Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 159029, 11 pages
doi:10.1155/2008/159029

Research Article

Starlike and Convex Properties for Hypergeometric Functions

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Received 13 February 2008; Accepted 18 June 2008

Recommended by Linda Sons

The purpose of the present paper is to give some characterizations for a (Gaussian) hypergeometric function to be in various subclasses of starlike and convex functions. We also consider an integral operator related to the hypergeometric function.

1. Introduction

Let $\mathcal{T}$ be the class consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{1.1}$$

that are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{T}$ consisting of starlike and convex functions of order α $(0 \leq \alpha < 1)$, respectively [1].

Recently, Bharati et al. [2] introduced the following subclasses of starlike and convex functions.

Definition 1.1. A function f of the form (1.1) is in $\mathcal{S}_p\mathcal{T}(\alpha,\beta)$ if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta, \quad \alpha \geq 0,\ 0 \leq \beta < 1, \tag{1.2}$$

and $f \in \mathcal{UCT}(\alpha,\beta)$ if and only if $zf' \in \mathcal{S}_p\mathcal{T}(\alpha,\beta)$.

Definition 1.2. A function f of the form (1.1) is in $\mathcal{PT}(\alpha)$ if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}+\alpha \geq \left|\frac{zf'(z)}{f(z)}-\alpha\right|, \quad 0<\alpha<\infty, \tag{1.3}$$

and $f \in \mathcal{CPT}(\alpha)$ if and only if $zf' \in \mathcal{PT}(\alpha)$.

Bharati et al. [2] showed that $\mathcal{S}_p\mathcal{T}(\alpha,\beta)=\mathcal{T}^*((\alpha+\beta)/(1+\alpha))$, $\mathcal{UCT}(\alpha,\beta)=\mathcal{C}((\alpha+\beta)/(1+\alpha))$, $\mathcal{PT}(\alpha)=\mathcal{T}^*(1-\alpha)$ $(0<\alpha\leq 1)$, and $\mathcal{CPT}(\alpha)=\mathcal{C}(1-\alpha)$ $(0<\alpha\leq 1)$. In particular, we note that $\mathcal{UCT}(1,0)$ is the class of uniformly convex functions given by Goodman [3] (also see [4–6]).

Let $F(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n, \tag{1.4}$$

where $c\neq 0,-1,-2,\ldots$, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n=\begin{cases}1 & \text{if } n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n\in\mathbb{N}=\{1,2,\ldots\}.\end{cases} \tag{1.5}$$

We note that $F(a,b;c;1)$ converges for $\operatorname{Re}(c-a-b)>0$ and is related to the Gamma function by

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.6}$$

Silverman [7] gave necessary and sufficient conditions for $zF(a,b;c;z)$ to be in $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, and also examined a linear operator acting on hypergeometric functions. For the other interesting developments for $zF(a,b;c;z)$ in connection with various subclasses of univalent functions, the readers can refer to the works of Carlson and Shaffer [8], Merkes and Scott [9], and Ruscheweyh and Singh [10].

In the present paper, we determine necessary and sufficient conditions for $zF(a,b;c;z)$ to be in $\mathcal{S}_p\mathcal{T}(\alpha,\beta)$, $\mathcal{UCT}(\alpha,\beta)$, $\mathcal{PT}(\alpha)$, and $\mathcal{CPT}(\alpha)$. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Results

To establish our main results, we need the following lemmas due to Bharati et al. [2].

Lemma 2.1. (i) *A function f of the form* (1.1) *is in $\mathcal{S}_p\mathcal{T}(\alpha,\beta)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}\left(n(1+\alpha)-(\alpha+\beta)\right)a_n\leq 1-\beta. \tag{2.1}$$

(ii) *A function f of the form* (1.1) *is in $\mathcal{UCT}(\alpha,\beta)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}n\left(n(1+\alpha)-(\alpha+\beta)\right)a_n\leq 1-\beta. \tag{2.2}$$

Lemma 2.2. (i) *A function f of the form* (1.1) *is in $\mathcal{PT}(\alpha)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}(n-1+\alpha)a_n \leq \alpha. \tag{2.3}$$

(ii) *A function f of the form* (1.1) *is in $\mathcal{CPT}(\alpha)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}n(n-1+\alpha)a_n \leq \alpha. \tag{2.4}$$

Theorem 2.3. (i) *If $a,b>-1,\ c>0$, and $ab<0$, then $zF(a,b;c;z)$ is in $\mathcal{S}_p\mathcal{T}(\alpha,\beta)$ if and only if*

$$c \geq a+b+1-\frac{(1+\alpha)ab}{(1-\beta)}. \tag{2.5}$$

(ii) *If $a,b>0$ and $c>a+b+1$, then $F_1(a,b;c;z)=z(2-F(a,b;c;z))$ is in $\mathcal{S}_p\mathcal{T}(\alpha,\beta)$ if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\frac{(1+\alpha)ab}{(1-\beta)(c-a-b-1)}\right) \leq 2. \tag{2.6}$$

Proof. (i) Since

$$zF(a,b;c;z)=z+\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n = z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n, \tag{2.7}$$

according to (i) of Lemma 2.1, we must show that

$$\sum_{n=2}^{\infty}(n(1+\alpha)-(\alpha+\beta))\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\beta). \tag{2.8}$$

Noting that $(\lambda)_n=\lambda(\lambda+1)_{n-1}$ and then applying (1.6), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}((n+2)(1+\alpha)-(\alpha+\beta))\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}\\
&\quad=(1+\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(1-\beta)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\
&\quad=(1+\alpha)\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+(1-\beta)\frac{c}{ab}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right).
\end{aligned} \tag{2.9}$$

Hence, (2.8) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\alpha+(1-\beta)\frac{c-a-b-1}{ab}\right) \le (1-\beta)\left(\frac{c}{|ab|}+\frac{c}{ab}\right)=0. \tag{2.10}$$

Thus, (2.10) is valid if and only if $1+\alpha+(1-\beta)(c-a-b-1)/(ab) \le 0$ or, equivalently, $c \ge a+b+1-(1+\alpha)ab/(1-\beta)$.

(ii) Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n, \tag{2.11}$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}\left(n(1+\alpha)-(\alpha+\beta)\right)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le 1-\beta. \tag{2.12}$$

Now,

$$\begin{aligned}\sum_{n=2}^{\infty}\left(n(1+\alpha)-(\alpha+\beta)\right)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} &= (1+\alpha)\sum_{n=1}^{\infty}\frac{n(a)_n(b)_n}{(c)_n(1)_n}(1-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\ &= \frac{(1+\alpha)ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}+(1-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(1+\alpha)ab}{c-a-b-1}+1-\beta\right)-(1-\beta).\end{aligned} \tag{2.13}$$

But this last expression is bounded above by $1-\beta$ if and only if (2.6) holds. □

Theorem 2.4. (i) *If* $a,b>-1$, $ab<0$, *and* $c>a+b+2$, *then* $zF(a,b;c;z)$ *is in* $\mathcal{UCT}(\alpha,\beta)$ *if and only if*

$$(1+\alpha)(a)_2(b)_2+(3+2\alpha-\beta)ab(c-a-b-2)+(1-\beta)(c-a-b-2)_2 \ge 0. \tag{2.14}$$

(ii) *If* $a,b>0$ *and* $c>a+b+2$, *then* $F_1(a,b;c;z)=z(2-F(a,b;c;z))$ *is in* $\mathcal{UCT}(\alpha,\beta)$ *if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(1+\alpha)(a)_2(b)_2}{(1-\beta)(c-a-b-2)_2}+\left(\frac{3+2\alpha-\beta}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)+1\right) \le 2. \tag{2.15}$$

Proof. (i) Since zF has the form (2.7), we see from (ii) of Lemma 2.1 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n\big(n(1+\alpha)-(\alpha+\beta)\big)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|}(1-\beta). \tag{2.16}$$

Writing $(n+2)((n+2)(1+\alpha)-(\alpha+\beta)) = (1+\alpha)(n+1)^2+(2+\alpha-\beta)(n+1)+(1-\beta)$, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)\big((n+2)(1+\alpha)-(\alpha+\beta)\big)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad= (1+\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(2+\alpha-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\qquad+(1-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad= \frac{(1+\alpha)(a+1)(b+1)}{c+1}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+(3+2\alpha-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\qquad+(1-\beta)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&\quad= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\bigg((1+\alpha)(a+1)(b+1)+(3+2\alpha-\beta)(c-a-b-2) \\
&\qquad\qquad+\frac{1-\beta}{ab}(c-a-b-2)_2\bigg)-\frac{(1-\beta)c}{ab}.
\end{aligned} \tag{2.17}$$

This last expression is bounded above by $(1-\beta)c/|ab|$ if and only if

$$(1+\alpha)(a+1)(b+1)+(3+2\alpha-\beta)(c-a-b-2)+\frac{1-\beta}{ab}(c-a-b-2)_2 \leq 0, \tag{2.18}$$

which is equivalent to (2.14).

(ii) In view of (ii) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} n\big(n(1+\alpha)-(\alpha+\beta)\big)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1-\beta. \tag{2.19}$$

Now,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)\big((n+2)(1+\alpha)-(\alpha+\beta)\big)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&\quad= (1+\alpha)\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}-(\alpha+\beta)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.
\end{aligned} \tag{2.20}$$

Writing $n+2=(n+1)+1$, we have

$$\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}},$$

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}&=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\&=\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.\end{aligned}\tag{2.21}$$

Substituting (2.21) into the right-hand side of (2.20), we obtain

$$(1+\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}}+(3+2\alpha-\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+(1-\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.\tag{2.22}$$

Since $(a)_{n+k}=(a)_k(a+k)_n$, we write (2.22) as

$$\begin{aligned}&\frac{(1+\alpha)(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+(3+2\alpha-\beta)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\&\quad+(1-\beta)\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right).\end{aligned}\tag{2.23}$$

By simplification, we see that the last expression is bounded above by $1-\beta$ if and only if (2.15) holds. □

Theorem 2.5. (i) *If $a,b>-1$, $c>0$, and $ab<0$, then $zF(a,b;c;z)$ is in $\mathcal{PT}(\alpha)$ if and only if*

$$c\ge a+b+1-\frac{ab}{\alpha}.\tag{2.24}$$

(ii) *If $a,b>0$ and $c>a+b+1$, then $F_1(a,b;c;z)=z(2-F(a,b;c;z))$ is in $\mathcal{PT}(\alpha)$ if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\frac{ab}{\alpha(c-a-b-1)}\right)\le 2.\tag{2.25}$$

Proof. (i) Since

$$zF(a,b;c;z)=z+\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n=z-\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,\tag{2.26}$$

according to (i) of Lemma 2.2, we must show that

$$\sum_{n=2}^{\infty}(n-1+\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\le\frac{c}{|ab|}\alpha.\tag{2.27}$$

Noting that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ and then applying (1.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1+\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} &= \sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \alpha\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\ &= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \alpha\frac{c}{ab}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right).\end{aligned} \tag{2.28}$$

Hence, (2.27) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\alpha\frac{c-a-b-1}{ab}\right) \le \alpha\left(\frac{c}{|ab|}-\frac{c}{ab}\right) = 0. \tag{2.29}$$

Thus, (2.29) is valid if and only if $1+\alpha(c-a-b-1)/ab \le 0$ or, equivalently, $c \ge a+b+1-ab/\alpha$.

(ii) Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n, \tag{2.30}$$

by (i) of Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty}(n-1+\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le \alpha. \tag{2.31}$$

Now,

$$\begin{aligned}\sum_{n=2}^{\infty}(n-1+\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} &= \sum_{n=1}^{\infty}\frac{n(a)_n(b)_n}{(c)_n(1)_n} + \alpha\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\ &= \frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + \alpha\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{ab}{c-a-b-1}+\alpha\right)-\alpha.\end{aligned} \tag{2.32}$$

But this last expression is bounded above by α if and only if (2.25) holds. □

Theorem 2.6. (i) *If $a,b > -1$, $ab < 0$, and $c > a+b+2$, then $zF(a,b;c;z)$ is in $\mathcal{CPT}(\alpha)$ if and only if*

$$(a)_2(b)_2 + (2+\alpha)ab(c-a-b-2) + \alpha(c-a-b-2)_2 \ge 0. \tag{2.33}$$

(ii) *If $a,b > 0$ and $c > a+b+2$, then $F_1(a,b;c;z) = z(2-F(a,b;c;z))$ is in $\mathcal{CPT}(\alpha)$ if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(a)_2(b)_2}{\alpha(c-a-b-2)_2} + \left(\frac{2+\alpha}{\alpha}\right)\left(\frac{ab}{c-a-b-1}\right)+1\right) \le 2. \tag{2.34}$$

Proof. (i) Since zF has the form (2.26), we see from (ii) of Lemma 2.2 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(n-1+\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \le \frac{c}{|ab|}\alpha. \tag{2.35}$$

Writing $(n+2)(n+1+\alpha) = (n+1)^2 + (1+\alpha)(n+1) + \alpha$, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)(n+1+\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad = \sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1+\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \alpha\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad = \frac{(a+1)(b+1)}{c+1}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + (2+\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \alpha\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&\quad = \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\left((a+1)(b+1) + (2+\alpha)(c-a-b-2) + \frac{\alpha}{ab}(c-a-b)_2\right) - \frac{\alpha c}{ab}.
\end{aligned} \tag{2.36}$$

This last expression is bounded above by $\alpha c/|ab|$ if and only if $(a+1)(b+1) + (2+\alpha)(c-a-b-2) + (\alpha/ab)(c-a-b-1)_2 \le 0$, which is equivalent to (2.33).

(ii) In view of (ii) of Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} n(n-1+\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le \alpha. \tag{2.37}$$

Now,

$$\sum_{n=0}^{\infty}(n+2)\big(n+2-(1-\alpha)\big)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} = \sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - (1-\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \tag{2.38}$$

Substituting (2.21) into the right-hand side of (2.38), we obtain

$$\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + (2+\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \alpha\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{2.39}$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (2.39) as

$$\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} + (2+\alpha)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \alpha\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right). \tag{2.40}$$

By simplification, we see that the last expression is bounded above by α if and only if (2.34) holds. □

3. An integral operator

In the next theorems, we obtain similar-type results in connection with a particular integral operator $G(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G(a,b;c;z)=\int_0^z F(a,b;c;t)dt. \tag{3.1}$$

Theorem 3.1. *Let* $a,b>-1$, $ab<0$, *and* $c>\max\{0,a+b\}$. *Then,*

(i) $G(a,b;c;z)$ *defined by* (3.1) *is in* $\mathcal{S}_p\mathcal{T}(\alpha,\beta)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(1+\alpha)}{ab}-\frac{(\alpha+\beta)(c-a-b)}{(a-1)_2(b-1)_2}\right)+\frac{(\alpha+\beta)(c-1)_2}{(a-1)_2(b-1)_2}\leq 0; \tag{3.2}$$

(ii) $G(a,b;c;z)$ *defined by* (3.1) *is in* $\mathcal{PT}(\alpha)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{1}{ab}+\frac{(\alpha-1)(c-a-b)}{(a-1)_2(b-1)_2}\right)-\frac{(\alpha-1)(c-a)_2}{(a-1)_2(b-1)_2}\leq 0. \tag{3.3}$$

Proof. (i) Since

$$G(a,b;c;z)=z-\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}z^n, \tag{3.4}$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}\big(n(1+\alpha)-(\alpha+\beta)\big)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\leq(1-\beta)\frac{c}{|ab|}. \tag{3.5}$$

Now,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big((n+2)(1+\alpha)-(\alpha+\beta)\big)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}\\
&\quad=(1+\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}-(\alpha+\beta)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}\\
&\quad=\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{1+\alpha}{ab}-\frac{(\alpha+\beta)(c-a-b)}{(a-1)_2(b-1)_2}\right)\\
&\qquad+\frac{(\alpha+\beta)(c-1)_2}{(a-1)_2(b-1)_2}-(1-\beta)\frac{c}{ab}\\
&\quad\leq(1-\beta)\frac{c}{|ab|},
\end{aligned} \tag{3.6}$$

which is equivalent to (3.2).

(ii) According to (i) of Lemma 2.2, it is sufficient to show that

$$\sum_{n=2}^{\infty}(n-1+\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \le \alpha\frac{c}{|ab|}. \tag{3.7}$$

Now,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+1+\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\
&\quad = \sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (\alpha-1)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\
&\quad = \frac{c}{ab}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right) \\
&\qquad + (\alpha-1)\frac{c}{ab}\left(\frac{(c-1)}{(a-1)(b-1)}\left(\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right)-1\right) \\
&\quad = \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{1}{ab}+\frac{(\alpha-1)(c-a-b)}{(a-1)_2(b-1)_2}\right) - \frac{(\alpha-1)(c-1)_2}{(a-1)_2(b-1)_2} - \alpha\frac{c}{ab} \\
&\quad \le \alpha\frac{c}{|ab|},
\end{aligned} \tag{3.8}$$

which is equivalent to (3.3).

Now, we observe that $G(a,b;c;z) \in \mathcal{UCT}(\alpha,\beta)(\mathcal{CPT}(\alpha))$ if and only if $zF(a,b;c;z) \in \mathcal{S}_p\mathcal{T}(\alpha,\beta)(\mathcal{PT}(\alpha))$. Thus, any result of functions belonging to the class $\mathcal{S}_p\mathcal{T}(\alpha,\beta)(\mathcal{PT}(\alpha))$ about zF leads to that of functions belonging to the class $\mathcal{UCT}(\alpha,\beta)(\mathcal{CPT}(\alpha))$. Hence, we obtain the following analogues to Theorems 2.3 and 2.5. □

Theorem 3.2. *Let* $a,b > -1$, $ab < 0$, *and* $c > a+b+2$. *Then,*

(i) $G(a,b;c;z)$ *defined by* (3.1) *is in* $\mathcal{UCT}(\alpha,\beta)$ *if and only if*

$$c \ge a+b+1-\frac{(1+\alpha)ab}{(1-\beta)}; \tag{3.9}$$

(ii) $G(a,b;c;z)$ *defined by* (3.1) *is in* $\mathcal{CPT}(\alpha)$ *if and only if*

$$c \ge a+b+1-\frac{ab}{\alpha}. \tag{3.10}$$

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