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Research Article
On Classical Quotient Rings of Skew Armendariz Rings

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Let R be a ring, α an automorphism, and δ an α-derivation of R. If the classical quotient ring Q of R exists, then R is weak α-skew Armendariz if and only if Q is weak $\widetilde{\alpha}$-skew Armendariz.

1. Introduction

For a ring R with a ring endomorphism, $\alpha : R\to R$ and an α-derivation δ of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$, we denote by $R[x;\alpha,\delta]$ the skew polynomial ring whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$.

Rege and Chhawchharia [1] called a ring R an *Armendariz* ring if whenever any polynomial $f(x)=a_0+a_1x+\cdots+a_nx^n$, $g(x)=b_0+b_1x+\cdots+b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. This nomenclature was used by them since it was Armendariz [2, Lemma 1] who initially showed that a *reduced* ring (i.e., a ring without nonzero nilpotent elements) always satisfies this condition. A number of papers have been written on the Armendariz property of rings. For basic and other results on Armendariz rings, see, for example, [1–11].

The Armendariz property of rings was extended to skew polynomial rings with skewed scalar multiplication in [7].

For an endomorphism α of a ring R, R is called an *α-skew Armendariz* ring (or, a skew Armendariz ring with the endomorphism α) if for $p = \sum_{i=0}^{m} a_ix^i$ and $q = \sum_{j=0}^{n} b_jx^j \in R[x;\alpha]$, $pq = 0$ implies $a_i\alpha^i(b_j) = 0$ for all $0 \le i \le m$ and $0 \le j \le n$.

Recall that an endomorphism α of a ring R is called *rigid* (see [5] and [12]) if $a\alpha(a)=0$ implies $a=0$ for $a\in R$. R is called an *α-rigid ring* [12] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced by [12, Propositions 5 and 6].

If R is an α-rigid ring, then for $p=\sum_{i=0}^{m} a_i x^i$ and $q=\sum_{j=0}^{n} b_j x^j \in R[x;\alpha,\delta]$, $pq=0$ if and only if $a_i b_j=0$ for all $0\le i\le m$ and $0\le j\le n$ [12, Proposition 6].

Various properties of the Ore extensions have been investigated by many authors; (see [1–13]). Most of these have worked either with the case $\delta=0$ and α-automorphism or the case where α is the identity. However, the recent surge of interest in quantum groups and quantized algebras has brought renewed interest in general skew polynomial rings, due to the fact that many of these quantized algebras and their representations can be expressed in terms of iterated skew polynomial rings. This development calls for a thorough study of skew polynomial rings.

Anderson and Camillo [3] assert that for a semiprime left and right Noetherian ring R, R is Armendariz if and only if the classical right quotient ring $Q(R)$ of R is reduced. Anderson and Camillo [3, Theorem 7] proved that if R is a prime ring which is left and right Noetherian, then R is Armendariz if and only if R is reduced. Kim and Lee in [9] obtained this result under a weaker condition. They proved that if R is a semiprime right and left Goldie ring, then R is Armendariz if and only if R is reduced. Kim and Lee also proved that if there exists the classical right quotient ring $Q(R)$ of a ring R, then R is reduced if and only if $Q(R)$ is reduced.

In this paper, we obtain a generalized result of [3, Theorem 7] and [9, Theorem 16], for reduced rings, onto α-skew Armendariz rings [9, Proposition 18 and Corollary 19] as corollaries.

2. The results

We first give the following definition of α-skew Armendariz ring, and notice that our definition is compatible with Hong et al.'s [7], assuming δ to be the zero mapping.

Definition 2.1. Let R be a ring with a ring endomorphism α and an α-derivation δ. R is an *α-skew Armendariz* ring (or, a skew Armendariz ring with the endomorphism α) if for $p=\sum_{i=0}^{m} a_i x^i$ and $q=\sum_{j=0}^{n} b_j x^j \in R[x;\alpha,\delta]$, $pq=0$ implies $a_i\alpha^i(b_j)=0$ for all $0\le i\le m$ and $0\le j\le n$.

We notice that to extend Armendariz property to the Ore extension $R[x;\alpha,\delta]$, we do not need any more conditions. In the case $\delta=0$, several examples of α-skew Armendariz rings are obtained in [7], and in the same method, one can provide similar results as in [7], for the more general cases, of the Ore extension $R[x;\alpha,\delta]$. For instance, it is easy to prove that α-rigid rings are α-skew Armendariz.

Definition 2.2. Let R be a ring with a ring endomorphism α and an α-derivation δ. R is a *weak α-skew Armendariz* ring, if for linear polynomials $f(x)=a_0+a_1x$ and $g(x)=b_0+b_1x\in R[x;\alpha,\delta]$, $f(x)g(x)=0$ implies $a_i\alpha^i(b_j)=0$ for all $0\le i,\ j\le 1$.

Note that an α-skew Armendariz ring is trivially a weak α-skew Armendariz ring and a subring of an α-skew Armendariz ring is also α-skew Armendariz; while for the identity endomorphism I_R of a ring R, R is Armendariz if and only if R is I_R-skew Armendariz and $\delta = 0$.

Let R be a ring with the classical right (left) quotient ring Q. Then each injective endomorphism α and α-derivation δ of R extends to Q, respectively, by setting $\widetilde{\alpha}(c^{-1}r) = \alpha(c)^{-1}\alpha(r)$ and $\widetilde{\delta}(c^{-1}r) = \alpha(c)^{-1}(\delta(r) - \delta(c)c^{-1}r)$, for each $r, c \in R$ with c regular.

A ring R is called right Ore given $a, b \in R$ with b regular if there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists the classical right quotient ring of R.

Note that every Ore domain is an α-skew Armendariz ring for every automorphism α and α-derivation δ.

Now we obtain a generalized result of [9, Theorem 16], for reduced rings, onto α-skew Armendariz rings, by showing that the weak α-skew Armendariz condition extends to its classical quotient ring.

Theorem 2.3. *Let R be a ring, α an automorphism, and δ an α-derivation of R. If the classical quotient ring Q of R exists, then R is weak α-skew Armendariz if and only if Q is weak $\widetilde{\alpha}$-skew Armendariz.*

Proof. Let $f(x) = c_0^{-1}a_0 + c_1^{-1}a_1x$ and $g(x) = s_0^{-1}b_0 + s_1^{-1}b_1x \in Q[x;\widetilde{\alpha},\widetilde{\delta}]$ such that $f(x)g(x) = 0$. Then there exist $a'_i, b'_j \in R$ and regular elements $c, s \in R$ such that $c_i^{-1}a_i = c^{-1}a'_i$ and $s_j^{-1}b_j = s^{-1}b'_j$. We then have $(c^{-1}a'_0 + c^{-1}a'_1x)(s^{-1}b'_0 + s^{-1}b'_1x) = 0$. So $(a'_0 + a'_1x)s^{-1}(b'_0 + b'_1x) = 0$ and hence $[a'_0s^{-1} + a'_1\widetilde{\delta}(s^{-1}) + a'_1\widetilde{\alpha}(s^{-1})x](b'_0 + b'_1x) = 0$. Then $[a'_0s^{-1} - a'_1\alpha(s)^{-1}\delta(s)s^{-1} + a'_1\alpha(s)^{-1}x](b'_0 + b'_1x) = 0$. Now there exist $d, s_2 \in R$, with s_2 regular, such that $\delta(s)s^{-1} = s_2^{-1}d$. Thus $[a'_0s^{-1} - a'_1\alpha(s)^{-1}s_2^{-1}d + a'_1\alpha(s)^{-1}x](b'_0 + b'_1x) = 0$. There exist $a''_0, a''_1, a'''_1, s_3, s_4, s_5 \in R$, with s_3, s_4, s_5 regulars, such that $a'_0s^{-1} = s_3^{-1}a''_0, a'_1\alpha(s)^{-1}s_2^{-1} = s_4^{-1}a''_1, a'_1\alpha(s)^{-1} = s_5^{-1}a'''_1$. So $[s_3^{-1}a''_0 - s_4^{-1}a''_1d + s_5^{-1}a''_1x](b'_0 + b'_1x) = 0$. There exist $t, d_0, d_1, d_2 \in R$, with t regular, such that $s_3^{-1}a''_0 = t^{-1}d_0, s_4^{-1}a''_1 = t^{-1}d_1, s_5^{-1}a'''_1 = t^{-1}d_2$. Hence $t^{-1}(d_0 - d_1d + d_2x)(b'_0 + b'_1x) = 0$. Now the Armendariz condition implies the following equations:

$$\begin{aligned}
(d_0 - d_1d)b'_0 &= 0,\\
(d_0 - d_1d)b'_1 &= 0,\\
d_2\alpha(b'_0) &= 0,\\
d_2\alpha(b'_1) &= 0.
\end{aligned} \tag{2.1}$$

We have $(d_0 - d_1d + d_2x)(b'_0 + b'_1x) = 0$, so $(d_0 - d_1d)b'_0 + d_2\delta(b'_0) + ((d_0 - d_1d)b'_1 + d_2\alpha(b'_0) + d_2\delta(b'_1))x + d_2\alpha(b'_1)x^2 = 0$. Thus we get the following equations:

$$\begin{gathered}
(d_0 - d_1d)b'_0 + d_2\delta(b'_0) = 0,\\
(d_0 - d_1d)b'_1 + d_2\alpha(b'_0) + d_2\delta(b'_1) = 0,\\
d_2\alpha(b'_1) = 0.
\end{gathered} \tag{2.2}$$

These equations and (2.1) imply that

$$d_2\delta(b_0') = 0,$$
$$d_2\delta(b_1') = 0. \tag{2.3}$$

Now we have

$$(d_0 - d_1 d)b_0' = 0 \Longleftrightarrow t^{-1}(d_0 - d_1 d)b_0' = 0 \Longleftrightarrow (s_3^{-1}a_0'' - s_4^{-1}a_1'' d)b_0' = 0$$
$$\Longleftrightarrow (a_0' s^{-1} - a_1'\alpha(s)^{-1}s_2^{-1}d)b_0' = 0 \Longleftrightarrow (a_0' s^{-1} - a_1'\alpha(s)^{-1}\delta(s)s^{-1})b_0' = 0$$
$$\Longleftrightarrow (a_0' s^{-1} + a_1' \ \widetilde{\delta}(s^{-1}))b_0' = 0. \tag{2.4}$$

A similar argument shows that

$$(d_0 - d_1 d)b_1' = 0 \Longleftrightarrow (a_0' s^{-1} + a_1'\widetilde{\delta}(s^{-1}))b_1' = 0. \tag{2.5}$$

Also we have

$$d_2\delta(b_0') = 0 \Longleftrightarrow t^{-1}d_2\delta(b_0') = 0 \Longleftrightarrow s_5^{-1}a_1'''\delta(b_0') = 0 \Longleftrightarrow a_1'\alpha(s)^{-1}\delta(b_0') = 0$$
$$\Longleftrightarrow a_1' \ \widetilde{\alpha}(s^{-1}) \ \widetilde{\delta} \ (b_0') = 0. \tag{2.6}$$

A similar argument shows that

$$d_2\delta(b_1') = 0 \Longleftrightarrow a_1' \ \widetilde{\alpha}(s^{-1}) \ \widetilde{\delta}(b_1') = 0, \tag{2.7}$$
$$d_2\alpha(b_0') = 0 \Longleftrightarrow a_1' \ \widetilde{\alpha}(s^{-1}) \ \widetilde{\alpha}(b_0') = 0, \tag{2.8}$$
$$d_2\alpha(b_1') = 0 \Longleftrightarrow a_1' \ \widetilde{\alpha}(s^{-1}) \ \widetilde{\alpha}(b_1') = 0. \tag{2.9}$$

Now take $h(x) = a_0' + a_1' x$ and $k(x) = s^{-1}b_1' \in Q[x;\widetilde{\alpha},\widetilde{\delta}]$, and using (2.9), (2.5), (2.7), then we get

$$h(x)k(x) = (a_0' + a_1' x)(s^{-1}b_1') = a_0' s^{-1}b_1' + a_1'\widetilde{\alpha}(s^{-1}b_1')x + a_1'\widetilde{\delta}\ (s^{-1}b_1')$$
$$= a_0' s^{-1}b_1' + a_1'\widetilde{\delta}\ (s^{-1}b_1') + a_1'\widetilde{\alpha}\ (s^{-1})\ \widetilde{\alpha}\ (b_1')x$$
$$= a_0' s^{-1}b_1' + a_1'\widetilde{\delta}\ (s^{-1})b_1' + a_1'\widetilde{\alpha}\ (s^{-1})\ \widetilde{\delta}\ (b_1')$$
$$= (a_0' s^{-1} + a_1'\widetilde{\delta}(s^{-1}))b_1' + a_1'\widetilde{\alpha}\ (s^{-1})\ \widetilde{\delta}(b_1') = 0. \tag{2.10}$$

Therefore, we have $(a_0' + a_1' x)(s^{-1}b_1') = 0$. Now there exist $m, n \in R$ with n regular, we get $s^{-1}b_1' = mn^{-1}$. Thus $(a_0' + a_1' x)mn^{-1} = 0$, hence $(a_0' + a_1' x)m = 0$. Since R is weak skew-Armendariz, we can deduce that $a_0' m = a_1'\delta(m) = 0$. But

$$a_0' m = 0 \Longleftrightarrow a_0' mn^{-1} = 0 \Longleftrightarrow a_0' s^{-1}b_1' = 0. \tag{2.11}$$

Equations (2.5) and (2.11) imply that

$$a_1' \widetilde{\delta}(s^{-1})b_1' = 0. \tag{2.12}$$

Now take $p(x) = a_0' + a_1'x$ and $q(x) = s^{-1}b_0'$, using (2.8), (2.4), (2.6) then we get

$$\begin{aligned} p(x)q(x) &= (a_0' + a_1'x)s^{-1}b_0' = a_0's^{-1}b_0' + a_1' \widetilde{\alpha}(s^{-1}b_0')x + a_1' \widetilde{\delta}(s^{-1}b_0') \\ &= a_0's^{-1}b_0' + a_1' \widetilde{\delta}(s^{-1})b_0' + a_1' \widetilde{\alpha}(s^{-1}) \widetilde{\delta}(b_0') + a_1' \widetilde{\alpha}(s^{-1}) \widetilde{\alpha}(b_0')x \\ &= (a_0's^{-1} + a_1' \widetilde{\delta}(s^{-1}))b_0' + a_1' \widetilde{\alpha}(s^{-1}) \widetilde{\delta}(b_0') = 0. \end{aligned} \tag{2.13}$$

So $(a_0' + a_1'x)s^{-1}b_0' = 0$. But there exist $u, v \in R$ with v regular such that $s^{-1}b_0' = uv^{-1}$. Thus $(a_0' + a_1'x)uv^{-1} = 0$ and hence $(a_0' + a_1'x)u = 0$. The Armendariz condition implies that $a_0'u = 0$, and so $a_0'uv^{-1} = 0$. So we get

$$a_0's^{-1}b_0' = 0. \tag{2.14}$$

By (2.14) and (2.4), we have

$$a_1' \widetilde{\delta}(s^{-1})b_0' = 0. \tag{2.15}$$

By (2.14),

$$a_0's^{-1}b_0' = 0 \Longleftrightarrow c^{-1}a_0's_0^{-1}b_0 = 0 \Longleftrightarrow c_0^{-1}a_0s_0^{-1}b_0 = 0. \tag{2.16}$$

By (2.11),

$$a_0's^{-1}b_1' = 0 \Longleftrightarrow c^{-1}a_0's_1^{-1}b_1 = 0 \Longleftrightarrow c_0^{-1}a_0s_1^{-1}b_1 = 0. \tag{2.17}$$

By (2.6) and (2.15), we have $a_1' \widetilde{\delta}(s^{-1})b_0' = 0 = a_1' \widetilde{\alpha}(s^{-1}) \widetilde{\delta}(b_0')$. Thus

$$\begin{aligned} a_1'(\widetilde{\delta}(s^{-1})b_0' + \widetilde{\alpha}(s^{-1}) \widetilde{\delta}(b_0')) = 0 \Longleftrightarrow a_1' \widetilde{\delta}(s^{-1}b_0') = 0 \Longleftrightarrow c^{-1}a_1' \widetilde{\delta}(s^{-1}b_0') = 0 \\ \Longleftrightarrow c_1^{-1}a_1 \widetilde{\delta}(s_0^{-1}b_0) = 0. \end{aligned} \tag{2.18}$$

By (2.12) and (2.7), we have $a_1'\widetilde{\delta}(s^{-1})b_1' = 0 = a_1'\widetilde{\alpha}(s^{-1})\widetilde{\delta}(b_1')$. Thus

$$a_1'\widetilde{\delta}(s^{-1})b_1' + a_1'\widetilde{\alpha}(s^{-1})\widetilde{\delta}(b_1') = 0 \Longleftrightarrow a_1'\widetilde{\delta}(s^{-1}b_1') = 0 \Longleftrightarrow c_1^{-1}a_1\widetilde{\delta}(s_1^{-1}b_1) = 0. \tag{2.19}$$

Using $(c_0^{-1}a_0 + c_1^{-1}a_1x)(s_0^{-1}b_0 + s_1^{-1}b_1x) = 0$ and (2.16), (2.17), (2.18), (2.19), we also have $c_1^{-1}a_1\widetilde{\alpha}(s_0^{-1}b_0) = 0$ and $c_1^{-1}a_1\widetilde{\alpha}(s_1^{-1}b_1) = 0$. Therefore Q is a weak $\widetilde{\alpha}$-skew Armendariz ring. □

Now we show that skew-Armendariz rings are Abelian (i.e., every idempotent is central).

Lemma 2.4. *Every weak α-skew Armendariz ring is Abelian.*

Proof. Let R be a weak α-skew Armendariz ring and let $e^2 = e$, $a \in R$. Consider the polynomials $f(x) = e - ea(1-e)x$ and $g(x) = 1 - e + ea(1-e)x \in R[x;\alpha,\delta]$. Then we have $f(x)g(x) = 0$. Since R is weak skew Armendariz, $eea(1-e) = 0$. So $ea = eae$. Next let $h(x) = 1 - e - (1-e)aex$ and $k(x) = e + (1-e)aex \in R[x;\alpha,\delta]$. We have $h(x)k(x) = 0$ and since R is weak skew Armendariz, it implies that $(1-e)(1-e)ae = 0$. Thus $ae = eae$ and so $ae = ea$ which implies that R is Abelian. □

Corollary 2.5. *Let R be a semiprime Goldie ring and α-automorphism and δ an α-derivation of R. Then the following are equivalent:*

(1) *R is weak α-skew Armendariz;*
(2) *R is α-skew Armendariz;*
(3) *Q is $\widetilde{\alpha}$-skew Armendariz;*
(4) *Q is weak $\widetilde{\alpha}$-skew Armendariz;*
(5) *R is α-rigid;*
(6) *Q is $\widetilde{\alpha}$-rigid.*

Proof. The proof follows by Theorem 2.3. For the implication $2 \Rightarrow 5$, notice that when R is a weak α-skew Armendariz ring, then by Theorem 2.3, Q is weak $\widetilde{\alpha}$-skew Armendariz and hence Q is Abelian by Lemma 2.4 so Q is an aAbelian semisimple ring and hence is reduced. Now, suppose that $a\widetilde{\alpha}(a) = 0$ for $a \in Q$. So we have $\widetilde{\delta}(a)\widetilde{\alpha}(a) = \widetilde{\alpha}(a)\widetilde{\delta}(\widetilde{\alpha}(a)) = 0$. Now, let $h(x) = \widetilde{\alpha}(a) - \widetilde{\alpha}(a)x$ and $k(x) = a + \widetilde{\alpha}(a)x \in Q[x;\widetilde{\alpha},\widetilde{\delta}]$. Then $h(x)k(x) = 0$. Since Q is weak $\widetilde{\alpha}$-skew Armendariz, we have $\widetilde{\alpha}(a)\widetilde{\alpha}(a) = 0$. But Q is reduced and $\widetilde{\alpha}$ is a monomorphism, therefore $a = 0$. Thus Q is $\widetilde{\alpha}$-rigid, so R is α-rigid. □

By Corollary 2.5, it is shown that a semiprime right Goldie ring R with an automorphism α is weak α-skew Armendariz if and only if it is reduced.

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References

[1] M. B. Rege and S. Chhawchharia, "Armendariz rings," *Proceedings of the Japan Academy, Series A*, vol. 73, no. 1, pp. 14–17, 1997.

[2] E. P. Armendariz, "A note on extensions of Baer and p.p.-rings," *Journal of the Australian Mathematical Society, Series A*, vol. 18, pp. 470–473, 1974.

[3] D. D. Anderson and V. Camillo, "Armendariz rings and Gaussian rings," *Communications in Algebra*, vol. 26, no. 7, pp. 2265–2272, 1998.

[4] E. Hashemi and A. Moussavi, "Polynomial extensions of quasi-Baer rings," *Acta Mathematica Hungarica*, vol. 107, no. 3, pp. 207–224, 2005.

[5] Y. Hirano, "On annihilator ideals of a polynomial ring over a noncommutative ring," *Journal of Pure and Applied Algebra*, vol. 168, no. 1, pp. 45–52, 2002.

[6] C. Y. Hong, T. K. Kwak, and S. T. Rizvi, "Extensions of generalized Armendariz rings," *Algebra Colloquium*, vol. 13, no. 2, pp. 253–266, 2006.

[7] C. Y. Hong, N. K. Kim, and T. K. Kwak, "On skew Armendariz rings," *Communications in Algebra*, vol. 31, no. 1, pp. 103–122, 2003.

[8] C. Huh, Y. Lee, and A. Smoktunowicz, "Armendariz rings and semicommutative rings," *Communications in Algebra*, vol. 30, no. 2, pp. 751–761, 2002.

[9] N. K. Kim and Y. Lee, "Armendariz rings and reduced rings," *Journal of Algebra*, vol. 223, no. 2, pp. 477–488, 2000.

[10] T.-K. Lee and T.-L. Wong, "On Armendariz rings," *Houston Journal of Mathematics*, vol. 29, no. 3, pp. 583–593, 2003.

[11] A. Moussavi and E. Hashemi, "On (α, δ)-skew Armendariz rings," *Journal of the Korean Mathematical Society*, vol. 42, no. 2, pp. 353–363, 2005.

[12] C. Y. Hong, N. K. Kim, and T. K. Kwak, "Ore extensions of Baer and p.p.-rings," *Journal of Pure and Applied Algebra*, vol. 151, no. 3, pp. 215–226, 2000.

[13] A. Moussavi and E. Hashemi, "Semiprime skew polynomial rings," *Scientiae Mathematicae Japonicae*, vol. 64, no. 1, pp. 91–95, 2006.

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