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Research Article

Nonlinear Integrodifferential Equations of Mixed Type in Banach Spaces

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We prove two existence theorems for the integrodifferential equation of mixed type: $x'(t) = f(t,x(t),\int_0^t k_1(t,s)g(s,x(s))ds,\int_0^a k_2(t,s)h(s,x(s))ds)$, $x(0) = x_0$, where in the first part of this paper f, g, h, x are functions with values in a Banach space E and integrals are taken in the sense of Henstock-Kurzweil (HK). In the second part f, g, h, x are weakly-weakly sequentially continuous functions and integrals are taken in the sense of Henstock-Kurzweil-Pettis (HKP) integral. Additionally, the functions f, g, h, x satisfy some conditions expressed in terms of the measure of noncompactness or the measure of weak noncompactness.

1. Introduction

The Henstock-Kurzweil integral encompasses the Newton, Riemann, and Lebesgue integrals [1–3]. A particular feature of this integral is that integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0,1]$ and $F(0) = 0$, can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957–1958 and has since proved useful in the study of ordinary differential equations [4–7]. In the paper, [8] Cao defined the Henstock integral in a Banach space, which is a generalization of the Bochner integral. The Pettis integral is also a generalization of the Bochner integral [9]. This notion is strictly relative to weak topologies in Banach spaces.

In [10], Cichon et al.generalized both concepts of integral introducing the Henstock-Kurzweil-Pettis integral.

The paper is divided into two main sections. In Section 1, we prove some existence theorem for the problem

$$x'(t) = f\left(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\,ds, \int_0^a k_2(t,s)h(s,x(s))\,ds\right),$$
$$x(0) = x_0, \quad x_0 \in E,\ t \in I_a = [0,a],\ a \in \mathbb{R}_+, \tag{1.1}$$

where E is a Banach space with the norm $\|\cdot\|$, f, g, h, x are functions with values in a Banach space E, k_j, $j = 1,2$ are real-valued functions, and integrals are taken in the sense of HL.

In Section 2, we prove some existence theorem for the problem (1.1), where f, g, h, x are functions with values in a Banach space E, weakly-weakly sequentially continuous, and k_j, $j = 1,2$ are real-valued functions. The integrals are taken in the sense of Henstock-Kurzweil-Pettis.

We should mention that an extensive work has been done in the study of the solutions of particular cases of (1.1) (see, e.g., [11–18]).

Our fundamental tools are the Kuratowski measure of noncompactness [19] and the measure of weak noncompactness developed by De Blasi [20].

For any bounded subset A of E, we denote by $\alpha(A)$ *the Kuratowski measure of noncompactness of* A, that is, the infimum of all $\varepsilon > 0$, such that there exists a finite covering of A by sets of diameter smaller than ε.

The *De Blasi measure of weak noncompactness* $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\}, \tag{1.2}$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E.

The properties of the measure of noncompactness $\alpha(A)$ are as follows:

(i) if $A \subset B$ then $\alpha(A) \le \alpha(B)$;
(ii) $\alpha(A) = \alpha(\overline{A})$, where $\overline{A}$ denotes the closure of A;
(iii) $\alpha(A) = 0$ if and only if A is relatively compact;
(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;
(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $(\lambda \in \mathbb{R})$;
(vi) $\alpha(A + B) \le \alpha(A) + \alpha(B)$;
(vii) $\alpha(\operatorname{conv}(A)) = \alpha(A)$, where $\operatorname{conv}(A)$ denotes the convex extension of A.

The properties of the weak measure of noncompactness β are analogous to the properties of the measure of noncompactness $\alpha(A)$ (see [21]).

We now gather some well-known definitions and results from the literature, which we will use throughout this paper.

Definition 1.1. A function $f : I_a \times E_1 \to E_2$, where E_1 and E_2 are Banach spaces, is L^1-*Carathéodory*, if the following conditions hold:

(i) the map $s \mapsto f(s,x)$ is measurable for all $x \in E_1$;
(ii) the map $x \mapsto f(s,x)$ is continuous for almost all $s \in I_a$.

Definition 1.2. A function $f : I_a \to E$ is said to be *weakly continuous* if it is continuous from I_a to E endowed with its weak topology. A function $g : E \to E_1$, where E and E_1 are Banach spaces, is said to be *weakly-weakly sequentially continuous* if for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1.

When the sequence x_n tends weakly to x_0 in E, we will write $x_n \xrightarrow{\omega} x_0$.

Definition 1.3 [1, 3]. A family $\mathcal{F}$ of functions F is said to be *uniformly absolutely continuous in the restricted sense on $A \subseteq [a,b]$ or in short uniformly $AC_*(A)$* if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every F in $\mathcal{F}$ and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i,b_i]\}$ with $a_i, b_i \in A$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F,[a_i,b_i]) < \varepsilon$, where ω denotes the oscillation of F over $[a_i,b_i]$ (i.e., $\omega(F,[a_i,b_i]) = \sup\{|F(r) - F(s)| : r,s \in [a_i,b_i]\}$).

A family $\mathcal{F}$ of functions F is said to *be uniformly generalized absolutely continuous in the restricted sense on $[a,b]$ or uniformly ACG_** if $[a,b]$ is the union of a sequence of closed sets A_i such that on each A_i the family $\mathcal{F}$ is uniformly $AC_*(A_i)$.

In the proof of the main theorem in Section 1, we will apply the following fixed point theorem.

Theorem 1.4 [22]. *Let D be a closed convex subset of E, and let F be a continuous map from D into itself. If for some $x \in D$ the implication that*

$$\overline{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V)) \Longrightarrow V \text{ is relatively compact} \tag{1.3}$$

holds for every countable subset V of D, then F has a fixed point.

In Section 2 we will apply the following theorem.

Theorem 1.5 [23]. *Let X be a metrizable locally convex topological vector space. Let D be a closed convex subset of X, and let F be a weakly-weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication that*

$$\overline{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V)) \Longrightarrow V \text{ is relatively weakly compact} \tag{1.4}$$

holds for every subset V of D, then F has a fixed point.

2. Henstock-Kurzweil and Henstock-Kurzweil-Pettis integrals in Banach spaces

In this part, we present the definitions of Henstock-Kurzweil and Henstock-Kurzweil-Pettis integrals and properties of these integrals which we will use in the proof of the main theorems.

For more details, you can see [1, 3, 24].

Definition 2.1. Let δ be a positive function defined on the interval $[a,b]$. A tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $x \in [c,d]$. The tagged interval $(x,[c,d])$ is subordinate to δ if $[c,d] \subset [x - \delta(x), x + \delta(x)]$.

The letter P will be used to denote finite collections of nonoverlapping tagged intervals. Let

$$P = \{s_i, [c_i, d_i] : 1 \le i \le n\}, \quad n \in \mathbb{N}, \tag{2.1}$$

be such a collection in $[a,b]$. Then,

(i) the points $\{s_i : 1 \le i \le n\}$ are called the tags of P;
(ii) the intervals $\{[c_i, d_i] : 1 \le i \le n\}$ are called the intervals of P;
(iii) if $(s_i, [c_i, d_i])$ is subordinate to δ for each i, then we write P is sub δ;
(iv) if $[a,b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then P is called a tagged partition of $[a,b]$;
(v) if P is a tagged partition of $[a,b]$ and if P is sub δ, then we write P is sub δ on $[a,b]$;
(vi) if $f : [a,b] \to E$, then $f(P) = \sum_{i=1}^{n} f(s_i)(d_i - c_i)$;
(vii) if F is defined on the subintervals of $[a,b]$, then

$$F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = \sum_{i=1}^{n} (F(d_i) - F(c_i)). \tag{2.2}$$

If $F : [a,b] \to E$, then F can be treated as a function of intervals by defining $F([d,c]) = F(d) - F(c)$. For such a function, $F(P) = F(b) - F(a)$ if P is a tagged partition of $[a,b]$.

Definition 2.2 [1, 3]. A function $f : [a,b] \to \mathbb{R}$ is *Henstock-Kurzweil integrable on* $[a,b]$ if there exists a real number L with the following property: for each $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that $|f(P) - L| < \varepsilon$ whenever P is a tagged partition of $[a,b]$ that is subordinate to δ.

The function f is *Henstock-Kurzweil integrable on a measurable set* $A \subset [a,b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a,b]$. The number L in Definition 2.2 is called *the Henstock-Kurzweil integral of* f and we will denote it by $(HK)\int_a^b f(t)dt$.

Definition 2.3 [8]. A function $f : [a,b] \to E$ is *Henstock-Kurzweil integrable* on $[a,b]$ ($f \in HK([a,b],E)$) if there exists a vector $z \in E$ with the following property: for every $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is a tagged partition of $[a,b]$ sub δ. The function f is Henstock-Kurzweil integrable on a measurable set $A \subset [a,b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a,b]$. The vector z is *the Henstock-Kurzweil integral of* f.

We remark that this definition includes the generalized Riemann integral defined by Gordon [25]. In a special case, when δ is a constant function, we get the Riemann integral.

Definition 2.4 [8]. A function $f : [a,b] \to E$ is *HL integrable* on $[a,b]$ ($f \in HL([a,b],E)$) if there exists a function $F : [a,b] \to E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that if $P = \{s_i, [c_i, d_i] : 1 \le i \le n\}$ is a tagged partition of $[a,b]$ sub δ, then

$$\sum_{i=1}^{n} \| f(s_i)(d_i - c_i) - F([c_i, d_i]) \| < \varepsilon. \tag{2.3}$$

Remark 2.5. We note by triangle inequality that

$$f \in HL([a,b],E) \text{ implies } f \in HK([a,b],E). \tag{2.4}$$

In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

Definition 2.6 [9]. The function $f : I_a \to E$ is *Pettis integrable* (P integrable for short) if

(i) $\forall_{x^* \in E^*}\, x^* f$ is Lebesgue integrable on I_a,

(ii) $\forall_{\substack{A \subset I_a, \\ A\text{-measurable}}}\, \exists_{g \in E}\, \forall_{x^* \in E^*}\, x^* g = (L)\int_A x^* f(s)ds$,

where $(L)\int_A$ denotes the Lebesgue integral over A.

Now, we present a definition of an integral which is a generalization for both Pettis and Henstock-Kurzweil integrals.

Definition 2.7 [26]. The function $f : I_a \to E$ is *Henstock-Kurzweil-Pettis integrable* (HKP integrable for short) if there exists a function $g : I_a \to E$ with the following properties:

(i) $\forall_{x^* \in E^*}\, x^* f$ is Henstock-Kurzweil integrable on I_a;

(ii) $\forall_{t \in I_a}\, \forall_{x^* \in E^*}\, x^* g(t) = (HK)\int_0^t x^* f(s)ds$.

This function g will be called a *primitive of* f and by $g(a) = \int_0^a f(t)dt$ we will denote the *Henstock-Kurzweil-Pettis integral* of f on the interval I_a.

Theorem 2.8 (Mean value theorem). *If the function $f : I_a \to E$ is HK (or HKP) integrable, then*

$$\int_I f(t)dt \in |I| \cdot \overline{\operatorname{conv}} f(I), \tag{2.5}$$

where I is an arbitrary subinterval of I_a and $|I|$ is the length of I.

If the integral is taken in the sense of HL, then the proof is similar to that of [27, Lemma 2.1.3]. The proof for HKP integral is presented in [24].

Theorem 2.9 [8]. *Let $f : [a,b] \to E$ be HL integrable on $[a,b]$ and let $F(x) = \int_a^x f(t)dt$ for each $x \in [a,b]$. Then*

(i) *F is continuous on $[a,b]$,*

(ii) *F is differentiable almost everywhere on $[a,b]$ and $F' = f$,*

(iii) *f is measurable.*

Theorem 2.10 [28, Theorem 5]. *Suppose that $f_n : [a,b] \to E$, $n = 1,2,\dots$, is a sequence of HL integrable functions satisfying the following conditions:*

(i) *$f_n(x) \to f(x)$ almost everywhere in $[a,b]$, as $n \to \infty$;*

(ii) *the set of primitives of f_n, $\{F_n(x)\}$, where $F_n(x) = \int_a^x f_n(s)ds$, is uniformly ACG_* in n;*

(iii) *the primitives F_n are equicontinuous on $[a,b]$.*

Then, f is HL integrable on $[a,b]$ and $\int_a^x f_n \to \int_a^x f$ uniformly on $[a,b]$, as $n \to \infty$.

We remark that this theorem for Denjoy-Bochner integrals is mentioned in [28] without proof. It is also true for HL integrals. The proof is similar to that of [3, Theorem 7.6] (see also [29, Theorem 1.8]).

Theorem 2.11 [26]. *Let $f : I_a \to E$ and assume that $f_n : I_a \to E$, $n \in N$, are HKP integrable on I_a. For each $n \in N$, let F_n be a primitive of f_n. If it is assumed that*

(i) $\forall x^* \in E^* x^*(f_n(t)) \to x^*(f(t))$ *a.e. on* I_a,

(ii) *for each* $x^* \in E^*$, *the family* $G = (x^* F_n : n = 1,2,\dots)$ *is uniformly* ACG_* *on* I_a *(i.e., weakly uniformly* ACG_* *on* I_a*),*

(iii) *for each* $x^* \in E^*$, *the set* G *is equicontinuous on* I_a,

then f *is HKP integrable on* I_a *and* $\int_0^t f_n(s)ds$ *tends weakly in* E *to* $\int_0^t f(s)ds$ *for each* $t \in I_a$.

3. An existence result for integrodifferential equations

It is well known that Henstock's lemma plays an important role in the theory of the Henstock-Kurzweil integral in the real-valued case. On the other hand, in connection with the Henstock-Kurzweil integral for Banach-space-valued functions, Cao pointed out in [8] that Henstock's lemma holds for the case of finite dimensions, but it does not always hold in infinite dimensions.

In this section, we will use the HL integral which satisfies Henstock's lemma and which is more general than the Bochner integral.

Our fundamental tool is a Kuratowski measure of noncompactness α. It is necessary to remark that the following lemma is true.

Lemma 3.1 [30]. *Let $H \subset C(I_a,E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a$, $H(t) = \{h(t) \in E,\ h \in H\}$. Then, $\alpha(H(I_a)) = \sup_{t\in I_a} \alpha(H(t))$ and the function $t \mapsto \alpha(H(t))$ is continuous.*

Observe that the problem (1.1) is equivalent to the integral equation [31]:

$$x(t) = x_0 + \int_0^t f\left(z,x(z),\int_0^z k_1(z,s)g(s,x(s))\,ds,\int_0^a k_2(z,s)h(s,x(s))\,ds\right)dz \quad \text{for } t \in I_a. \tag{3.1}$$

To obtain the existence result, it is necessary to define a notion of a solution.

Definition 3.2. An ACG_* function $x : I_a \to E$ is said to be a solution of the problem (1.1) if it satisfies the following conditions:

(i) $x(0) = x_0$;

(ii) $x'(t) = f(t,x(t),\int_0^t k_1(t,s)g(s,x(s))ds,\int_0^a k_2(t,s)h(s,x(s))ds)$ for a.e. $t \in I_a$.

Definition 3.3. A continuous function $x : I_a \to E$ is said to be a solution of the problem (3.1) if $x(t) = x_0 + \int_0^t f(z,x(z),\int_0^z k_1(z,s)g(s,x(s))ds,\int_0^a k_2(z,s)h(s,x(s))ds)dz$ for every $t \in I_a$.

For $x \in C(I_a,E)$, we define the norm of x by: $\|x\|_C = \sup\{\|x(t)\|,\ t \in I_a\}$.

Let $B(p) = \{x \in C(I_a,E) :\ \|x\|_C \le \|x_0\|_C + p\}$, $p > 0$. Note that these sets are closed and convex.

Define the operator $F : C(I_a,E) \to C(I_a,E)$ by

$$F(x)(t) = x_0 + \int_0^t f\left(z,x(z),\int_0^z k_1(z,s)g(s,x(s))\,ds,\int_0^a k_2(z,s)h(s,x(s))\,ds\right)dz, \quad t \in I_a,\ x \in B(p), \tag{3.2}$$

where integrals are in the sense of HL.

Let

$$\Gamma(p) = \{F(x) \in C(I_a, E) : x \in B(p)\} \quad \text{for each } p > 0. \tag{3.3}$$

Let $r(K)$ be the spectral radius of the integral operator K defined by

$$K(u)(t) = \int_0^c k(t,s)u(s)ds, \tag{3.4}$$

where the kernel $k \in C(I_a \times I_a; \mathbb{R})$, $u \in C(I_a; E)$ and c denotes any fixed value in I_a, $a > 0$.

Theorem 3.4. *Assume that for each ACG_* function $x : I_a \to E$, functions $g(\cdot, x(\cdot))$, $f(\cdot, x(\cdot))$, $\int_0^{(\cdot)}(k_1(\cdot,s)g(s,x(s)))ds$, $\int_0^a k_2(\cdot,s)h(s,x(s))ds)$ are HL integrable, f, g, and h are L^1-Carathéodory functions. Let $k_1, k_2 : I_a \times I_a \to \mathbb{R}_+$ be measurable functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous.*

Assume that there exist $p_0 > 0$ and positive constants L, L_1, and d_1, such that

$$\begin{gathered}
\alpha(g(I,X)) \le L\alpha(X) \quad \text{for } I \subset I_a, \text{ for every } X \subset B(p_0), \\
\alpha(h(I,X)) \le L_1\alpha(X) \quad \text{for } I \subset I_a, \text{ for every } X \subset B(p_0), \\
\alpha(f(t,A,C,D)) \le d_1 \cdot \max\{\alpha(A), \alpha(C), \alpha(D)\} \quad \text{for every } A, C, D \subset B(p_0),\ t \in I_a,
\end{gathered} \tag{3.5}$$

where $g(I,X) = \{g(t,x(t)) : t \in I, x \in X\}$, $h(I,X) = \{h(t,x(t)) : t \in I, x \in X\}$, $f(t,A,C,D) = \{f(t,x_1,x_2,x_3) : (x_1,x_2,x_3) \in A \times C \times D\}$ and α denotes the Kuratowski measure of noncompactness.

Moreover, let $\Gamma(p_0)$ be equicontinuous, equibounded, and uniformly ACG_* on I_a. Then, there exists at least one solution of the problem (1.1) on I_c, for some $0 < c \le a$, such that $d_1 \cdot c \cdot L \cdot r(K) < 1$ and $d_1 \cdot c < 1$.

Proof. By equicontinuity and equiboundedness of $\Gamma(p_0)$ there exists a number c, $0 < c \le a$ such that

$$\left\| \int_0^t f\left(z, x(z), \int_0^z k_1(z,s)g(s,x(s))ds, \int_0^c k_2(z,s)h(s,x(s))ds\right)dz \right\| \le p_0, \quad \text{where } p_0 > 0,\ x \in B(p_0),\ t \in I_c. \tag{3.6}$$

By our assumptions the operator F is well defined and maps $B(p_0)$ into $B(p_0)$:

$$\begin{aligned}
\|F(x)(t)\| &= \left\| x_0 + \int_0^t f\left(z, x(z), \int_0^z k_1(z,s)g(s,x(s))ds, \int_0^c k_2(z,s)h(s,x(s))ds\right)dz \right\| \\
&\le \|x_0\| + \left\| \int_0^t f\left(z, x(z), \int_0^z k_1(z,s)g(s,x(s))ds, \int_0^c k_2(z,s)h(s,x(s))ds\right)dz \right\| \\
&\le \|x_0\| + p_0, \quad t \in I_c,\ x_0 \in E.
\end{aligned} \tag{3.7}$$

Using Theorem 2.10, we deduce that F is continuous.

Suppose that $V \subset B(p_0)$ satisfies the condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$, for some $x \in B(p_0)$. We will prove that V is relatively compact, thus (1.3) is satisfied. Theorem 1.4 will ensure that F has a fixed point.

Let, for $t \in I_c$, $V(t) = \{v(t) \in E : v \in V\}$. Since $V \subset B(p_0)$, $F(V) \subset \Gamma(p_0)$. Then, $V \subset \overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ is equicontinuous. By Lemma 3.1, $t \mapsto \nu(t) = \alpha(V(t))$ is continuous on I_c.

For fixed $t \in I_c$ we divide the interval $[0,t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m$, $i = 0,1,\dots,m$.

Let $V([t_i,t_{i+1}]) = \{u(s) : u \in V,\ t_i \le s \le t_{i+1}\}$, $i = 0,1,\dots,m-1$. By Lemma 3.1 and the continuity of ν there exists $s_i \in I_i = [t_i,t_{i+1}]$, such that

$$\alpha(V([t_i,t_{i+1}])) = \sup\{\alpha(V(s)) : t_i \le s \le t_{i+1}\} := \nu(s_i). \tag{3.8}$$

For fixed $z \in I_c$ we divide the interval $[0,z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m$, $j = 0,1,\dots,m$.

Let $V([z_j,z_{j+1}]) = \{u(s) : u \in V,\ z_j \le s \le z_{j+1}\}$, $j = 0,1,\dots,m-1$. By Lemma 3.1 and the continuity of ν there exists $s_j \in I_j = [z_j,z_{j+1}]$ such that

$$\alpha(V([z_j,z_{j+1}])) = \sup\{\alpha(V(s)) : z_j \le s \le z_{j+1}\} := \nu(s_j). \tag{3.9}$$

Furthermore, we divide the interval $[0,c]$ into m parts: $0 = r_0 < r_1 < \cdots < r_m = c$, where $r_k = kc/m$, $k = 0,1,\dots,m$.

Let $V([r_k,r_{k+1}]) = \{u(s) : u \in V,\ r_k \le s \le r_{k+1}\}$, $k = 0,1,\dots,m-1$. By Lemma 3.1 and the continuity of ν there exists $s_k \in I_k = [r_k,r_{k+1}]$, such that

$$\alpha(V([r_k,r_{k+1}])) = \sup\{\alpha(V(s)) : r_k \le s \le r_{k+1}\} := \nu(s_k). \tag{3.10}$$

By Definition 2.7 and the properties of the HL integral, we have

$$\begin{aligned}
F(x)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f\Bigg(z, x(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k_1(z,s) g(s,x(s))\,ds,& \\
\sum_{k=0}^{m-1} \int_{r_k}^{r_{k+1}} k_2(z,s) h(s,x(s))\,ds\Bigg) dz \in x_0& \\
+ \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{\text{conv}} f\Bigg(I_i, V(I_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j) \overline{\text{conv}}(k_1(I_i,I_j) g(I_j, V(I_j))),& \\
\sum_{k=0}^{m-1} (r_{k+1} - r_k) \overline{\text{conv}}(k_2(I_i,I_k) h(I_k, V(I_k)))\Bigg),&
\end{aligned} \tag{3.11}$$

where $k(I,J) = \{k(t,s) : t \in I,\ s \in J\}$ and $g(I,V(I)) = \{g(t,x(t)) : t \in I,\ x \in V\}$.

Using (3.5) and the properties of measure of noncompactness α, we have

$$\begin{aligned}
&\alpha(F(V)(t)) \\
&\quad \leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\overline{\text{conv}}\alpha\Bigg(f\Bigg(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j))), \\
&\qquad\qquad \sum_{k=0}^{m-1}(r_{k+1}-r_k)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Bigg)\Bigg) \\
&\quad \leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)d_1\max\Bigg(\alpha(V(I_i)), \alpha\Bigg(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\Bigg), \\
&\qquad\qquad \alpha\Bigg(\sum_{k=0}^{m-1}(r_{k+1}-r_k)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Bigg)\Bigg).
\end{aligned} \tag{3.12}$$

Let us observe that if

$$\begin{aligned}
\alpha(V(I_i)) = \max\Bigg(&\alpha(V(I_i)), \alpha\Bigg(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\Bigg), \\
&\alpha\Bigg(\sum_{k=0}^{m-1}(r_{k+1}-r_k)\overline{\text{conv}}k_2(I_i,I_k)h(I_k,V(I_k))\Bigg)\Bigg),
\end{aligned} \tag{3.13}$$

then

$$\alpha(V(t)) = \alpha(\overline{\text{conv}}(\{x(t)\}\cup F(V(t))) \leq \alpha(F(V(t))) < d_1\cdot c\cdot\alpha(V(t))) \quad \text{for every } t\in I_c. \tag{3.14}$$

Because $d_1\cdot c<1$, so $\alpha(V(t))<\alpha(V(t))$, a contradiction.

If

$$\begin{aligned}
&\alpha\Bigg(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\Bigg) \\
&\quad = \max\Bigg(\alpha(V(I_i)), \alpha\Bigg(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\Bigg), \\
&\qquad\qquad \alpha\Bigg(\sum_{k=0}^{m-1}(r_{k+1}-r_k)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Bigg)\Bigg),
\end{aligned} \tag{3.15}$$

then

$$\begin{aligned}\alpha(F(V)(t)) &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(g(I_j,V(I_j)))\\ &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(V(I_j))\\ &\leq d_1\cdot L\cdot\frac{c}{m}\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\alpha(V(I_j))\sum_{i=0}^{m-1}k_1(I_i,I_j).\end{aligned}\tag{3.16}$$

For $j=0,1,\ldots,m-1$ there exists $q_j=0,1,\ldots,m-1$ such that $k_1(I_i,I_j)\leq k_1(I_{q_j},I_j)$. So

$$\alpha(F(V)(t))\leq d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(s_j),\quad \text{where } s_j\in I_j.\tag{3.17}$$

Hence,

$$\begin{aligned}\alpha(F(V)(t))\leq{}& d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)(v(s_j)-v(p_j))\\ &+d_1\cdot L\cdot c\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(p_j).\end{aligned}\tag{3.18}$$

By the continuity of v we have $v(s_j)-v(p_j)<\varepsilon_1$ and $\varepsilon_1\to 0$ as $m\to\infty$.

Therefore,

$$\alpha(F(V)(t))\leq d_1\cdot c\cdot L\cdot\int_0^c k_1(t,s)v(s)ds\quad\text{for } t\in I_c.\tag{3.19}$$

Since $V=\overline{\text{conv}}(\{x\}\cup F(V))$, by the property of the measure of noncompactness, we have $\alpha(V(t))\leq\alpha(F(V)(t))$, so $v(t)\leq d_1\cdot c\cdot L\cdot\int_0^c k_1(t,s)v(s)ds$ for $t\in I_c$. Because this inequality holds for every t$\in I_c$ and $L\cdot d_1\cdot c\cdot r(K)<1$, so by applying Gronwall's inequality, we conclude that $\alpha(V(t))=0$ for $t\in I_c$. Hence Arzela-Ascoli's theorem implies that the set V is relatively compact. Consequently, by Theorem 1.4, F has a fixed point which is a solution of the problem (1.1).

Similarly, if

$$\begin{aligned}&\alpha\left(\sum_{k=0}^{m-1}(r_{k+1}-r_k)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\right)\\ &\quad=\max\left(\alpha(V(I_i)),\alpha\left(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\right),\right.\\ &\qquad\qquad\left.\alpha\left(\sum_{k=0}^{m-1}(r_{k+1}-r_k)\overline{\text{conv}}\,k_2(I_i,I_k)h(I_k,V(I_k))\right)\right),\end{aligned}\tag{3.20}$$

then we prove that $\alpha(F(V)(t)) \leq d_1 \cdot c \cdot L_1 \cdot \int_0^c k_2(t,s)v(s)ds$ and we conclude that the set V is relatively compact. By Theorem 1.4, F has a fixed point which is a solution of the problem (1.1). $\square$

4. An existence result for integrodifferential equations in weak sense

In this part, we prove a theorem for the existence of pseudosolutions to the Cauchy problem

$$x'(t) = f\left(t,x(t),\int_0^t k_1(t,s)g(s,x(s))ds,\int_0^a k_2(t,s)h(s,x(s))ds\right), \quad x(0) = x_0 \tag{4.1}$$

in Banach spaces. Functions f, g, h, x will be assumed Henstock-Kurzweil-Pettis integrable but this assumption is not sufficient for the existence of solutions. We impose a weak compactness-type conditions expressed in terms of measures of weak noncompactness. Throughout this part, $(E,\|\cdot\|)$ will denote a real Banach space, E^* the dual space. Unless otherwise stated, we assume that "$\int$" denotes the Henstock-Kurzweil-Pettis integral.

Fix $x^* \in E^*$ and consider the equation

$$(x^*x)'(t) = x^*f\left(t,x(t),\int_0^t k_1(t,s)g(s,x(s))ds,\int_0^a k_2(t,s)h(s,x(s))ds\right), \quad t \in I_a. \tag{4.2}$$

Now, we can introduce the following definition.

Definition 4.1. A function $x : I_a \to E$ is said to be a pseudosolution of the Cauchy problem (1.1) if it satisfies the following conditions:

(i) $x(\cdot)$ is absolutely continuous;
(ii) $x(0) = x_0$;
(iii) for each $x^* \in E^*$ there exists a negligible set $A(x^*)$ (i.e., mes $A(x^*) = 0$) such that for each $t \notin A(x^*)$,

$$(x^*x)'(t) = x^*f\left(t,x(t),\int_0^t k_1(t,s)g(s,x(s))ds,\int_0^a k_2(t,s)h(s,x(s))ds\right). \tag{4.3}$$

In this part, we use a weak measure of noncompactness of De Blasi β.

It is necessary to remark that the following lemma is true.

Lemma 4.2 [32]. *Let $H \subset C(I_a,E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a, H(t) = \{h(t) \in E,\ h \in H\}$. Then, $\beta(H(I_a)) = \sup_{t\in I_a}\beta(H(t))$ and the function $t \mapsto \beta(H(t))$ is continuous.*

Theorem 4.3. *Assume that for each ACG_* function $x : I_a \to E$, functions: $g(\cdot,x(\cdot))$, $h(\cdot,x(\cdot))$, $f(\cdot,x(\cdot))$, $\int_0^{(\cdot)}(k_1(\cdot,s)g(s,x(s)))ds$, $\int_0^a k_2(\cdot,s)h(s,x(s))ds)$ are Henstock-Kurzweil-Pettis integrable, f, g, and h are weakly-weakly sequentially continuous functions. Let k_1, $k_2 : I_a \times I_a \to \mathbb{R}_+$ be measurable functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous.*

Assume that there exist $p_0 > 0$ and positive constants L, L_1, and d such that

$$\begin{gathered}\beta(g(I,X)) \leq L\beta(X) \quad \text{for } I \subset I_a, \text{ for every } X \subset B(p_0),\\ \beta(h(I,X)) \leq L_1\beta(X) \quad \text{for } I \subset I_a, \text{ for every } X \subset B(p_0),\\ \beta(f(t,A,C,D)) \leq d \cdot \max\{\beta(A),\beta(C),\beta(D)\} \quad \text{for every } A,C,D \subset B(p_0),\ t \in I_a,\end{gathered} \tag{4.4}$$

where the sets $g(I,X)h(I,X)$ and $f(t,A,C,D)$ are defined as in Theorem 3.4 and β denotes the De Blasi measure of weak noncompactness.

Moreover, let $\Gamma(p_0)$ be equicontinuous and uniformly ACG_* on I_a. Then, there exists at least one pseudosolution of the problem (1.1) on I_c, for some $0 < c \leq a$, such that $d \cdot c \cdot L \cdot r(K) < 1$ and $d \cdot c < 1$.

Proof. By equicontinuity of $\Gamma(p_0)$, there exists a number c, $0 < c \leq a$, such that

$$\left\|\int_0^t f\left(z,x(z),\int_0^z k_1(z,s)g(s,x(s))\,ds,\int_0^c k_2(z,s)h(s,x(s))\,ds\right)dz\right\| \leq p_0, \quad \text{where } x \in B(p_0),\ t \in I_c. \tag{4.5}$$

Indeed, for any $x^* \in E^*$, such that $\|x^*\| \leq 1$ and for any $x \in B(p_0)$, we have

$$\begin{aligned}&|x^*F(x)(t)|\\ &\quad = |x^*x_0| + \left|x^*\int_0^t f\left(z,x(z),\int_0^z k_1(z,s)g(s,x(s))\,ds,\int_0^c k_2(z,s)h(s,x(s))\,ds\right)dz\right|\\ &\quad \leq \|x^*\|\|x_0\| + \|x^*\|\left\|\int_0^t f\left(z,x(z),\int_0^z k_1(z,s)g(s,x(s))\,ds,\int_0^c k_2(z,s)h(s,x(s))\,ds\right)dz\right\|\\ &\quad \leq \|x_0\| + p_0.\end{aligned} \tag{4.6}$$

From here

$$\begin{gathered}\sup\{|x^*F(x)(t)| : x^* \in E^*,\ \|x^*\| \leq 1\} \leq \|x_0\| + p_0,\\ \|F(x)(t)\| \leq \|x_0\| + p_0,\end{gathered} \tag{4.7}$$

so $F(x)(t) \in B(p_0)$.

We will show, that the operator F is weakly-weakly sequentially continuous.

By [32, Lemma 9], a sequence $x_n(\cdot)$ is weakly convergent in $C(I_c,E)$ to $x(\cdot)$ if and only if $x_n(t)$ tends weakly to $x(t)$ for each $t \in I_c$. Because $g(s,\cdot)$ and $h(s,\cdot)$ are weakly-weakly sequentially continuous, so if $x_n \xrightarrow{\omega} x$ in $(C(I_c,E),\omega)$, then $g(s,x_n(s)) \xrightarrow{\omega} g(s,x(s))$ and $h(s,x_n(s)) \xrightarrow{\omega} h(s,x(s))$in E for $t \in I_c$ and by Theorem 2.11 we have

$$\lim_{n\to\infty}\int_0^z k_1(t,s)g(s,x_n(s))\,ds = \int_0^z k_1(t,s)g(s,x(s))\,ds \tag{4.8}$$

weakly in E for each $t \in I_c$ and

$$\lim_{n\to\infty}\int_0^c k_2(t,s)h(s,x_n(s))\,ds = \int_0^c k_2(t,s)h(s,x(s))\,ds \tag{4.9}$$

weakly in E for each $t \in I_c$. Moreover, because f is weakly-weakly sequentially continuous, so $f(t,x_n(t))$, $\int_0^t k_1(t,s)g(s,x_n(s))ds$, $\int_0^c k_2(t,s)h(s,x_n(s))ds)$ tends to $f(t,x(t)$, $\int_0^t k_1(t,s)g(s,x(s))ds$, $\int_0^c k_2(t,s)h(s,x(s))ds)$ weakly in E for each $t \in I_c$.

Suppose that $V \subset B(p_0)$ satisfies the condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ for some $x \in B(p_0)$. We will prove that V is relatively compact, thus (1.4) is satisfied. Theorem 1.5 will ensure that F has a fixed point.

Let, for $t \in I_c$, $V(t) = \{v(t) \in E : v \in V\}$. Since $V \subset B(p_0)$, $F(V) \subset \Gamma(p_0)$. Then $V \subset \overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ is equicontinuous. By Lemma 4.2, $t \mapsto v(t) = \beta(V(t))$ is continuous on I_c.

Therefore, as in Theorem 3.4, we prove that $\beta(V(t)) = 0$, for t$\in I_c$, so that the set V is relatively weakly compact. Consequently, by Theorem 1.5 F has a fixed point which is a pseudosolution of the problem (1.1). □

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