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Research Article
Sufficient Conditions for Janowski Starlikeness

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Let $A,B,D,E \in [-1,1]$ and let $p(z)$ be an analytic function defined on the open unit disk, $p(0)=1$. Conditions on A, B, D, and E are determined so that $1+\beta zp'(z)$ being subordinated to $(1+Dz)/(1+Ez)$ implies that $p(z)$ is subordinated to $(1+Az)/(1+Bz)$. Similar results are obtained by considering the expressions $1+\beta(zp'(z)/p(z))$ and $1+\beta(zp'(z)/p^2(z))$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $S^*[A,B]$ denote the class of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1). \tag{1.1}$$

Functions in $S^*[A,B]$ are called the *Janowski starlike functions* ([1, 2]). Certain well-known subclasses of starlike functions are special cases of the class $S^*[A,B]$ for suitable choices of the parameters A and B. For example, when $0 \leq \alpha < 1$, $S^*[1-2\alpha,-1] =: S^*_\alpha$ is the familiar class of starlike functions of order α and $S^*[1-\alpha,0] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1-\alpha \quad (z \in U)\} =: S^*(\alpha)$. For $0 < \alpha \leq 1$, let $S^*[\alpha,-\alpha] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \alpha|zf'(z)/f(z)+1| \quad (z \in U)\} =: S^*[\alpha]$.

Silverman [3], Obradowič and Tuneski [4], and many others (see [5–9]) have studied properties of functions defined in terms of the quotient $(1+zf''(z)/f'(z))/(zf'(z)/f(z))$. In fact, Silverman [3] has obtained the order of starlikeness for the functions in the class

G_b defined by

$$G_b := \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, 0 < b \leq 1,\ z \in U \right\} \tag{1.2}$$

and Obradowič and Tuneski [4] improved the result of Silverman [3] by showing $G_b \subset S^*[0, -b] \subset S^*(2/(1+\sqrt{1+8b}))$. Later, Tuneski [10] obtained conditions for the inclusion $G_b \subset S^*[A,B]$ to hold. If we let $zf'(z)/f(z) =: p(z)$, then $G_b \subset S^*[A,B]$ becomes

$$1 + \frac{zp'(z)}{p(z)^2} \prec 1 + bz \Longrightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{1.3}$$

Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. Frasin and Darus [11] have shown that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{2-\alpha} \Longrightarrow \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha. \tag{1.4}$$

Again by writing $z^2 f'(z)/(f(z))^2$ as $p(z)$, we see that the above implication is special case of

$$1 + \beta \frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Longrightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{1.5}$$

Another special case of the above implications was considered by Ponnusamy and Rajasekaran [12].

Nunokawa et al. [13] have shown that if $p(z)$ is analytic in U, $p(0) = 1$ and $1 + zp'(z) \prec 1 + z$, then $p(z) \prec 1 + z$. Using this, they have obtained a criterion for a normalized analytic function to be univalent. In this paper, we extend the result by replacing the subordinate function $1 + z$ by a function of the form $(1 + Dz)/(1 + Ez)$. In fact, we determine conditions on $A, B, D, E \in [-1, 1]$ so that

$$1 + \beta zp'(z) \prec \frac{1+Dz}{1+Ez} \Longrightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{1.6}$$

Similar results are obtained by considering the expressions $1 + \beta(zp'(z)/p^2(z))$, $1 + \beta(zp'(z)/p(z))$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

2. Differential subordination

Lemma 2.1. *Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$, and $\beta \neq 0$. Assume that*

$$(A - B)|\beta| \geq (D - E)(1 + B^2) + |2B(D - E) - E\beta(A - B)|. \tag{2.1}$$

If $p(z)$ is analytic in U with $p(0)=1$ and

$$1+\beta z p'(z) \prec \frac{1+Dz}{1+Ez}, \tag{2.2}$$

then

$$p(z) \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

Proof. Define the function $P(z)$ by

$$P(z) := 1+\beta z p'(z), \tag{2.4}$$

and the function $w(z)$ by

$$w(z) := \frac{p(z)-1}{A-Bp(z)}, \tag{2.5}$$

or equivalently by

$$p(z) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.6}$$

Then $w(z)$ is meromorphic in U and $w(0)=0$. We need to show that $|w(z)|<1$ in U. By a computation, we get

$$P(z) = \frac{(1+Bw(z))^2 + (A-B)\beta z w'(z)}{(1+Bw(z))^2}. \tag{2.7}$$

Therefore

$$\frac{P(z)-1}{D-EP(z)} = \frac{(A-B)\beta z w'(z)}{(D-E)(1+Bw(z))^2 - E(A-B)\beta z w'(z)}. \tag{2.8}$$

Assume that there exists a point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1. \tag{2.9}$$

Then by [14, Lemma 1.3, page 28], there exists $k \ge 1$ such that $z_0 w'(z_0) = k w(z_0)$. Let $w(z_0)=e^{i\theta}$. For this z_0, we have

$$\begin{aligned} \left| \frac{P(z_0)-1}{D-EP(z_0)} \right| &= \frac{(A-B)k|\beta|}{\left[I^2+(H-J)^2+4HJt^2+4I(H+J)t\right]^{1/2}} \\ &\ge \frac{(A-B)k|\beta|}{\max_{-1\le t\le 1}\left\{\left[I^2+(H-J)^2+4HJt^2+4I(H+J)t\right]^{1/2}\right\}}, \end{aligned} \tag{2.10}$$

where $I := 2B(D-E) - k\beta E(A-B)$, $J := (D-E)B^2$, $H := (D-E)$, and $t := \cos\theta$. A computation shows that

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{(A-B)|\beta|k}{H+|I|+J}. \tag{2.11}$$

Yet another calculation shows that the function $\psi(k) := (A-B)|\beta|k/(H+|I|+J)$ is an increasing function of k. Since $k \geq 1$, we have $\psi(k) \geq \psi(1)$ and therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{(A-B)|\beta|}{(D-E)(1+B^2)+|2B(D-E)-E\beta(A-B)|}, \tag{2.12}$$

which by (2.1) is greater than or equal to 1. This contradicts $P(z) \prec (1+Dz)/(1+Ez)$ and completes the proof. □

Remark 2.2. When $\beta = 1$, $E = 0 = B$, and $D = 1 = A$, Lemma 2.1 reduces to [13, Lemma 1, page 1035]. Further if $p(z) = z^2 f'(z)/f(z)^2$, Lemma 2.1 reduces to [13, Theorem 1, page 1036].

By taking $p(z) = zf'(z)/f(z)$ in Lemma 2.1, we have the following result.

THEOREM 2.3. *Let the conditions of Lemma 2.1 hold. If $f \in \mathcal{A}$ satisfies*

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+Dz}{1+Ez}, \tag{2.13}$$

then $f \in S^*[A,B]$.

By taking $\beta = 1$, $A = \alpha = -B$, and $D = -E = \delta$ $(0 < \alpha, \delta \leq 1)$ in Theorem 2.3, we have the following result.

COROLLARY 2.4. *Let $0 < \alpha \leq 1$ and $\delta = \alpha/(1+\alpha)^2$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \delta\left|2+\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|, \tag{2.14}$$

then $f(z) \in \mathcal{S}^*[\alpha]$.

By taking $\beta = 1$ $A = 1-2\alpha$, $B = -1$, $D = (1-\alpha)/2$, and $E = 0$ $(0 \leq \alpha < 1)$ in Theorem 2.3, we have the following result.

COROLLARY 2.5. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{1-\alpha}{3} \quad (0 \leq \alpha < 1), \tag{2.15}$$

then $f(z) \in \mathcal{S}^*_\alpha$.

By replacing $p(z)$ by $1/p(z)$, $\beta = -1$, A by $-B$, and B by $-A$ in Lemma 2.1, we have the following result.

Lemma 2.6. *Let* $-1 \le B < A \le 1$, $-1 \le E < D \le 1$. *Assume that*

$$(A-B) \ge (D-E)\left(1+A^2\right) + \left|E(A-B) - 2A(D-E)\right|. \tag{2.16}$$

If $p(z)$ *is analytic in* U *with* $p(0) = 1$ *and*

$$1 + \frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez}, \tag{2.17}$$

then

$$p(z) \prec \frac{1+Az}{1+Bz}. \tag{2.18}$$

When $p(z) = zf'(z)/f(z)$, in Lemma 2.6, we have the following theorem.

Theorem 2.7. *Let* $-1 \le B < A \le 1$, $-1 \le E < D \le 1$. *Assume that (2.16) holds. If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \frac{1+Dz}{1+Ez}, \tag{2.19}$$

then $f \in S^*[A,B]$.

Example 2.8. If $f \in G_{1-\alpha/(2-\alpha)^2}$ $(0 \le \alpha < 1)$, then $f \in S^*(\alpha)$. If $f \in \mathcal{A}$ satisfies

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \beta \left|1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}\right| \quad \left(\beta = \frac{\alpha}{1+3\alpha+\alpha^2},\ 0 < \alpha \le 1\right), \tag{2.20}$$

then $f \in S^*[\alpha]$. Similarly if (2.20) holds with $\beta = (1-\alpha)/[1+(1-2\alpha)^2 + |5\alpha - 3|]$ $(0 \le \alpha < 1)$, then $f \in S^*_\alpha$.

Remark 2.9. When $E = 0$ and $D = b$ $(0 < b \le 1)$, Corollary 2.5 reduces to [10, Corollary 2.6, page 203]. When $A = 0 = E$ and $D = -B = b$ $(0 < b \le 1)$, Corollary 2.5 reduces to [4, Theorem 1, page 61]. When $A = 0 = E$ and $D = -B = 1$, Corollary 2.5 reduces to [3, Corollary 1, page 76].

Lemma 2.10. *Let* $-1 \le B < A \le 1$, $-1 \le E < D \le 1$, $AB \ge 0$, *and* $\beta \ne 0$. *Assume that*

$$|\beta|(A-B) \ge (D-E)(1+AB) + \left|(D-E)(A+B) - E\beta(A-B)\right|. \tag{2.21}$$

Let $p(z)$ *be analytic in* U *with* $p(0) = 1$ *and*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}, \tag{2.22}$$

then

$$p(z) \prec \frac{1+Az}{1+Bz}. \tag{2.23}$$

Proof. The proof is similar to the proof of Lemma 2.1. □

Remark 2.11. When $E\beta \leq 0$, $AB \leq 0$, Lemma 2.10 is valid provided the following conditions hold:

$$(1-A\beta)^2\{2E\beta(A+B)(D-E)-(A-B)[(D-E)^2+(E\beta)^2]\} \geq 4\beta^2(A-B)AB \tag{2.24}$$

instead of (2.21).

Remark 2.12. When $\beta = -1$, $A = \lambda = E$, and $D = B = 0(|\lambda| \leq 1)$, Lemma 2.10 reduces to [12, Theorem 1(iii), page 195].

Example 2.13. By taking $\beta = 1$, $B = 0$, $D = A/(1+A)$, and $E = 0$ in Lemma 2.10, we have the following result. Let $0 < A \leq 1$. Let $p(z)$ be analytic in U with $p(0) = 1$. If $|zp'(z)/p(z)| < A/(1+A)$, then $p(z) \prec 1+Az$. When $p(z) = zf'(z)/f(z)$, $A = 1-\alpha$, we have the following result.

If $f(z) \in \mathcal{A}$ satisfies

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| < \frac{1-\alpha}{2-\alpha} \quad (0 \leq \alpha < 1), \tag{2.25}$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

By taking $p(z) = z^2f'(z)/f^2(z)$ in Lemma 2.10, we have the following result.

THEOREM 2.14. *Let the conditions of Lemma 2.10 hold. If $f \in \mathcal{A}$ satisfies*

$$1+\beta\left(\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right) \prec \frac{1+Dz}{1+Ez}, \tag{2.26}$$

then

$$\frac{z^2f'(z)}{f^2(z)} \prec \frac{1+Az}{1+Bz}. \tag{2.27}$$

Remark 2.15. When $\beta = 1$, $A = \alpha$, $B = 0$, $E = 0$, and $D = (1-\alpha)/(2-\alpha)$ $(0 \leq \alpha < 1)$, Theorem 2.14 reduces to [11, Theorem 2.4, page 307].

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