}$$

Without lose of generality, suppose $A \neq 0$, then we have

$$x^{(p-1)(1-\lambda)} f^p(x) = \frac{1}{Ax}, \tag{2.16}$$

which contradicts the fact that $0 < \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx < \infty$, hence (2.13) takes the form of strict inequality, so we obtain (2.11).

Assume that the constant factor $(p^2+q^2)/2\lambda^2$ in (2.11) is not the best possible, then there exists a positive number k (with $k<(p^2+q^2)/2\lambda^2$) such that (2.11) is still valid if one replaces $(p^2+q^2)/2\lambda^2$ by k. In particular, for $0<\varepsilon<q\lambda/2p$, setting $\widetilde{f}$ and $\widetilde{g}$ as $\widetilde{f}(x)=\widetilde{g}(x)=0$ for $x\in(0,1)$, $\widetilde{f}(x)=x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p}$, $\widetilde{g}(x)=x^{-[1+\varepsilon+(q-1)(1-\lambda)]/q}$ for $x\in[1,\infty)$, then we have

$$k\left(\int_0^\infty x^{(p-1)(1-\lambda)}\widetilde{f}^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{(q-1)(1-\lambda)}\widetilde{g}^q(y)dy\right)^{1/q}=\frac{k}{\varepsilon}. \tag{2.17}$$

By using Lemma 2.2, we find

$$\begin{aligned}&\iint_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+\left|x^\lambda-y^\lambda\right|}\widetilde{f}(x)\widetilde{g}(y)dx\,dy\\ &\quad=\iint_1^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+\left|x^\lambda-y^\lambda\right|}x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p}y^{-[1+\varepsilon+(q-1)(1-\lambda)]/q}dx\,dy\\ &\quad>\frac{p^2+q^2}{2\lambda^2\varepsilon}\left[1+o(1)\right]-O(1).\end{aligned} \tag{2.18}$$

Therefore, we get

$$\frac{p^2+q^2}{2\lambda^2\varepsilon}\left[1+o(1)\right]-O(1)<\frac{k}{\varepsilon} \tag{2.19}$$

or

$$\frac{p^2+q^2}{2\lambda^2}\left[1+o(1)\right]-\varepsilon O(1)<k. \tag{2.20}$$

For $\varepsilon\to0^+$, it follows that $(p^2+q^2)/2\lambda^2\le k$. This contradicts the fact that $k<(p^2+q^2)/2\lambda^2$. Hence, the constant factor in (2.11) is the best possible. Theorem 2.3 is proved. □

Theorem 2.4. *Let* $0<p<1$, $1/p+1/q=1$, $\lambda>0$, $f(x)\ge0$, $g(y)\ge0$. *If* $0<\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx<\infty$, $0<\int_0^\infty y^{(q-1)(1-\lambda)}g^q(y)dy<\infty$, *then one has*

$$\begin{aligned}&\iint_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+\left|x^\lambda-y^\lambda\right|}f(x)g(y)dx\,dy\\ &\quad>\frac{p^2+q^2}{2\lambda^2}\left(\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{(q-1)(1-\lambda)}g^q(y)dy\right)^{1/q},\end{aligned} \tag{2.21}$$

where the constant factor $(p^2+q^2)/2\lambda^2$ *is the best possible. In particular, for* $\lambda=1$, *the inequality reduces to*

$$\iint_0^\infty \frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)g(y)dx\,dy>\frac{p^2+q^2}{2}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}. \tag{2.22}$$

Proof. Applying reverse Hölder's inequality and the same arguments as before, we have (2.21).

If the constant factor $(p^2+q^2)/2\lambda^2$ in (2.21) is not the best possible, then there exists a positive number h (with $h > (p^2+q^2)/2\lambda^2$), such that (2.21) is still valid if one replaces $(p^2+q^2)/2\lambda^2$ by h. In particular, for $0 < \varepsilon < q\lambda/2p$, setting $\widetilde{f}$ and $\widetilde{g}$ as in Theorem 2.3, we have

$$h\left(\int_0^\infty x^{(p-1)(1-\lambda)}\widetilde{f}^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{(q-1)(1-\lambda)}\widetilde{g}^q(y)dy\right)^{1/q} = \frac{h}{\varepsilon}. \tag{2.23}$$

By using Lemma 2.2, we find

$$\begin{aligned}
&\iint_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}\widetilde{f}(x)\widetilde{g}(y)dx\,dy \\
&= \iint_1^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p}y^{-[1+\varepsilon+(q-1)(1-\lambda)]/q}dx\,dy \\
&< \frac{p^2+q^2}{2\lambda^2\varepsilon}\left[I+o(1)\right].
\end{aligned} \tag{2.24}$$

Therefore, we get

$$\frac{p^2+q^2}{2\lambda^2\varepsilon}\left[1+o(1)\right] > \frac{h}{\varepsilon} \quad \text{or} \quad \frac{p^2+q^2}{2\lambda^2}\left[1+o(1)\right] > h \tag{2.25}$$

for $\varepsilon \to 0^+$, and it follows that $(p^2+q^2)/2\lambda^2 \geq h$. This contradicts the fact that $h > (p^2+q^2)/2\lambda^2$. Hence, the constant factor in (2.21) is the best possible. Theorem 2.4 is proved. □

Theorem 2.5. *If $p>1$, $1/p+1/q=1$, $\lambda>0$, $f(x)\geq 0$, $0<\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx<\infty$, then one has*

$$\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}f(x)dx\right]^p dy < \left(\frac{p^2+q^2}{2\lambda^2}\right)^p\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx, \tag{2.26}$$

where the constant factor $((p^2+q^2)/2\lambda^2)^p$ is the best possible. Inequality (2.26) *is equivalent to* (2.11).

Proof. Setting

$$g(y) = y^{\lambda-1}\left[\int_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}f(x)dx\right]^{p-1}, \tag{2.27}$$

then by (2.11), we find

$$\begin{aligned}
&\int_0^\infty y^{(q-1)(1-\lambda)}g^q(y)dy \\
&= \int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}f(x)dx\right]^p dy = \iint_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}f(x)g(y)dx\,dy \\
&\leq \frac{p^2+q^2}{2\lambda^2}\left(\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{|\ln x-\ln y|}{x^\lambda+y^\lambda+|x^\lambda-y^\lambda|}f(x)dx\right]^p dy\right)^{1/q}.
\end{aligned} \tag{2.28}$$

Hence, we obtain

$$\int_0^\infty y^{(q-1)(1-\lambda)} g^q(y)dy \le \left(\frac{p^2+q^2}{2\lambda^2}\right)^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx, \tag{2.29}$$

Thus, by (2.11), both (2.28) and (2.29) keep the form of strict inequalities, then we have (2.26).

Applying Hölder's inequality, we have

$$\begin{aligned}
&\iint_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|} f(x)g(y)dx\,dy \\
&= \int_0^\infty \left[y^{(\lambda-1)/p}\int_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|} f(x)dx\right]\left[y^{(1-\lambda)/p} g(y)\right]dy \\
&\le \left\{\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|} f(x)dx\right]^p dy\right\}^{1/p} \times \left\{\int_0^\infty y^{(q-1)(1-\lambda)} g^q(y)dy\right\}^{1/q}.
\end{aligned} \tag{2.30}$$

Therefore, by (2.26) we have (2.11). It follows that inequality (2.26) is equivalent to (2.11), and the constant factors in (2.26) are the best possible. The theorem is proved. □

Theorem 2.6. *If* $0 < p < 1$, $1/p + 1/q = 1$, $\lambda > 0$, $f(x) \ge 0$, $0 < \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx < \infty$, *then one has*

$$\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{|\ln x - \ln y|}{x^\lambda + y^\lambda + |x^\lambda - y^\lambda|} f(x)dx\right]^p dy > \left(\frac{p^2+q^2}{2\lambda^2}\right)^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx, \tag{2.31}$$

where the constant factor $((p^2+q^2)/2\lambda^2)^p$ *is the best possible. Inequality* (2.31) *is equivalent to* (2.21).

The proof of Theorem 2.6 is similar to that of Theorem 2.5, so we omit it.

Acknowledgment

The author would like to thank the anonymous referees for their suggestions and corrections.

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