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Research Article

On Semiabelian π-Regular Rings

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A ring R is defined to be *semiabelian* if every idempotent of R is either right semicentral or left semicentral. It is proved that the set $N(R)$ of nilpotent elements in a π-regular ring R is an ideal of R if and only if $R/J(R)$ is abelian, where $J(R)$ is the Jacobson radical of R. It follows that a semiabelian ring R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular, which extends the fundamental result of Badawi (1997). Moreover, several related results and examples are given.

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1. Introduction

Throughout this paper, rings are associative with unity and modules are unitary. Given a ring R, we use the symbol $\mathrm{Id}(R)$ to denote the set of idempotents in R, $U(R)$ its unit group. The Jacobson radical, the prime radical, and the set of nilpotent elements of a ring R are denoted by $J(R)$, $P(R)$, and $N(R)$, respectively. The symbol $\mathrm{Max}(R)$ (resp., $\mathrm{Max}_r(R)$) stands for the set of maximal (resp., maximal right) ideals of a ring R. As usual, the symbol $M_n(R)$ denotes the ring of $n\times n$ matrices over a ring R, $\mathrm{UTM}_n(R)$ denotes the ring of $n\times n$ upper triangular matrices over R, and E_{ij} $(1\le i,\ j\le n)$ denotes the $n\times n$ matrix units over R. Let M be an R-R bimodule and $A=(a_{ij})_{n\times n}\in M_n(R)$, we write $MA=\{(ma_{ij})_{n\times n}\mid m\in M\}$, and write $V=\sum_{i=1}^{n-1}E_{i,i+1}$ for $n\ge 2$. And we use the symbol $T_n(R,M)$ to denote the ring of $n\times n$ upper triangular matrices whose principal diagonal elements are identical and belong to R and the other elements belong to M, and write $V_n(R,M)=RI_n+MV+\cdots+MV^{n-1}$ for $n\ge 2$ where I_n is the $n\times n$ identity matrix over R. Moreover, we use the symbol $\mathbb{Z}_p$ to denote the ring of integers modulo a prime p.

Following [1], an idempotent e in a ring R is called right (resp., left) semicentral if for every $x\in R$, $ex=exe$ (resp., $xe=exe$). And the set of right (resp., left) semicentral idempotents of R is denoted by $S_r(R)$ (resp., $S_l(R)$). We define a ring R to be *semiabelian* if $\mathrm{Id}(R) = S_r(R) \cup S_l(R)$, this notion is a proper generalization of that of an abelian ring.

Recall that a ring R is called π-regular if for every $x \in R$, there exist an element $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. In the case of $n = 1$ for all $x \in R$, then R is regular. An element a in a ring R is strongly π-regular if there exist $b \in R$ and a positive integer n such that $a^n = a^{n+1}b$ with $ab = ba$. And a ring R is strongly π-regular if every element of R is strongly π-regular. Clearly, a strongly π-regular ring is a π-regular ring. A ring is called right (resp., left) quasiduo if every maximal right (resp., left) ideal is an ideal. And a ring is quasiduo if it is right and left quasiduo. A ring R is called an exchange ring if for every $a \in R$, there exists $e \in \mathrm{Id}(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. It is known that a π-regular ring is an exchange ring (see [2, Example 2.3]). A ring is reduced if it has no nonzero nilpotent elements. And a ring is abelian if every idempotent is central. It is well known that a reduced ring is an abelian ring. For the above notions we refer the reader to [3, 4].

In [5], Badawi studied abelian π-regular rings and obtained some interesting results. The fundamental result is that an abelian ring R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. In this paper, we study semiabelian π-regular rings, extending some of the main results of [5]. It is proved that for every such ring R, $N(R)$ is an ideal of R if and only if $R/J(R)$ is abelian. It follows that if R is a semiabelian ring, then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. Moreover, several related results and examples are given.

2. Extensions of semiabelian rings

We start this section with the following definition.

Definition 2.1. A ring R is called *semiabelian* if $\mathrm{Id}(R) = S_r(R) \cup S_l(R)$.

Clearly, an abelian ring is semiabelian. But the converse is not true in general as the following example shows.

Example 2.2. Let R be any ring for which $\mathrm{Id}(R) = \{0, 1\}$ (e.g., a local ring). Then $\mathrm{UTM}_2(R)$ is a semiabelian ring which is not abelian.

Proof. Clearly, $\mathrm{UTM}_2(R)$ is not abelian. And it is quite easy to check that

$$\mathrm{Id}(\mathrm{UTM}_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \mid a, b \in R \right\}, \tag{2.1}$$

and that $\left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$ is a left semicentral idempotent and $\left(\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right)$ is a right semicentral idempotent for any $a, b \in R$. Hence, $\mathrm{UTM}_2(R)$ is semiabelian.

One may expect that the conclusion of Example 2.2 is true for $n \geq 3$, but this is not the case. In fact, for any ring R, idempotent $E_{11} + E_{33}$ is neither right nor left semicentral in $\mathrm{UTM}_3(R)$. This implies that for any $n \geq 3$, $\mathrm{UTM}_n(R)$ is not semiabelian. Also the direct sum of two nonabelian semiabelian rings is not semiabelian. Now let R_1 and R_2 be semiabelian rings which are not abelian. Take $e_1 \in R_1$ to be a right semicentral idempotent which is not central and $e_2 \in R_2$ to be a left semicentral idempotent which is not central, then the idempotent (e_1, e_2) is neither right nor left semicentral in $R_1 \bigoplus R_2$.

In view of this situation, it is necessary for us to study how to obtain more examples of nonabelian semiabelian rings from a given nonabelian semiabelian ring. Clearly, the direct sum of an abelian ring and a nonabelian semiabelian ring is a semiabelian ring which is not abelian. Next we consider several extensions of semiabelian rings. □

THEOREM 2.3. *A ring R is semiabelian if and only if the ring $R[[x]]$ of formal power series over R is semiabelian.*

Proof. Assume that R is semiabelian and $f(x) \in \mathrm{Id}(R[[x]])$. Then $f(x) = e + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ with $e \in \mathrm{Id}(R)$. Now we prove that if $e \in S_r(R)$, then $f(x) \in S_r(R[[x]])$. Since $f(x)^2 = f(x)$, we have $ea_1 + a_1e = a_1$ by comparing the coefficients of x in the equation $f(x)^2 = f(x)$. Multiplying two sides of the equation $ea_1 + a_1e = a_1$ by e, we have $ea_1e = 0$, which gives $ea_1e = ea_1 = 0$ and so $a_1 = a_1e$. Assume that $ea_i = 0$ and $a_i = a_ie$ hold for all $1 \le i \le n-1$. We claim that $ea_n = 0$ and $a_ne = a_n$. In fact, comparing the coefficients of x^n in the equation $f(x)^2 = f(x)$, we have $ea_n + a_1a_{n-1} + \cdots + a_ia_{n-i} + \cdots + a_{n-1}a_1 + a_ne = a_n$. Since $a_i = a_ie$ for $1 \le i \le n-1$ in the above expression, we have $a_ia_{n-i} = a_iea_{n-i}e = 0$. It follows that $ea_n + a_ne = a_n$. Multiplying both sides of this equation by e, then $ea_ne = ea_n = 0$, which gives $a_n = a_ne$. By induction, we have $f(x) = f(x)e$ and $ef(x) = e$. Now for any $g(x) \in R[[x]]$, then $eg(x)e = eg(x)$ since $e \in S_r(R)$, and hence $f(x)g(x)f(x) = f(x)eg(x)f(x) = f(x)eg(x)ef(x) = f(x)eg(x)e = f(x)eg(x) = f(x)g(x)$. Similarly, if $e \in S_l(R)$, then $f(x) \in S_l(R[[x]])$ holds. Hence $R[[x]]$ is semiabelian. The only if part of the proof is trivial since the subring of a semiabelian ring is semiabelian. And the proof is complete. □

COROLLARY 2.4. *A ring R is semiabelian if and only if the ring $R[x]$ of polynomials over R is semiabelian.*

It is known by [6, Propositions 2.4 and 2.5] that if $f(x) = e + \sum_{i=1}^{\infty} a_ix^i \in S_l(R[[x]])$, then $e \in S_l(R)$, $ef(x) = f(x)$, and $f(x)e = e$. This is true, in particular, for a polynomial $f(x) = e + \sum_{i=1}^{n} a_ix^i \in S_l(R[[x]])$. Similarly, if $f(x) = e + \sum_{i=1}^{\infty} a_ix^i \in S_r(R[[x]])$, then $e \in S_r(R)$, $f(x)e = f(x)$, and $ef(x) = e$. And this is true especially when $f(x) \in S_r(R[x])$.

From the proof of Theorem 2.3 and the above argument, we obtain a characterization of left (resp., right) semicentral idempotents in $R[[x]]$ and $R[x]$.

PROPOSITION 2.5. *Let $f(x)$ be in $R[[x]]$ (resp., $R[x]$) with the constant term e. Then one has the following conclusions:*

(1) *$f(x) \in S_l(R[[x]])$ (resp., $S_l(R[x])$) if and only if $e \in S_l(R)$, $ef(x) = f(x)$ and $f(x)e = e$;*
(2) *$f(x) \in S_r(R[[x]])$ (resp., $S_r(R[x])$) if and only if $e \in S_r(R)$, $f(x)e = f(x)$ and $ef(x) = e$.*

Similar to the proof of Theorem 2.3, it is easy to prove the next theorem.

THEOREM 2.6. *A ring R is semiabelian if and only if the group ring RC_∞ is semiabelian, where C_∞ is the infinite cyclic group.*

THEOREM 2.7. *A ring R is semiabelian if and only if $T_n(R,M)$ is semiabelian, where M is an R-R bimodule.*

Proof. Assume that R is a semiabelian ring and $E_n \in \mathrm{Id}(T_n(R,M))$. Then

$$E_n = \begin{pmatrix} e & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & e & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & e & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & e \end{pmatrix}, \tag{2.2}$$

where $e \in \mathrm{Id}(R)$ and $a_{ij} \in M$. We claim that if $e \in S_r(R)$, then $E_n \in S_r(T_n(R,M))$. First we prove that $E_n = E_n e$ is true by induction on n. This is trivial in the case of $n = 1$. Assume that $E_{n-1} = E_{n-1}e$ holds for any $n \geq 2$. In the case of n, then $E_n = \left(\begin{smallmatrix} E_{n-1} & \alpha \\ 0 & e \end{smallmatrix}\right)$ where $\alpha = (a_{1n}, a_{2n}, \ldots, a_{n-1,n})^T$. Since E_n is an idempotent, we have $E_{n-1}\alpha + \alpha e = \alpha$, which gives $E_{n-1}\alpha e = 0$. On the other hand, since $E_{n-1}\alpha e = E_{n-1}e\alpha e = E_{n-1}e\alpha = E_{n-1}\alpha$, then $E_{n-1}\alpha = 0$. Therefore $\alpha = \alpha e$; this implies $E_n = E_n e$. Using this fact, we prove that $E_n B_n E_n = E_n B_n$ is true for any $B_n \in T_n(R,M)$. This is trivial in the case of $n = 1$. Assume that $E_{n-1}B_{n-1}E_{n-1} = E_{n-1}B_{n-1}$ holds for any $n \geq 2$ and $B_{n-1} \in T_{n-1}(R,M)$. In the case of n, we write $B_n = \left(\begin{smallmatrix} B_{n-1} & \beta \\ 0 & b_{nn} \end{smallmatrix}\right)$ where $B_{n-1} \in T_{n-1}(R,M)$ and $b_{nn} \in R$. Hence we have the following equations:

$$\begin{gathered} E_n B_n = \begin{pmatrix} E_{n-1}B_{n-1} & E_{n-1}\beta + \alpha b_{nn} \\ 0 & e b_{nn} \end{pmatrix}, \\ E_n B_n E_n = \begin{pmatrix} E_{n-1}B_{n-1}E_{n-1} & E_{n-1}B_{n-1}\alpha + E_{n-1}\beta e + \alpha b_{nn} e \\ 0 & e b_{nn} e \end{pmatrix}. \end{gathered} \tag{2.3}$$

By the assumption, $E_{n-1}B_{n-1}E_{n-1} = E_{n-1}B_{n-1}$ and $eb_{nn}e = eb_{nn}$ hold. Also, $E_n = E_n e$ implies $\alpha = \alpha e$. It follows that $\alpha b_{nn} e = \alpha e b_{nn} e = \alpha e b_{nn} = \alpha b_{nn}$ and $E_{n-1}\beta e = E_{n-1}e\beta e = E_{n-1}e\beta = E_{n-1}\beta$. Moreover, we have $E_{n-1}B_{n-1}\alpha = E_{n-1}B_{n-1}E_{n-1}\alpha = 0$ since $E_{n-1}\alpha = 0$. Hence, $E_n B_n E_n = E_n B_n$ and so $E_n \in S_r(T_n(R,M))$. Similarly, it can be proved that if $e \in S_l(R)$, then $E_n \in S_l(T_n(R,M))$. Therefore $T_n(R,M)$ is semiabelian. The only if part of the proof is trivial. □

Corollary 2.8. *A ring R is semiabelian if and only if the trivial extension $T_2(R,M)$ is semiabelian, where M is an R-R bimodule.*

Corollary 2.9. *A ring R is semiabelian if and only if $R[x]/(x^n)$ is semiabelian, where (x^n) is an ideal generated by x^n in $R[x]$.*

Proof. It is trivial in the case of $n = 1$. If $n \geq 2$, then there exists a ring isomorphism θ: $V_n(R,R) = RI_n + RV + \cdots + RV^{n-1} \to R[x]/(x^n)$ defined by $\theta(r_0 I_n + r_1 V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} + (x^n)$. By Theorem 2.7, $V_n(R,R)$ is semiabelian, so $R[x]/(x^n)$ is semiabelian. □

3. Semiabelian π-regular rings

For convenience of the reader, we list some known facts which are necessary for the study of π-regularity of rings.

Lemma 3.1 (see [7, Lemma 2.1]). *For an idempotent e in a ring R, the following conditions are equivalent:*
(1) $e \in S_r(R)$;
(2) $1 - e \in S_l(R)$;
(3) $(1 - e)Re = 0$.

Lemma 3.2 (see [4, Theorem 23.2]). *The following conditions are equivalent for a ring R:*
(1) *R is strongly* π*-regular;*
(2) *every prime factor ring of R is strongly* π*-regular;*
(3) $R/P(R)$ *is strongly* π*-regular;*

The next theorem extends [8, Theorem 1].

Theorem 3.3. *Let R be a semiabelian exchange ring. Then R/P is a local ring for every prime ideal of R.*

Proof. According to [9, Theorem 1], an exchange ring with only two idempotents is a local ring. And by [10, Lemma 4.2], a prime ring is semicentral reduced, that is, it has only 0 and 1 for its semicentral idempotents. Because R/P is a prime semiabelian exchange ring, R/P is a local ring. □

Corollary 3.4 (see [8, Theorem 1]). *Let R be an abelian exchange ring. Then R/P is a local ring for every prime ideal of R.*

Corollary 3.5. *Let R be a semiabelian exchange ring. Then R/P is a division ring for every right (resp., left) primitive ideal of R.*

Proof. Since R/P is a local ring and $J(R/P) = 0$, R/P is a division ring. □

Stock [2, Lemma 4.10] proved that if R is an exchange ring with $J(R) = 0$, then R is abelian if and only if it is reduced. Since a semiprime semiabelian ring is abelian (see [1, page 569]), we get the following lemma immediately.

Lemma 3.6. *Let R be an exchange ring with* $J(R) = 0$. *Then the following conditions are equivalent:*
(1) *R is reduced;*
(2) *R is abelian;*
(3) *R is semiabelian.*

Lemma 3.7. *Let R be a semiabelian exchange ring. Then so is every homomorphic image of R, and* $R/J(R)$ *is an abelian exchange ring.*

Proof. The first assertion is easy to prove since any homomorphic image of an exchange ring is also an exchange ring and idempotents can be lifted modulo every right ideal of R (cf. [11]). Now since $R/J(R)$ is a semiabelian exchange ring with the Jacobson radical zero, it is an abelian exchange ring by Lemma 3.6. □

Theorem 3.8. *Let R be an exchange ring with* $J(R)$ *nil. Then* $N(R)$ *is an ideal of R if and only if* $R/J(R)$ *is an abelian ring.*

Proof. ($\Rightarrow$) Since $N(R)$ is a nil ideal of R, $N(R) \subseteq J(R)$ holds. On the other hand, we have $J(R) \subseteq N(R)$ by the assumption. It follows that $J(R) = N(R)$ and so $R/J(R)$ is reduced and hence it is abelian.

($\Leftarrow$) Because R is an exchange ring and $R/J(R)$ is abelian, $R/J(R)$ is an abelian exchange ring and so it is reduced by Lemma 3.6. Hence $N(R) \subseteq J(R)$. On the other hand, $J(R) \subseteq N(R)$ by the assumption. So $N(R) = J(R)$ is an ideal of R.

It is known and easy to prove that the Jacobson radical of a π-regular ring is nil. Hence Theorem 3.8 implies that for a π-regular ring R, $N(R)$ is an ideal of R if and only if $R/J(R)$ is abelian. And [2, Example 4.16] shows that the class of exchange rings with $J(R)$ nil properly contains the class of π-regular rings.

Badawi [5, Theorem 2] proved that if R is an abelian π-regular ring, then $N(R)$ is an ideal of R. In fact, the similar result is true for a right (resp., left) quasiduo π-regular ring. □

Corollary 3.9. *If R is a right (resp., left) quasiduo π-regular ring, then $N(R)$ is an ideal of R.*

Proof. Since R is a right (resp., left) quasiduo ring, $R/J(R)$ is reduced by [12, Corollary 2] and hence it is abelian. And since R is π-regular, it is an exchange ring with $J(R)$ nil. Hence, $N(R)$ is an ideal of R by Theorem 3.8. □

Corollary 3.10. *If R is a semiabelian π-regular ring, then $N(R)$ is an ideal of R.*

Proof. Clearly, R is a semiabelian exchange ring with $J(R)$ nil, and $R/J(R)$ is abelian by Lemma 3.7. By Theorem 3.8, $N(R)$ is an ideal of R. □

Theorem 3.11. *Let R be a semiabelian ring. Then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular.*

Proof. ($\Rightarrow$) Suppose that R is π-regular. By Corollary 3.10, $N(R)$ is an ideal of R and so $R/N(R)$ is reduced and π-regular. Let $\bar{x} \in R/N(R)$. Then there exist $\bar{y} \in R/N(R)$ and a positive integer n such that $\bar{x}^n = \bar{x}^n\bar{y}\bar{x}^n$. Write $\bar{e} = \bar{x}^n\bar{y}$. Then $\bar{e} \in \mathrm{Id}(R/N(R))$ and $[(\bar{1} - \bar{e})\bar{x}]^n = \bar{0} = (\bar{1} - \bar{e})\bar{x}$ since $R/N(R)$ is reduced. Hence $R/N(R)$ is regular.

($\Leftarrow$) Assume that $N(R)$ is an ideal of R and $\widetilde{R} = R/N(R)$ is regular. Then $\widetilde{R}$ is abelian regular (and hence unit regular) since it is reduced. To prove R is π-regular, it is sufficient to prove that R/P is strongly π-regular for every prime ideal P of R by Lemma 3.2. For any $x \in R$, then $\widetilde{x} = x + N(R) \in \widetilde{R}$ is unit regular. So we have $\widetilde{x} = \widetilde{e}\widetilde{u} = \widetilde{u}\widetilde{e}$ with $e \in \mathrm{Id}(R)$ and $u \in U(R)$ since idempotents and units of $\widetilde{R}$ can be lifted modulo $N(R)$. Hence, $x = eu + w_1 = ue + w_2$ where $w_1, w_2 \in N(R)$, which implies $ex = e(u + w_1)$ and $xe = (u + w_2)e$, and $(1 - e)x = x - ex = (1 - e)w_1 \in N(R)$, $x(1 - e) = x - xe = w_2(1 - e) \in N(R)$. So there exists a positive integer n such that $[(1 - e)x]^n = [x(1 - e)]^n = 0$. Now if $e \in S_l(R)$, then $1 - e \in S_r(R)$ by Lemma 3.1. Equation $[(1 - e)x]^n = 0$ implies $(1 - e)x^n = 0$, and hence $x^n = ex^n$. If $e \in P$, then $x^n \in P$ and $\bar{x} = x + P \in N(R/P)$, so $\bar{x}$ is strongly π-regular in R/P. If $e \notin P$, then $0 = eR(1 - e) \subseteq P$ by Lemma 3.1, which gives $1 - e \in P$ and so $\bar{e} = \bar{1}$ in R/P. This implies $\bar{x} = \bar{e}\bar{x} = \overline{e(u + w_1)} = \overline{u + w_1}$ in R/P. Hence, $\bar{x}$ is a unit and so it is a strongly π-regular element in R/P. If $e \in Sr(R)$, then $1 - e \in S_l(R)$. Equation $[x(1 - e)]^n = 0$ implies $x^n(1 - e) = 0$, and hence $x^n = x^n e$. Note that $xe = (u + w_2)e$. Similar to the above proof, it can be shown that $\bar{x}$ is a nilpotent element or a unit in R/P. And the proof is completed. □

It is known by Example 2.2 that $\mathrm{UTM}_2(R)$ is a nonabelian semiabelian π-regular ring for any π-regular local ring (e.g., an artinian local ring by Lemma 3.2) R. From this we can construct more nonabelian semiabelian π-regular rings by using Theorems 2.7 and 3.11.

Corollary 3.12 (see [5, Theorem 3]). *Let R be an abelian ring. Then R is π-regular if and only if $N(R)$ is an ideal of R and $R/(R)$ is regular.*

The following corollary is an immediate result of Theorem 3.11.

Corollary 3.13. *Let R be a semiabelian π-regular ring. Then for any prime ideal P of R, every element in R/P is either a nilpotent element or a unit, and hence R is strongly π-regular with $J(R) = N(R)$.*

In light of Theorem 3.11, we naturally ask the following question.

Question 3.14. Let R be any ring. If $N(R)$ is an ideal of R and $R/N(R)$ is regular, then is R π-regular?

There are many partially positive solutions to this question (see [13–15] for the details). For a ring R with bounded index (i.e., there exists a positive integer n such that $a^n = 0$ for all $a \in N(R)$), the answer is also positive.

Proposition 3.15. *Let R be a ring with bounded index. If $N(R)$ is an ideal of R and $R/N(R)$ is regular, then R is strongly π-regular.*

Proof. It is proved in [16, Lemma 11] that if I is a right ideal of a ring R and n is a positive integer such that $a^n = 0$ for all $a \in I$, then $a^{n-1}Ra^{n-1} = 0$. Now since $R/N(R)$ is reduced and regular, it is strongly π-regular. By Lemma 3.2, it is sufficient to prove that $N(R) = P(R)$. Let m be the bounded index (the least positive integer m such that $a^m = 0$ for all $a \in N(R)$) of R. If $m = 1$, then $P(R) = N(R) = 0$. If $m \geq 2$, then $N(R) \neq 0$. We claim that $P(R) = N(R)$ is also true. If not, then $N(R)/P(R)$ is a nonzero nil ideal of $\overline{R} = R/P(R)$ with the bounded index $n \geq 2$. Thus there exists a nonzero element $\bar{a} \in N(R)/P(R)$ such that $\bar{a}^n = \bar{0}$ and $\bar{a}^{n-1} \neq 0$, so $\overline{R}\bar{a}^{n-1}\overline{R} \neq \bar{0}$. By [16, Lemma 11], $\bar{a}^{n-1}\overline{R}\bar{a}^{n-1} = \bar{0}$ and so $(\overline{R}\bar{a}^{n-1}\overline{R})^2 = \bar{0}$, which is impossible since $R/P(R)$ is a semiprime ring. So $P(R) = N(R)$, and the proof is completed. □

Theorem 3.16. *Let R be a semiabelian ring. Then R is π-regular if and only if there exists a nil ideal I of R such that R/I is π-regular.*

Proof. (⇒) If R is π-regular, then $I = N(R)$ is an ideal of R and R/I is regular by Theorem 3.11, and so we are done.

(⇐) If R/I is π-regular for some nil ideal I of R, then R/I is semiabelian π-regular by Lemma 3.7. According to Theorem 3.11, $N(R/I) = N(R)/I$ is an ideal of R/I. So $N(R)$ is a nil ideal of R. Since R/I is π-regular, $R/N(R)$ is π-regular. And since $R/N(R)$ is reduced and π-regular, $R/N(R)$ is regular by [4, Proposition 23.5]. Therefore R is π-regular by Theorem 3.11. □

A consequence of the above theorem is the following corollary.

Corollary 3.17. *Let R be a semiabelian ring. Then R is π-regular if and only if $R/P(R)$ is π-regular.*

Recall [17] that a ring R is said to have stable range one if whenever $aR+bR=R$ for $a,b\in R$, there exists $y\in R$ such that $a+by\in U(R)$. In [18], a ring R is said to satisfy the unit 1-stable condition if for any $a,b,c\in R$ with $ab+c=1$, there exists $u\in U(R)$ such that $au+c\in U(R)$. Combining [18, Corollary 4.2] with Corollary 3.5, we have the following proposition which extends [5, Theorem 6].

Proposition 3.18. *For a semiabelian exchange ring (in particular, a semiabelian π-regular ring) R, the following statements are equivalent:*
(1) *every element of R is a sum of two units;*
(2) *R satisfies the unit 1-stable range condition;*
(3) *for any factor ring R_1 of R, every element of R_1 is a sum of two units;*
(4) *$\mathbb{Z}_2$ is not a homomorphic image of R.*

4. Some remarks

In the final section, we give some remarks upon the previous results.

Remark 4.1. Every semiabelian exchange ring (in particular, a semiabelian π-regular ring) R is a quasiduo ring.

Proof. According to Theorem 3.3, a semiabelian exchange ring R is a right pm-ring in the sense that every prime ideal of R is contained in a unique maximal right ideal, equivalently, every prime ideal is contained in a unique maximal ideal. By [19], if R is a right pm-ring, then $\mathrm{Max}(R)=\mathrm{Max}_r(R)$ and hence R is a right quasiduo ring. And by [3, Theorem 4.6], an exchange ring R is right quasiduo if and only if it is left quasiduo. So every semiabelian exchange ring R is a quasiduo ring. But the converse is not true in general. □

Remark 4.2. There exists a quasiduo π-regular ring R which is not a semiabelian π-regular ring.

Proof. Let $R_1=R_2=\mathrm{UTM}_2(\mathbb{Z}_2)$ and $R=R_1\bigoplus R_2$. Then R is clearly π-regular. Since $R/J(R)\cong R_1/J(R_1)\bigoplus R_2/J(R_2)=\mathbb{Z}_2\bigoplus\mathbb{Z}_2\bigoplus\mathbb{Z}_2\bigoplus\mathbb{Z}_2$ is a commutative ring, it is a quasiduo ring and so is R. But R is not semiabelian. In fact, (E_{11},E_{22}) is neither right nor left semicentral idempotent in R where E_{11} and E_{22} are the 2×2 matrix units over $\mathrm{UTM}_2(\mathbb{Z}_2)$.

In [20], a ring R is called unit π-regular if for every $a\in R$, there exist $u\in U(R)$ and a positive integer n such that $a^n=a^nua^n$. By [21, page 3584], a strongly π-regular ring is unit π-regular, but the converse is not true in general. □

Remark 4.3. There exists a unit regular ring R which is not a strongly π-regular ring.

Proof. Let F be a field and $R=\prod_{n=1}^{\infty}M_n(F)$. Then R is unit regular since every $M_n(F)$ is unit regular. We prove that R is not strongly π-regular. Assume to the contrary, then $a=(a_1,a_2,\dots,a_n,\dots)$ is strongly π-regular, where for any positive integer n, $a_n=(a_{ij})_{n\times n}\in M_n(F)$ with $a_{ij}=0$ when $i\geq j$, and $a_{ij}=1$ when $i<j$. Hence there exist $b\in R$ and a

positive integer m such that $a^m = a^{2m}b$. It follows that $a^m_{m+1} \neq 0$ and $a^{2m}_{m+1} = 0$, which is impossible.

Ara proved in [17] that a strongly π-regular ring has stable range one. In light of Remark 4.3, we naturally ask the following question with which we conclude this paper. □

Question 4.4. Does a unit π-regular ring R have stable range one?

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