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Research Article

Use of a Strongly Nonlinear Gambier Equation for the Construction of Exact Closed Form Solutions of Nonlinear ODEs

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We establish an analytical method leading to a more general form of the exact solution of a nonlinear ODE of the second order due to Gambier. The treatment is based on the introduction and determination of a new function, by means of which the solution of the original equation is expressed. This treatment is applied to another nonlinear equation, subjected to the same general class as that of Gambier, by constructing step by step an appropriate analytical technique. The developed procedure yields a general exact closed form solution of this equation, valid for specific values of the parameters involved and containing two arbitrary (free) parameters evaluated by the relevant initial conditions. We finally verify this technique by applying it to two specific sets of parameter values of the equation under consideration.

1. Introduction

The class of equations solved by elliptic functions, like (2.1) we first examine here (Section 2), triggered off the problem of the classification of the general nonlinear differential equation to special categories with respect to the character of the singular points of the solutions. The investigation was undertaken by E. Picard, P. Painlevé, B. Gambier, and their associates around the beginning of the 20th century, and led to the production of a large number of memoirs, listed by Davis in [1, Bibliography]. We refer here to [2–9]. The problem was also extensively presented in 1927 by Ince in [10].

The French analysts mentioned above adopted in their study the polynomial form of a second-order nonlinear differential equation [1, Chapter 8, Section 1, equation (1); Section 5, equation (3)] and tried to establish conditions under which the critical points of a solution, that is to say branch points and essential singularities, would be fixed points, while the functions representing the solutions of these equations would have only poles as movable singularities. Their work resulted in the discovery of 50 types of equations (recorded by Davis in [1, Appendix 1]) possessing the above property. Of these, all but 6 are integrable in terms of classical and (or) transcendental functions. The remaining six equations, known as *Painlevé equations* (due to both Painlevé and Gambier), require the introduction of new transcendental functions for their solution, the so-called *Painlevé transcendents*.

In addition, many nonlinear ODEs of the second- as well as of the first-order, governing various problems in Mechanics and Physics, do not accept analytical solutions in terms of known functions. As an example, we mention the nonlinear oscillator equations (Rayleigh, Van der Pol, Duffing) which have been investigated thoroughly in the literature (also by the present author, see [11]), as well as the most cases of Abel equations of the second kind (the few solvable cases are presented by Polyanin and Zaitsev [12]). Thus, the establishment of methods leading to the construction of new functions, by means of which exact analytical solutions can be extracted, would be a desirable advance in the theory of nonlinear differential equations.

In this work, aiming at this target, in Section 2 we treat a nonlinear equation due to Gambier (equation (2.1)), which under a specific functional transformation can be reduced to a form included in the group of 50 equations mentioned above. Then, in Section 3 we consider a method leading to a more general analytical solution for this specific equation, and in Section 4 we develop a general analytical technique applicable to another nonlinear ODE (equation (4.1)) not included in the above list of 50 equations. This technique is based on the introduction of a new function, called $\mathcal{P}$, by which we obtain an exact closed form solution depending on the parameters of the equation and containing two arbitrary constants which yield a special solution in the case of an initial value problem. In Section 5 we investigate the determination of the function $\mathcal{P}$. Finally, in Section 6 we apply the obtained results to two special cases of (4.1) (equations (6.1) and (6.2)) and extract their exact solutions' formulas, while in Section 7 we present the expressions and graphics concerning the new functions involved in the above obtained solutions.

We note that (2.1) and (4.1) (see later in this work) are special cases of a more general class of nonlinear ODEs of the form

$$y''_{xx} - q(x)y'_x + f(y)y'^2_x = g(x,y), \tag{1.1}$$

where $f(y)$ is a rational function of y, namely,

$$f(y) = \frac{\Gamma_m y^m + \Gamma_{m-1}y^{m-1} + \cdots + \Gamma_0}{y^n + A_{n-1}y^{n-1} + \cdots + A_0}, \quad m,n \in \mathbb{Z}^+, \tag{1.2}$$

$q(x)$ is an arbitrary function of x, while y'_x and y''_{xx} denote the first and the second derivatives of the function $y(x)$ with respect to x, that is, dy/dx and d^2y/dx^2, respectively. Furthermore, A_i, $i = 0,\dots,n-1$, and Γ_j, $j = 0,\dots,m$, in (1.2) are real constants and $g(x,y)$ in (1.1) is a function possessing continuous partial derivatives with respect to x and y. In their work, Polyanin and Zaitsev [12] present several equations subjected to the form (1.1) and give their solutions or basic transformations and the resulted reductions of the

Table 1.1. Cases of the function f (equation (1.1)).

$f(y)$	Equations in [12]
-1	(2.8.1.41)
a = arbitrary constant	(2.6.4.3-4), (2.8.1.48), (2.9.3.23), (2.9.3.32), (2.9.4.2)
$\mp\frac{1}{y}$	(2.8.1.65), (2.9.3.6-7)
$\frac{a}{y}$, a = arbitrary constant	(2.6.4.6), (2.6.4.8-9), (2.6.4.12), (2.8.1.60), (2.9.3.10)
ay^m, a = arbitrary constant, $m \neq -1$	(2.6.4.5), (2.6.4.7), (2.6.4.13)

considered equations. In most of them, the function $f(y)$ has a rather simple form, as shown in Table 1.1 (all the equations cited in this table refer to [12]).

We also mention Langmuir's equation [1, Chapter 7, Section 2, equation (9)] with $f(y) = (1/3)(1/y)$. In most of the above equations $g(x, y)$ is equal to 0, while in the others is a polynomial of y, the coefficients of which are constants or functions of x. Moreover, $q(x)$ has a specific form in all equations except in [12, equations (2.9.3.6-7), (2.9.3.10), (2.9.4.2)], where it is arbitrary. On the other hand, studying equations where $f(y)$ is an arbitrary function of y [12, equations (2.9.3.38), (2.9.4.13), (2.9.4.18)] with $g = 0$ and [12, equations (2.9.3.26), (2.9.3.31), (2.9.4.19)] with g an arbitrary function of y (multiplied by e^x in [12, equation (2.9.3.26)]), we see that the proposed analytical methods result in integral equations, where even all the involved integrals can be determined, we may not derive a function $y = y(x)$.

Furthermore, as far as (2.1) (Gambier's) as well as the 5th Painlevé transcendent [12, equation (2.8.2.16)] are concerned (both of the form (1.1)), we have

$$f(y) = -\frac{3}{4}\frac{2y-1}{y^2-y}, \quad q(x) \text{ arbitrary}, \quad g(x, y) = 0 \tag{1.3}$$

(equation (2.1)),

$$\begin{gathered} f(y) = \frac{1}{2}\frac{3y-1}{y-y^2}, \qquad q(x) = -\frac{1}{x}, \\ g(x, y) = \frac{(y-1)^2(ay^2+\beta)}{x^2 y} + \frac{y(1-y)+\delta x y(y+1)}{x(1-y)} \end{gathered} \tag{1.4}$$

(5th Painlevé transcendent).

We also refer to [12, equation (2.8.1.67)] with

$$f(y) = -\frac{y}{y^2-b^2}, \quad q(x) = \frac{x}{a^2-x^2}, \quad g(x, y) = 0. \tag{1.5}$$

In this paper we treat analytically the constrained case (equation (4.1))

$$f(y) = \frac{\Gamma_1 y + \Gamma_0}{y^2-b^2}, \quad q(x) \text{ arbitrary}, \quad g(x, y) = 0, \tag{1.6}$$

with b, Γ_1, Γ_0 arbitrary real parameters ($b > 0$, $\Gamma_1^2 + \Gamma_0^2 \neq 0$).

2. Existence of an analytical solution of an equation due to Gambier

Developing the theory of second-order differential equations, Davis [1, Chapter 7, Section 3] presents a solution for the following equation:

$$(y^2-y)y''_{xx}-q(x)(y^2-y)y'_x-\frac{3}{4}(2y-1)y'^2_x=0,\quad q \text{ arbitrary.} \tag{2.1}$$

The above equation is due to B. Gambier; and the solution is obtained by means of the transformation (see [1])

$$y(x)=h[\xi(x)], \tag{2.2}$$

where $\xi(x)$ is a solution of the equation

$$\xi''_{xx}=q(x)\xi'_x. \tag{2.3}$$

Successive integrations of the latter equation furnishe

$$\xi(x)=\overline{c}_1\int e^{\int q(x)dx}dx+\overline{c}_2, \tag{2.4}$$

with $\overline{c}_1$, $\overline{c}_2$ being integration constants (we can perfectly take the values 1 and 0 for $\overline{c}_1$ and $\overline{c}_2$, resp.). Thus, by differentiating twice (2.2) with respect to x, substituting the obtained expressions together with (2.2) into (2.1), and finally making use of (2.3), we manage to eliminate the first derivative and bring (2.1) under the form

$$h''_{\xi\xi}-\frac{3}{4}\left(\frac{1}{h}+\frac{1}{h-1}\right)h'^2_\xi=0, \tag{2.5}$$

where h'_ξ and $h''_{\xi\xi}$ denote the first and second derivatives of $h(\xi)$ with respect to ξ.

Moreover, the specific treatment presented in [1] introduces the substitution

$$h(\xi)=\left[1-Q^2(\xi)\right]^{-1}, \tag{2.6}$$

where $Q(\xi)=Q(\xi,-a_1,-a_0)$ is the elliptic function of Weierstrass, which possesses the property (see [1])

$$u=Q(\xi)\Longrightarrow \xi=Q^{-1}(u)=\int_u^{+\infty}\frac{ds}{\sqrt{4s^3+a_1s+a_0}}. \tag{2.7}$$

By use of (2.7) we easily derive the following relations:

$$Q'^2_\xi=4Q^3+a_1Q+a_0, \tag{2.8a}$$

$$Q''_{\xi\xi}=6Q^2+\frac{1}{2}a_1. \tag{2.8b}$$

Then, by differentiating successively (2.6) with respect to ξ and making use of the obtained expressions for h'_ξ and $h''_{\xi\xi}$, together with (2.6), (2.5) is reduced to

$$2(Q-Q^3)Q''_{\xi\xi}+(3Q^2-1)Q'^2_\xi=0. \tag{2.9}$$

Furthermore, introduction of (2.8) into the latter equation returns the polynomial form

$$2(4+a_1)Q^3+3a_0Q^2-a_0=0. \tag{2.10}$$

Finally, by requiring that (2.10) holds true for every value of $Q(\xi)$, we extract the values $a_1=-4$, $a_0=0$. Thus, taking into account (2.2), (2.4), and (2.6), we obtain the explicit solution of (2.1), namely,

$$y(x)=\frac{1}{1-Q^2(\xi,4,0)},\qquad \xi(x)=\bar{c}_1\int e^{\int q(x)dx}dx+\bar{c}_2. \tag{2.11}$$

3. Generalization of the analytical treatment: development of a constructive technique

The idea is to set

$$h(\xi)=\frac{1}{\varepsilon_2P^2(\xi)+\varepsilon_1P(\xi)+\varepsilon_0},\quad \kappa\neq 0, \tag{3.1}$$

instead of (2.6), where ε_2, ε_1, ε_0 are parameters and $P(\xi)$ is a function of ξ, the first derivative of which is assumed to satisfy the relation

$$P'^2_\xi=a_nP^n+a_{n-2}P^{n-2}+\cdots+a_0,\quad a_i=\text{constants},\ a_n\neq 0. \tag{3.2}$$

The latter equation yields

$$P''_{\xi\xi}=\frac{na_n}{2}P^{n-1}+\frac{(n-2)a_{n-2}}{2}P^{n-3}+\cdots+\frac{a_1}{2}. \tag{3.3}$$

Thus, evaluating h'_ξ and $h''_{\xi\xi}$ by means of (3.1), introducing the results together with (3.1) into (2.5), and finally substituting (3.2) and (3.3), after some algebra, we obtain a polynomial of P in a more general form than that presented in (2.10), namely,

$$4(3-n)\varepsilon_2^3a_nP^{n+4}+(24-10n)\varepsilon_2^2\varepsilon_1a_nP^{n+3}+\cdots=0. \tag{3.4}$$

In addition, the requirement that (3.4) holds true for every value of $P(\xi)$ inside an interval $P(J)$, with J being the domain of $\xi(x)$, results in a system of equations, formed by setting the coefficients of P^m, $m = n+4, n+3, \dots, 0$, equal to zero. Thus, by means of the P^{n+4} equation (a_n, $\varepsilon_2 \neq 0$) we determine n:

$$n = 3. \tag{3.5}$$

For this specific value of n, (3.2) and (3.3) become

$$P_{\xi}'^2 = a_3 P^3 + a_1 P + a_0, \quad a_3 \neq 0, \tag{3.6a}$$

$$P_{\xi\xi}'' = \frac{3}{2} a_3 P^2 + \frac{a_1}{2}. \tag{3.6b}$$

Therefore, the function $P(\xi) = u$ can be defined as the inverse function of the generalized integral

$$\xi = P^{-1}(u) = \int_u^{+\infty} \frac{ds}{\sqrt{a_3 s^3 + a_1 s + a_0}}, \tag{3.7a}$$

on condition that the integral converges, or can be defined as the inverse function of the indefinite integral

$$\xi = P^{-1}(u) = \int \frac{du}{\sqrt{a_3 u^3 + a_1 u + a_0}}. \tag{3.7b}$$

Note that $P(\xi)$ must be a continuous function inside the domain of $\xi(x)$.

Furthermore, after substituting $n = 3$ into the set of equations furnished from the coefficients of the P polynomial (3.4), then by using the P^6 equation

$$-6 a_3 \varepsilon_2^2 \varepsilon_1 = 0, \tag{3.8}$$

we obtain (a_3, $\varepsilon_2 \neq 0$)

$$\varepsilon_1 = 0. \tag{3.9}$$

Moreover, taking into account (3.9), the P^4 equation

$$12 a_0 \varepsilon_2^3 = 0 \tag{3.10}$$

results in

$$a_0 = 0. \tag{3.11}$$

Finally, after the obtained values $\varepsilon_1 = a_0 = 0$ have been substituted into all the equations of the system ($n = 3$), then the remaining equations corresponding to the odd powers of P become

$$\begin{gathered} \varepsilon_0 a_1 (\varepsilon_0 - 1) = 0, \\ (a_1 \varepsilon_2 + 5 a_3 \varepsilon_0)(\varepsilon_0 - 1) = 0, \\ 2 a_1 \varepsilon_2 + a_3 (3 - 5\varepsilon_0) = 0. \end{gathered} \tag{3.12}$$

The system of the above equations yields the following four cases of solution.

Case 1.

$$\varepsilon_0 = a_1 = a_3 = 0. \tag{3.13}$$

Case 2.

$$a_1 = a_3 = 0. \tag{3.14}$$

Case 3.

$$\varepsilon_0 = 1, \qquad a_3 = \varepsilon_2 a_1. \tag{3.15}$$

Case 4.

$$\varepsilon_0 = 1, \qquad a_1 = a_3 = 0. \tag{3.16}$$

It is obvious that Cases 1, 2, and 4 have to be rejected ($a_3 \neq 0$) and thus the solution of the system (3.12) is provided by Case 3 together with (3.9) and (3.11), namely,

$$\varepsilon_1 = a_0 = 0, \qquad \varepsilon_0 = 1, \qquad a_3 = \varepsilon_2 a_1. \tag{3.17}$$

Therefore, the solution of (2.1) is constructed by means of (2.2) and (3.1), where the parameters included in (3.1) and (3.6a) are as in (3.17). We write

$$y(x)0 = \frac{1}{\varepsilon_2 P^2[\xi(x)] + 1}, \qquad \xi(x) = \int e^{\int q(x)dx} dx \tag{3.18a}$$

$$P_{\xi}'^2 = a_1 P(\varepsilon_2 P^2 + 1), \qquad P_{\xi\xi}'' = \frac{a_1}{2}(3\varepsilon_2 P^2 + 1). \tag{3.18b}$$

Differentiating twice the above expression of $y(x)$, then introducing y, y_x', y_{xx}'' into (2.1) and taking into account the equation giving $\xi(x)$, that is,

$$\xi_{xx}'' - q(x)\xi_x' = 0, \tag{3.19}$$

as well as the expressions (3.18b), we see that (2.1) is verified. Thus, the developed technique is able to construct an exact analytical solution of (2.1).

We note here that the solution (3.18a) has a general form yielding to (2.11) as a special case, since if we take $a_3 = 4$ and assume that (3.7a) holds true, then $P(\xi)$ becomes the elliptic function of Weierstrass:

$$P(\xi) = Q(\xi, -a_1, -a_0) = Q\left(\xi, -\frac{4}{\varepsilon_2}, 0\right). \tag{3.20}$$

Moreover, by setting $\varepsilon_2 = -1$ we obtain solution (2.11).

4. Use of the technique for another nonlinear ODE

We apply now the technique developed above, in order to solve analytically the following equation:

$$(y^2-b^2)y''_{xx}-q(x)(y^2-b^2)y'_x+(\Gamma_1 y+\Gamma_0)y'^2_x=0, \tag{4.1}$$

where $q(x)$ is an arbitrary function of x and b, Γ_1, Γ_0 are parameters taking real values, with $b>0$ and $\Gamma_1^2+\Gamma_0^2\neq 0$. We observe that if in the nonlinear terms yy''_{xx} and yy'_x of (2.1) we replace y with b^2 and take the values $-3/2$ and $3/4$ for Γ_1 and Γ_0, respectively, then we obtain (4.1). Moreover, (4.1) is subjected to appropriate initial conditions, namely,

$$y(x_0)=y_0, \qquad y'(x_0)=y'_0. \tag{4.2}$$

Thus, we simply follow the steps of the above procedure, as it was generalized in Section 3. More precisely we perform the following steps.

(1) We use the transformation

$$y(x)=h[\xi(x)]. \tag{4.3}$$

Successive differentiations of (4.3) yield

$$y'_x=h'_\xi\xi'_x, \qquad y''_{xx}=h''_{\xi\xi}\xi'^2_x+h'_\xi\xi''_{xx}. \tag{4.4}$$

(2) Similarly to the treatment of (2.1) (see Section 2), we require that

$$\xi''_{xx}=q(x)\xi'_x, \tag{4.5a}$$

by means of which we compute

$$\xi(x)=\bar{c}_1\int e^{\int q(x)dx}dx+\bar{c}_2, \tag{4.5b}$$

with $\bar{c}_1$, $\bar{c}_2$ being constants of integration.

(3) By using (4.4) and (4.5a), we reduce (4.1) to

$$h''_{\xi\xi}+\frac{1}{2b}\left(\frac{\Gamma}{h-b}+\frac{\overline{\Gamma}}{h+b}\right)h'^2_\xi=0, \tag{4.6}$$

with $\Gamma=b\Gamma_1+\Gamma_0$ and $\overline{\Gamma}=b\Gamma_1-\Gamma_0$.

(4) We introduce a new function $\mathcal{P}(\xi)$ by setting

$$h(\xi)=\frac{1}{\kappa\mathcal{P}^2(\xi)+\lambda\mathcal{P}(\xi)+\mu}, \qquad \kappa\neq 0, \tag{4.7}$$

with κ, λ, μ being real parameters, while we assume that $\mathcal{P}'_\xi$ satisfies

$$\mathcal{P}'^2_\xi=b_n\mathcal{P}^n+b_{n-2}\mathcal{P}^{n-2}+\cdots+b_0, \quad b_i=\text{constants},\ b_n\neq 0. \tag{4.8}$$

Differentiation of (4.8) with respect to ξ yields

$$\mathcal{P}''_{\xi\xi} = \frac{nb_n}{2}\mathcal{P}^{n-1} + \frac{(n-2)b_{n-2}}{2}\mathcal{P}^{n-3} + \cdots + \frac{b_1}{2}. \tag{4.9}$$

(5) By using (4.7), (4.8), and (4.9), after some algebra, we finally transform (4.6) to a polynomial of $\mathcal{P}$, namely,

$$(n-6)\kappa^4 b_n \mathcal{P}^{n+6} + \left(\frac{7}{2}n - 18\right)\kappa^3 \lambda b_n \mathcal{P}^{n+5} + \cdots = 0. \tag{4.10}$$

(6) We form a system of equations by setting the coefficients of the $\mathcal{P}$ polynomial (4.10) equal to zero. Thus, by means of the $\mathcal{P}^{n+6}$ equation (b_n, $\kappa \neq 0$) we determine n:

$$n = 6. \tag{4.11}$$

For this specific value of n, (4.8) and (4.9) take the form

$$\mathcal{P}'^2_{\xi} = b_6\mathcal{P}^6 + b_4\mathcal{P}^4 + b_3\mathcal{P}^3 + b_2\mathcal{P}^2 + b_1\mathcal{P} + b_0, \quad b_6 \neq 0 \tag{4.12a}$$

$$\mathcal{P}''_{\xi\xi} = 3b_6\mathcal{P}^5 + 2b_4\mathcal{P}^3 + \frac{3}{2}b_3\mathcal{P}^2 + b_2\mathcal{P} + \frac{b_1}{2}. \tag{4.12b}$$

(7) We define the function $\mathcal{P}(\xi) = u$, as the inverse function of the indefinite integral

$$\xi = \mathcal{P}^{-1}(u) = S = \int \frac{du}{\sqrt{b_6u^6 + b_4u^4 + b_3u^3 + b_2u^2 + b_1u + b_0}}. \tag{4.13}$$

$\mathcal{P}(\xi)$ should be a continuous function inside the interval J in which $\xi(x)$ is taking values. Further, an investigation with respect to the determination of $\mathcal{P}(\xi)$ is presented below (Section 5).

(8) By making use of the set of equations established in step (6), we proceed to the evaluation of the constants and parameters involved in (4.7) and (4.12a). Thus, after substitution of $n = 6$ into all the equations of the system, the $\mathcal{P}^{11}$ equation

$$3b_6\kappa^3\lambda = 0 \tag{4.14}$$

yields (b_6, $\kappa \neq 0$)

$$\lambda = 0. \tag{4.15}$$

Taking now into account (4.15), the $\mathcal{P}^9$ equation

$$-3b_3\kappa^4 = 0 \tag{4.16}$$

furnishes ($\kappa \neq 0$)

$$b_3 = 0. \tag{4.17}$$

Furthermore, after substituting (4.15) and (4.17) into the equations of the system, then, by means of the resulted $\mathcal{P}^7$ equation

$$-5b_1\kappa^4 = 0, \tag{4.18}$$

we have ($\kappa \neq 0$)

$$b_1 = 0. \tag{4.19}$$

Finally, after all the obtained results, that is, $\lambda = b_3 = b_1 = 0$, have been substituted into the original system ($n = 6$), then the remaining equations corresponding to the zeroth and even powers of $\mathcal{P}$ take the following form:

$$\begin{aligned}
&b_0\mu(\mu^2 - r^2) = 0,\\
&b_0\kappa[(2\Gamma_1 + 3)r^2 + 2\Gamma_0 r^2\mu - \mu^2] + 2b_2\mu(\mu^2 - r^2) = 0,\\
&b_0\kappa(2\Gamma_0 r^2 - 5\mu) + 2b_2\kappa[(\Gamma_1 + 1)r^2 + \Gamma_0 r^2\mu + \mu^2] + 3b_4\mu(\mu^2 - r^2) = 0,\\
&-3b_0\kappa^3 + 2b_2\kappa^2(\Gamma_0 r^2 - \mu) + b_4\kappa[2r^2(\Gamma_0\mu + \Gamma_1) + r^2 + 5\mu^2] + 4b_6\mu(\mu^2 - r^2) = 0,\\
&-2b_2\kappa^2 + b_4\kappa(2\Gamma_0 r^2 + \mu) + 2b_6[r^2(\Gamma_0\mu + \Gamma_1) + 4\mu^2] = 0,\\
&-b_4\kappa + 2b_6(\Gamma_0 r^2 + 2\mu) = 0,
\end{aligned} \tag{4.20}$$

where

$$r = \frac{1}{b}. \tag{4.21}$$

After performing algebraic manipulations, the system of six equations (4.20) results in the cases of solution listed in Table 4.1, where

$$\overline{\Gamma}_0 = \frac{\Gamma_0}{b}. \tag{4.22}$$

We can see that these solutions refer to specific values of Γ_0 and Γ_1. We must also note here that κ and one of b_0, b_2, b_4, b_6 (in most cases b_0 or b_2) play the role of the "free" parameters involved in the final solution, the evaluation of which will be achieved by means of the initial conditions referring to (4.1).

5. Determination of the function $\mathcal{P}(\xi)$

In step (7) of the analytical procedure developed in Section 4, we have considered the function $\mathcal{P}(\xi) = u$ as the inverse function of the indefinite integral S ($\xi = S(u)$, expression (4.13)). Thus, after the determination of S (by elementary or special functions) we should solve explicitly or implicitly the obtained relation with respect to u, thus determining the function $\mathcal{P}(\xi)$, that is,

$$\xi = \mathcal{P}^{-1}(u) = S(u) \Longrightarrow u = \mathcal{P}(\xi). \tag{5.1}$$

Table 4.1. Solutions of the system (4.20).

μ	$(\overline{\Gamma}_0, \Gamma_1)$	b_6	b_4	b_2	b_0
0	$\left(\pm\frac{1}{2}, \frac{-3}{2}\right)$	$\mp b^3\kappa^3 b_0$	$-b^2\kappa^2 b_0$	$\pm b\kappa b_0$	—
0	$\left(\pm\frac{3}{2}, \frac{-3}{2}\right)$	$\pm b^3\kappa^3 b_0$	$3b^2\kappa^2 b_0$	$\pm 3b\kappa b_0$	—
$\pm\frac{1}{b}$	$(0, -1)$	$\pm\frac{b^3}{2}\kappa^3 b_0$	$2b^2\kappa^2 b_0$	$\pm\frac{5b}{2}\kappa b_0$	—
$\pm\frac{1}{b}$	$(\pm 1, -2)$	$\pm\frac{b^3}{8}\kappa^3 b_0$	$\frac{3b^2}{4}\kappa^2 b_0$	$\pm\frac{3b}{2}\kappa b_0$	—
$\pm\frac{1}{b}$	$\left(\pm\frac{1}{2}, -\frac{3}{2}\right)$	$\pm\frac{b^3}{4}\kappa^3 b_0$	$\frac{5b^2}{4}\kappa^2 b_0$	$\pm 2b\kappa b_0$	—
$\pm\frac{1}{b}$	$\left(\mp\frac{1}{2}, -\frac{1}{2}\right)$	$\pm b^3\kappa^3 b_0$	$3b^2\kappa^2 b_0$	$\pm 3b\kappa b_0$	—
0	$\left(\pm\frac{1}{2}, -\frac{1}{2}\right)$	$\pm b\kappa b_4$	—	0	0
0	$(0, -1)$	$-b^2\kappa^2 b_2$	0	—	0
0	$(\pm 1, -1)$	$b^2\kappa^2 b_2$	$\pm 2b\kappa b_2$	—	0
$\pm\frac{1}{b}$	$(\mp 2, -2)$	—	0	0	0
$\pm\frac{1}{b}$	$(\mp 1, -2)$	$\pm\frac{b}{2}\kappa b_4$	—	0	0
$\pm\frac{1}{b}$	$\left(\mp\frac{3}{2}, -\frac{3}{2}\right)$	$\pm b\kappa b_4$	—	0	0
$\pm\frac{1}{b}$	$(0, -2)$	$\frac{b^2}{4}\kappa^2 b_2$	$\pm b\kappa b_2$	—	0
$\pm\frac{1}{b}$	$(\mp 1, -1)$	$b^2\kappa^2 b_2$	$\pm 2b\kappa b_2$	—	0
$\pm\frac{1}{b}$	$\left(\mp\frac{1}{2}, -\frac{3}{2}\right)$	$\frac{b^2}{2}\kappa^2 b_2$	$\pm\frac{3b}{2}\kappa b_2$	—	0

If an explicit expression (unique or not) $u(\xi)$ cannot be obtained, then it should be investigated if a function $u(\xi)$ can be determined implicitly by means of the equation $\xi - S(u) = 0$. This means that we should examine the well-known conditions of the implicit function theorem (for this we need to locate a point (ξ_0, u_0) such that $\xi_0 = S(u_0)$ and then study the existence and continuity of the derivative $S'(u)$ inside an interval containing u_0 and evaluate $S'(u_0)$). If the conditions of the theorem are satisfied, then a unique implicit smooth function $u = \mathcal{P}(\xi)$ is defined inside a certain area of (ξ_0, u_0). Note that, in case of a singular point, a nonunique function may be defined. In any case of solution of (5.1), the formula (4.7) (with $\lambda = 0$) combined with (4.5b) (with $\overline{c}_1 = 1$, $\overline{c}_2 = 0$) as well as with the results given by Table 4.1 represents an exact solution of (4.1) inside a certain domain of x, where the free parameters are evaluated by the use of initial conditions (4.2).

Technically, by making use of the substitution (see [13, equation (2.291.2)])

$$u^2 = z, \tag{5.2}$$

the integral S ((4.13), $b_1 = b_3 = 0$),

$$S = \int \frac{du}{\sqrt{b_6u^6 + b_4u^4 + b_2u^2 + b_0}}, \tag{5.3}$$

is transformed to

$$S = \frac{1}{2}\int \frac{dz}{\sqrt{z\left(b_6z^3 + b_4z^2 + b_2z + b_0\right)}}, \quad u > 0, \tag{5.4}$$

where we have considered u to take positive values (in the opposite case ($u < 0$), a sign "−" appears in the right-hand side of (5.4)). Moreover, the integral (5.4) is determined in dependence on the case of solution (Table 4.1). Thus, as far as the cases with $b_0 = 0$ are concerned, S becomes

$$S = S_0 = \frac{1}{2}\int \frac{dz}{z\sqrt{R_2(z)}}, \quad R_2(z) = b_6z^2 + b_4z + b_2, \tag{5.5}$$

the various types of which (depending on the sign of b_2 (or b_6) and the discriminant Δ_2 of the quadratic R_2) are provided by [13, equation (2.266)], while for the cases where $b_0 \neq 0$, the integral S takes the form

$$S = S_1 = \frac{1}{2\sqrt{\pm b_6}}\int \frac{d\xi}{\sqrt{\pm zR_3(z)}}, \quad b_6 \gtrless 0, \tag{5.6}$$

with

$$R_3(z) = z^3 + c_4z^2 + c_2z + c_0, \tag{5.7a}$$

$$c_i = \frac{b_i}{b_6}, \quad i = 0, 2, 4. \tag{5.7b}$$

The sign "±" in (5.6) corresponds to the positive and negative signs of b_6, respectively. By using now the substitution

$$z = \omega - \frac{c_4}{3}, \tag{5.8}$$

the cubic form R_3 can be written as

$$R_3(\omega) = \omega^3 + p\omega + q, \tag{5.9}$$

with

$$p = -\frac{c_4^3}{3} + c_2, \qquad q = \frac{2}{27}c_4^3 - \frac{c_4c_2}{3} + c_0. \tag{5.10}$$

As it is well-known [14, Cardan solution], the roots of $R_3(\omega)$ depend on the sign of the discriminant

$$\Delta_3 = \frac{p^3}{27} + \frac{q^2}{4}. \tag{5.11}$$

Furthermore, by introducing (5.7b) and (5.10) into (5.11), we find that Δ_3 is equal to zero in all cases of solution recorded in Table 4.1. Therefore, according to the Cardan solution, $R_3(z)$ has three real roots (two of them are equal), provided by

$$\begin{gathered} \rho_i = e_i - \frac{c_4}{3}, \quad i = 1,2, \\ e_1 = 2\sqrt[3]{-\frac{q}{2}}, \qquad e_2 = -\sqrt[3]{-\frac{q}{2}}, \end{gathered} \tag{5.12}$$

where the relation (5.8) has been taken into account. Here ρ_2 is the root of double multiplicity. Thus, the integral S (formula (5.6)) becomes

$$S_1 = \frac{1}{2\sqrt{\pm b_6}} \int \frac{dz}{(z-\rho_2)\sqrt{\pm z^2 \mp \rho_1 z}}. \tag{5.13}$$

By introducing now the substitution (see [13, equation (2.281)] with $n = 1$)

$$t = \frac{1}{z-\rho_2}, \quad z > \rho_2, \tag{5.14}$$

the integral (5.13) is transformed to

$$S_1 = -\frac{1}{2\sqrt{\pm b_6}} \int \frac{dt}{\sqrt{\gamma t^2 + \beta t + a}}, \tag{5.15}$$

with

$$\begin{aligned} \gamma &= \mp \rho_2(\rho_1 - \rho_2), \\ \beta &= \mp(\rho_1 - 2\rho_2), \\ a &= \pm 1. \end{aligned} \tag{5.16}$$

Finally, the latter integral can be computed by means of [13, equation (2.261)] (the formulas by means of which the integral (5.15) is expressed depend on the sign of γ and the discriminant $\overline{\Delta}_2$ of the quadratic $\gamma t^2 + \beta t + a$). It must be noted here that all the expressions concerning both integrals, $\xi = S_0(u)$ (formula (5.5)) and $\xi = S_1(u)$ (formula (5.15)), can be solved explicitly with respect to $z = u^2$. Thus, the function $u = \mathcal{P}(\xi)$ can be determined by taking the positive square root of the obtained expressions, since we have considered u to be positive (see the integral (5.4) above).

A question emerging here concerns the form of the integral S when $\Delta_3 \neq 0$. Obviously, this is not the case of (4.1) but it is quite possible to occur in this step of the followed procedure in case that it is applied to another nonlinear equation. Thus, when the sign of the discriminant Δ_3 is positive ($R_3(z)$ has one real and two conjugate complex roots) or negative ($R_3(z)$ has three distinct real roots), then by developing the expression $\pm R_3(z)$ and taking into account the integral (5.6), we deduce that only generalized integrals of the second kind of the form

$$\int_a^b \frac{dt}{\sqrt{tR_3(t)}} \tag{5.17}$$

can be determined by explicit formulas. In the above integral either, a (or b) represents $z = u^2$ and b (or a) is equal to one of the real roots of R_3. More precisely, the integral (5.17) is expressed by means of elliptic integrals of the first kind

$$F[\overline{f}(z,p,q,c_4),\overline{\kappa}(p,q,c_4)], \tag{5.18}$$

where c_4 and p, q are as in (5.7b) and (5.10), respectively. As it is well known, F can be solved with respect to z by means of the Jacobi elliptic sine function. Therefore, in order to finally obtain an integral of the kind (5.17), the integral S should be defined as a definite integral of the form

$$S = \int_{a_0}^{b_0} \frac{ds}{\sqrt{b_6 s^6 + b_4 s^4 + b_2 s^2 + b_0}}, \tag{5.19}$$

with either a_0 (or b_0) being equal to u and b_0 (or a_0) being equal to the square root of one of the real roots of R_3.

6. Application of the method to special cases of (4.1)

We now consider two special cases of (4.1), for which the parameters of the solution are provided by Table 4.1. More precisely, we apply the analysis developed in Section 5 to the cases

(i) $\Gamma_0 = 0$, $\Gamma_1 = -1$:

$$(y^2 - b^2)y''_{xx} - q(x)(y^2 - b^2)y'_x - yy'^2_x = 0, \tag{6.1}$$

(ii) $\Gamma_0 = (3/2)b$, $\Gamma_1 = -3/2$:

$$(y^2 - b^2)y''_{xx} - q(x)(y^2 - b^2)y'_x - \frac{3}{2}(y-b)y'^2_x = 0. \tag{6.2}$$

Equation (6.1). Here, two possible solutions are recorded in Table 4.1.

Case 1. (We consider the "+" case)

$$\mu = \frac{1}{b}, \qquad b_6 = \frac{b^3}{2}\kappa^3 b_0, \qquad b_4 = 2b^2\kappa^2 b_0, \qquad b_2 = \frac{5b}{2}\kappa b_0. \tag{6.3}$$

Since $b_0 \neq 0$, by making use of the above relations and combining (5.7b) with (5.10), we obtain the roots of $R_3(z)$ ($\Delta_3 = 0$), namely,

$$\rho_1 = -\frac{2}{b\kappa}, \qquad \rho_2 = -\frac{1}{b\kappa}, \quad \rho_2 \text{ is double.} \tag{6.4}$$

Then, by substituting (6.4) into (5.16), the integral S ($S = S_1$) given by formula (5.15) becomes

$$\xi = S_1 = -\frac{1}{2\sqrt{\pm b_6}} \int \frac{dt}{\sqrt{\mp \rho_2^2 t^2 \pm 1}}, \qquad t = \frac{1}{z - \rho_2} > 0. \tag{6.5}$$

The signs in (6.5) correspond to the cases $b_6 > 0$ and $b_6 < 0$, respectively. According to [13, equation (2.261)], S_1 is determined as

$$\xi = -\frac{1}{2\sqrt{b_6}\rho_2} \arcsin \frac{\rho_2}{z-\rho_2}, \quad b_6 > 0 \tag{6.6a}$$

$$\xi = -\frac{1}{2\sqrt{-b_6}\rho_2} \ln \left(\frac{2\rho_2\sqrt{-z(z-2\rho_2)} + 2\rho_2^2}{z-\rho_2} \right), \quad b_6 < 0,\ \rho_2 > 0. \tag{6.6b}$$

Moreover, solution of (6.6) with respect to $z = u^2$, respectively, results in

$$u^2 = \rho_2 + \frac{\rho_2}{\sin\left(-2\sqrt{b_6}\rho_2\xi\right)}, \quad b_6 > 0, \tag{6.7a}$$

$$u^2 = \rho_2 + \frac{4\rho_2^2 e^{-2\sqrt{-b_6}\rho_2\xi}}{e^{-4\sqrt{-b_6}\rho_2\xi} + 4\rho_2^2}, \quad b_6 < 0. \tag{6.7b}$$

Since $\mathcal{P}(\xi) = u$ can be determined (as the positive square root of the right-hand side of (6.7)), the relations (4.3) and (4.7) ($\lambda = 0$, $\mu = 1/b$) yield

$$y(x) = b\sin\left[2\sqrt{b_6}\rho_2\xi(x)\right], \quad b_6 > 0, \tag{6.8a}$$

$$y(x) = -b\frac{e^{-4\sqrt{-b_6}\rho_2\xi(x)} + 4\rho_2^2}{4\rho_2 e^{-2\sqrt{-b_6}\rho_2\xi(x)}}, \quad b_6 < 0, \tag{6.8b}$$

where the expression of ρ_2 (equation (6.4)) has been taken into account. Differentiating twice (6.8) with respect to x, introducing the obtained expressions into (6.1), and taking into account (4.5a), we see that (6.1) *is satisfied* in both cases. Therefore, (6.8) represent an exact explicit solution of (6.1), where b_6 and ρ_2 stand for the free parameters instead of κ, b_0 (see relations (6.3), (6.4)). We now examine the second case.

Case 2.

$$\mu = 0, \qquad b_6 = -b^2\kappa^2 b_2, \qquad b_4 = 0, \qquad b_0 = 0. \tag{6.9}$$

Since in this case we have that $b_0 = 0$, the corresponding integral S ($S = S_0$) given now by (5.5) becomes

$$\xi = S_0 = \frac{1}{2}\int \frac{dz}{z\sqrt{b_6 z^2 + b_2}}. \tag{6.10}$$

According to [13, equation (2.266)], we have the following two cases:

$$\xi = \mp\frac{1}{2\sqrt{-b_2}} \arcsin \frac{1}{b\kappa z}, \quad b_2 < 0, \tag{6.11a}$$

where the signs correspond to $\kappa > 0$ and $\kappa < 0$, respectively (b_6 has been substituted from (6.9)), and

$$\xi = -\frac{1}{2\sqrt{b_2}} \ln \frac{2\sqrt{b_2(b_6 z^2 + b_2)} + 2b_2}{z}, \quad b_2 > 0. \tag{6.11b}$$

Furthermore, by solving (6.11) with respect to $z = u^2$ and substituting b_6 from (6.9), we obtain

$$u^2 = -\frac{1}{b|\kappa| \sin\left(2\sqrt{-b_2}\xi\right)}, \quad b_2 < 0, \tag{6.12a}$$

$$u^2 = \frac{4b_2 e^{-2\sqrt{b_2}\xi}}{e^{-4\sqrt{b_2}\xi} + 4b^2\kappa^2 b_2^2}, \quad b_2 > 0. \tag{6.12b}$$

Thus, substituting the above expressions of $\mathcal{P}(\xi) = u$ into (4.7) ($\lambda = \mu = 0$), then by (4.3), we extract

$$y(x) = \mp b \sin\left[2\sqrt{-b_2}\xi(x)\right], \quad b_2 < 0, \tag{6.13a}$$

$$y(x) = \frac{e^{-4\sqrt{b_2}\xi(x)} + 4b^2\kappa^2 b_2^2}{4\kappa b_2 e^{-2\sqrt{b_2}\xi(x)}}, \quad b_2 > 0. \tag{6.13b}$$

The sign "−" in (6.13a) holds true when $\kappa > 0$, while the sign "+" holds true when $\kappa < 0$. Moreover, introducing y as well as its derivatives (with respect to x) into (6.1) and taking into account (4.5a), we deduce that (6.1) *is also verified* in both cases. Comparing with the expressions (6.8) obtained in Case 1, we see that the explicit forms of the obtained solutions are qualitatively the same. We mention that $\xi(x)$ is given from (4.5b), namely,

$$\xi(x) = \bar{c}_1 \int e^{\int q(x)dx} dx + \bar{c}_2 \tag{6.14}$$

(we can take $\bar{c}_1 = 1$, $\bar{c}_2 = 0$). Moreover, the respective to the case under consideration, free parameters (b_6, ρ_2 or κ, b_2) can be evaluated by means of the initial conditions referring to original equation (6.1) (relations (4.2)). These conditions will also be responsible for the kind of solution ((6.13a) or (6.13b), (6.8a) or (6.8b)) that (6.1) will accept.

It is worth to be noted here that solution (6.13a) (or (6.8a)) is in perfect agreement with the solution of the same equation with $q(x) = x/(a^2 - x^2)$ given by Polyanin and Zaitsev [12, equation (2.8.1.67)], namely,

$$y(x) = b \sin\left(A_1 \arcsin\frac{x}{a} + A_2\right), \quad A_1, A_2 = \text{constants}. \tag{6.15}$$

In fact, for this specific form of $q(x)$, (6.14) yields

$$\xi(x) = \bar{c}_1 \arcsin\frac{x}{a} + \bar{c}_2, \tag{6.16}$$

and therefore (6.13a) becomes

$$y(x) = \mp b \sin\left(2\sqrt{-b_2\overline{c}_1} \arcsin\frac{x}{a} + 2\sqrt{-b_2\overline{c}_2}\right), \tag{6.17}$$

or setting $B_i = \mp 2\sqrt{-b_2\overline{c}_i}$, $i = 1,2$,

$$y(x) = b \sin\left(B_1 \arcsin\frac{x}{a} + B_2\right), \tag{6.18}$$

which is identical to (6.15).

Equation (6.2). Table 4.1 also provides two possible solutions concerning this equation, namely, the following.

Case 1.

$$\mu = 0, \qquad b_6 = b^3\kappa^3 b_0, \qquad b_4 = 3b^2\kappa^2 b_0, \qquad b_2 = 3b\kappa b_0. \tag{6.19}$$

This case ($b_0 \neq 0$), as the case (i) of (6.1), corresponds to the integral S ($S = S_1$) (expression (5.6)), involving the cubic form $R_3(\xi)$. By means now of (5.7b) and (5.10), the roots of R_3 ($\Delta_3 = 0$) are found:

$$\rho_1 = \rho_2 = -\frac{1}{b\kappa}, \quad \rho_2 \text{ is double}, \tag{6.20}$$

and moreover combining (6.20) with (5.16), the obtained above integral (5.15) takes the form

$$\begin{aligned}\xi = S_1 &= -\frac{1}{2\sqrt{\pm b_6}} \int \frac{dt}{\sqrt{\pm\rho_2 t \pm 1}}, \quad t = \frac{1}{z-\rho_2} > 0,\\ &= \mp\frac{1}{\rho_2\sqrt{\pm b_6}}\sqrt{\pm\frac{z}{z-\rho_2}},\end{aligned} \tag{6.21}$$

where the signs correspond to the positive and negative signs of b_6, respectively. Since $z/(z-\rho_2)$ is positive, we reject the case $b_6 < 0$, writing

$$\xi = -\frac{1}{\rho_2\sqrt{b_6}}\sqrt{\frac{z}{z-\rho_2}}, \tag{6.22}$$

and therefore,

$$z = u^2 = \frac{b_0\xi^2}{1 - b\kappa b_0\xi^2}, \quad \kappa b_0 > 0, \tag{6.23}$$

where b_6 and ρ_2 have been substituted from (6.19) and (6.20). Thus proceeding as in the previous case, we derive

$$y(x) = -b + \frac{1}{\kappa b_0 \xi^2(x)}, \quad \kappa b_0 > 0. \tag{6.24}$$

Moreover, (6.2) *is satisfied* by formula (6.24), where κ and b_0 represent here the free parameters.

Case 2.

$$\mu = \frac{1}{b}, \qquad b_6 = -b\kappa b_4, \qquad b_2 = b_0 = 0. \tag{6.25}$$

Here, the integral S is given by formula (5.5) ($b_0 = 0$). More precisely, we write

$$\xi = S_0 = \frac{1}{2}\int \frac{dz}{z\sqrt{b_6 z^2 + b_4 z}}. \tag{6.26}$$

Therefore, according to [13, equation (2.266)], we have

$$\xi = -\frac{\sqrt{b_6 z^2 + b_4 z}}{b_4 z}, \tag{6.27}$$

and moreover,

$$z = u^2 = \frac{1}{b_4 \xi^2 + b\kappa}, \tag{6.28}$$

where b_6 has been substituted from (6.25). Then, similarly as before, $y(x)$ is obtained:

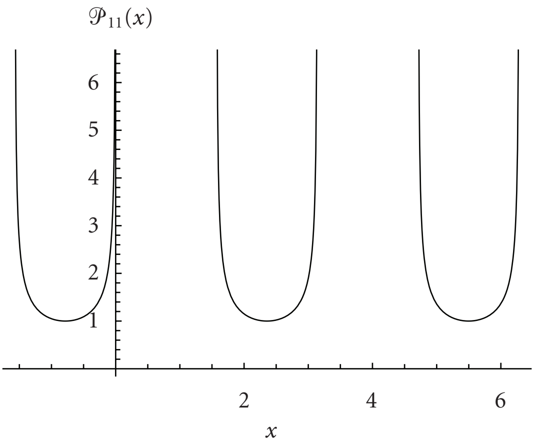
$$y(x) = -b - \frac{\kappa b^2}{b_4 \xi^2(x)}. \tag{6.29}$$

The latter formula *also verifies* (6.2). As in (6.1), the function $\xi(x)$ is provided by (6.14) ($\bar{c}_1 = 1$, $\bar{c}_2 = 0$), while the free parameters (κ, b_0 or κ, b_4) corresponding to Cases 1 and 2 are evaluated by the associated initial conditions. Finally, we also observe here the qualitative similarity between the expressions giving $y(x)$ in Cases 1 and 2.

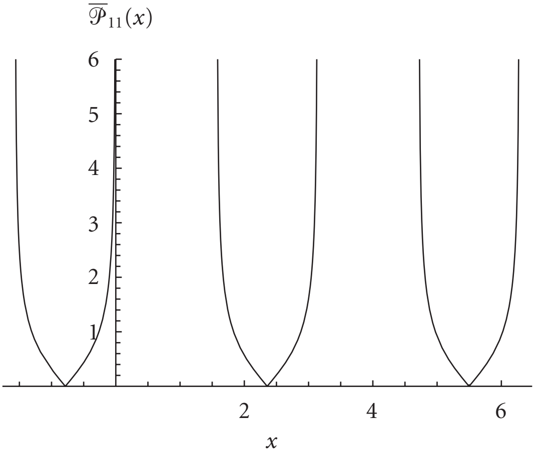
7. The functions $\mathcal{P}(x)$

According to the developed technique, the determination of the function $\mathcal{P}$ is achieved through the inversion of the integral (4.13), which in turn can be determined by means of classical procedures (Section 5). Considering now the functions $\mathcal{P}(x)$ involved in the obtained exact solutions of (6.1) and (6.2) (with $\xi(x)$ as independent variable), provided by the expressions (6.12), (6.7) and (6.28), (6.23), respectively (we take the positive square roots of the right-hand sides), we observe that their form is simply a specific combination of transcendental or (and) classical functions. On the other hand, we can see that they share the fundamental property of elliptic functions, that their only movable singularities are poles.

More specifically, taking arbitrary values for the respective free parameters, we have ($b > 0$).



(a) The function $\mathcal{P}_{11}$ $(b = 1,\ \kappa = \pm 1,\ b_2 = -1)$



(b) The function $\overline{\mathcal{P}}_{11}$ $(b = 1,\ \rho_2 = -1,\ b_6 = 1)$

Figure 7.1

Equation (6.1). Formula (6.12a), $b_2 < 0$:
Function $\mathcal{P}_{11}$ (Figure 7.1(a))

$$\mathcal{P}_{11}(x) = \left[-\frac{1}{b|\kappa|} \csc\left(2\sqrt{-b_2}x\right)\right]^{1/2}, \tag{7.1}$$

where $x \in (((2\lambda+1)/2\sqrt{-b_2})\pi, ((2\lambda+2)/2\sqrt{-b_2})\pi)$, $\lambda \in \mathbb{Z}$. $\mathcal{P}_{11}$ possesses obvious poles at

$$x = \frac{\lambda\pi}{2\sqrt{-b_2}}, \quad \lambda \in \mathbb{Z}. \tag{7.2}$$

Formula (6.7a), $b_6 > 0$:
Function $\overline{\mathcal{P}}_{11}$

$$\overline{\mathcal{P}}_{11}(x) = \left[\rho_2\left(1 - \csc\left(2\sqrt{b_6}\rho_2 x\right)\right)\right]^{1/2}, \quad \rho_2 = -\frac{1}{b\kappa}, \tag{7.3}$$

where $x \in (((2\lambda+1)/2\sqrt{b_6}|\rho_2|)\pi, ((2\lambda+2)/2\sqrt{b_6}|\rho_2|)\pi)$, $\lambda \in \mathbb{Z}$. $\overline{\mathcal{P}}_{11}$ possesses obvious poles at

$$x = \frac{\lambda\pi}{2\sqrt{b_6}\,|\rho_2|}, \quad \lambda \in \mathbb{Z}. \tag{7.4}$$

For $\rho_2 > 0$, $\overline{\mathcal{P}}_{11}$ presents a qualitatively similar graph to that of $\mathcal{P}_{11}$ in Figure 7.1(a) (displaced upwards), while for $\rho_2 < 0$ we obtain Figure 7.1(b). We observe that in this case, $\overline{\mathcal{P}}_{11}$ is not a smooth function inside its domain, since it accepts different values for its right-side and left-side derivatives at $x_0 = ((2\lambda+1)/2\sqrt{b_6}|\rho_2|)\pi + \pi/4\sqrt{b_6}|\rho_2|$, $\lambda \in \mathbb{Z}$, equal to $\sqrt{b_0}$ and $-\sqrt{b_0}$, respectively (in Figure 7.1(b), $b_0 = 2$). This result can also be derived from (4.12a) by setting $\mathcal{P}(x_0) = \overline{\mathcal{P}}_{11}(x_0) = 0$.

Formula (6.12b), $b_2 > 0$:
Function $\mathcal{P}_{12}$ (Figures 7.2(a), 7.2(b)),

$$\mathcal{P}_{12}(x) = \frac{2\sqrt{b_2}e^{-\sqrt{b_2}x}}{\left(e^{-4\sqrt{b_2}x} + 4b^2\kappa^2 b_2^2\right)^{1/2}}, \quad x \in \mathbb{R}. \tag{7.5}$$

$\mathcal{P}_{12}$ possesses no poles.

The function $\mathcal{P}$ corresponding to formula (6.7b) (defined everywhere in $\mathbb{R}$, possessing no poles, and valid only for $\rho_2 > 0$) presents also qualitatively similar graphs to that of $\mathcal{P}_{12}$, displaced upwards and stretched or compressed along x and y axes, depending on the values of parameters b_6, ρ_2.

Equation (6.2). Formula (6.28):
Function $\mathcal{P}_2$ (Figures 7.3, 7.4, 7.5)

$$\mathcal{P}_2(x) = \left(\frac{1}{b_4 x^2 + b\kappa}\right)^{1/2}. \tag{7.6}$$

We have the following three cases:

(i) $b_4 > 0\ \kappa > 0$:

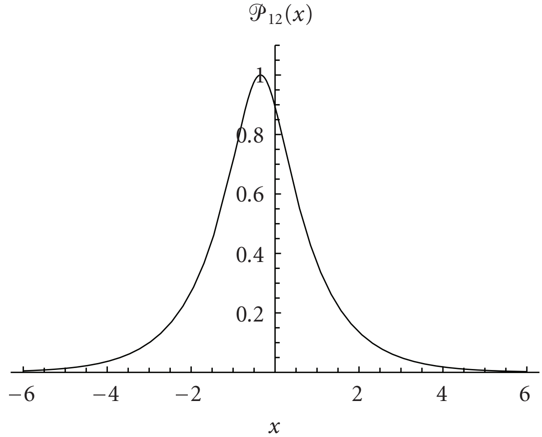
$$\mathcal{P}_2(x) = \mathcal{P}_{21}(x), \quad x \in \mathbb{R} \text{ (which possesses no poles)} \tag{7.7}$$

(ii) $b_4 > 0$, $\kappa < 0$:

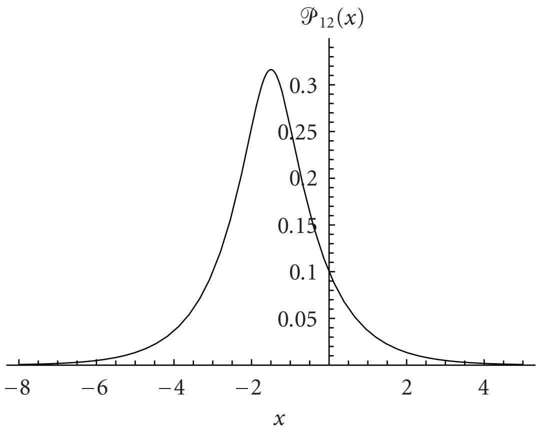
$$\mathcal{P}_2(x) = \mathcal{P}_{22}(x), \quad x \in \left(-\infty, -\sqrt{-b\kappa/b_4}\right) \cup \left(\sqrt{-b\kappa/b_4}, +\infty\right), \tag{7.8}$$

(iii) $b_4 < 0$, $\kappa > 0$:

$$\mathcal{P}_2(x) = \mathcal{P}_{23}(x), \quad x \in \left(-\sqrt{-b\kappa/b_4}, \sqrt{-b\kappa/b_4}\right). \tag{7.9}$$



(a) The function $\mathcal{P}_{12}$ $(b = 1,\ \kappa = \pm 1,\ b_2 = 1)$



(b) The function $\mathcal{P}_{12}$ $(b = 1,\ \kappa = \pm 10,\ b_2 = 1)$

Figure 7.2

Both $\mathcal{P}_{22}$ and $\mathcal{P}_{23}$ possess two poles at

$$x = -\sqrt{-\frac{b\kappa}{b_4}}, \sqrt{-\frac{b\kappa}{b_4}}. \tag{7.10}$$

Finally, in the case $b_4 < 0$, $\kappa < 0$, $\mathcal{P}_2$ does not accept real values for any $x \in \mathbb{R}$.
Formula (6.23):
Function $\overline{\mathcal{P}}_2$

$$\overline{\mathcal{P}}_2(x) = \left(\frac{b_0 x^2}{1 - b\kappa b_0 x^2}\right)^{1/2}, \quad \kappa b_0 > 0. \tag{7.11}$$

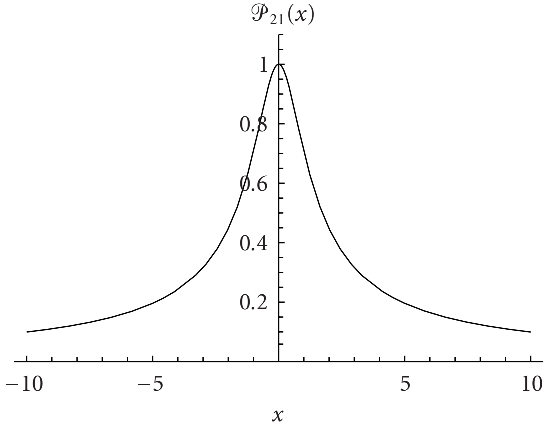


Figure 7.3. The function $\mathcal{P}_{21}$ ($b = 1$, $\kappa = 1$, $b_4 = 1$).

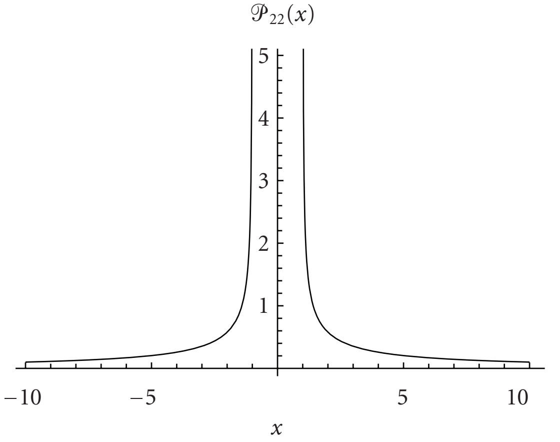


Figure 7.4. The function $\mathcal{P}_{22}$ ($b = 1$, $\kappa = -1$, $b_4 = 1$).

We have the following two cases:

(i) $b_0 < 0$:

$$\overline{\mathcal{P}}_2(x) = \overline{\mathcal{P}}_{22}(x), \quad x \in \left(-\infty, -\sqrt{\frac{1}{b\kappa b_0}}\right) \bigcup \left(\sqrt{\frac{1}{b\kappa b_0}}, \infty\right), \tag{7.12}$$

(ii) $b_0 > 0$:

$$\overline{\mathcal{P}}_2(x) = \overline{\mathcal{P}}_{23}(x), \quad x \in \left(-\sqrt{\frac{1}{b\kappa b_0}}, \sqrt{\frac{1}{b\kappa b_0}}\right). \tag{7.13}$$

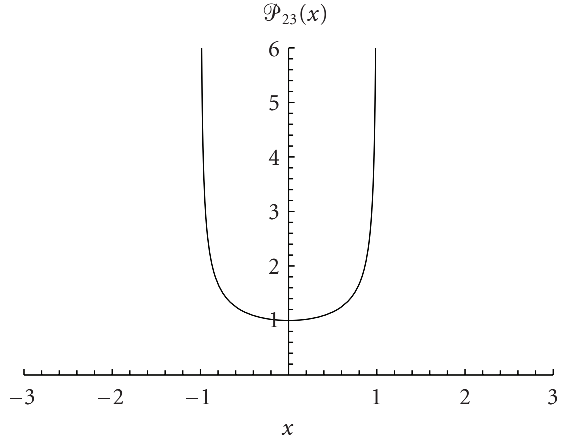


Figure 7.5. The function $\mathcal{P}_{23}$ ($b = 1$, $\kappa = 1$, $b_4 = -1$).

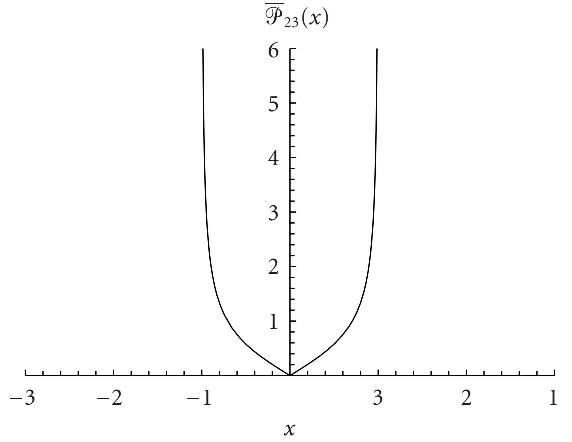


Figure 7.6. The function $\overline{\mathcal{P}}_{23}$ ($b = 1$, $\kappa = 1$, $b_0 = 1$).

Both $\overline{\mathcal{P}}_{22}$ and $\overline{\mathcal{P}}_{23}$ possess two poles at

$$x = -\sqrt{\frac{1}{b\kappa b_0}}, \sqrt{\frac{1}{b\kappa b_0}}. \tag{7.14}$$

The graph of $\overline{\mathcal{P}}_{22}$ is similar to that of $\mathcal{P}_{22}$ in Figure 7.4, while the respective graph to $\overline{\mathcal{P}}_{23}$ is presented in Figure 7.6. We see that $\overline{\mathcal{P}}_{23}$, as $\overline{\mathcal{P}}_{11}$ ($\rho_2 < 0$), is also a nonsmooth function since the right-side and the left-side derivatives at $x = 0$ take different values, equal to $\sqrt{b_0}$ and $-\sqrt{b_0}$, respectively.

8. Summary: conclusion

By the developed analytical procedure, we first obtained a more general form of the solution of Gambier's equation (2.1) than the existed explicit one. We then applied this method to another nonlinear ODE (equation (4.1)) extracting specific sets of values of the parameters involved, for which a closed form solution can be obtained. This solution is constructed through the inverse function of the integral S of the (algebraic) inverse of the square root of a polynomial of the 6th degree, the specific form of which allows the determination of S (Section 5). Thus, by inverting the extracted relations we determined explicitly or implicitly the function $\mathcal{P}$ introduced into the analysis and derived the respesctive (explicit or implicit) formula of the solution. Two arbitrary parameters contained in the obtained expressions provide special solutions in case of an initial value problem.

Furthermore, we note that in the special cases of (4.1) solved in this work as an application of the developed technique (Section 6), the corresponding functions $\mathcal{P}_{ij}$ and $\overline{\mathcal{P}}_{ij}$ (Section 7) as well as the corresponding (explicit) formulas of the function y (Section 6) represent specific combinations of elementary functions. Moreover, the most of the obtained functions $\mathcal{P}$ possess movable poles. We should also note the perfect agreement between the obtained solution and that given in [12] in case of (6.2). More generally speaking, we believe that the establishment of analytical techniques in the sense of constructing successive appropriate steps, as in the present work, can achieve the derivation of analytical solutions for several nonsolved analytically up today nonlinear ODEs or systems of ODEs.

References

[1] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, NY, USA, 1962.

[2] E. Picard, *Traité d'analyse*, vol. 2, Gauthier-Villars, Paris, France, 1893.

[3] E. Picard, *Traité d'analyse*, vol. 3, Gauthier-Villars, Paris, France, 1896.

[4] E. Picard, "Sur un classe d'equations differentielles," *Comptes Rendus*, vol. 104, pp. 41–43, 1887.

[5] P. Painlevé, "Sur la determination explicite des equations differentielles du second ordre à points critiques pixes," *Comptes Rendus*, vol. 126, pp. 1329–1332, 1898.

[6] P. Painlevé, "Sur les equations differentielles du second ordre à points critiques fixes," *Comptes Rendus*, vol. 126, pp. 1185–1188, 1699–1700, 1898.

[7] P. Painlevé, "Sur les equations differentielles du second ordre à points critiques fixes," *Comptes Rendus*, vol. 127, pp. 541–544, 945–948, 1898.

[8] P. Painlevé, "Sur les equations differentielles du second ordre à points critiques fixes," *Comptes Rendus*, vol. 129, pp. 750–753, 949–952, 1899.

[9] B. Gambier, "Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est a points critiques fixes," *Acta Mathematica*, vol. 33, no. 1, pp. 1–55, 1910.

[10] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, NY, USA, 1927.

[11] M. P. Markakis, "On the reduction of non-linear oscillator-equations to Abel forms," *Applied Mathematics and Computation*, vol. 157, no. 2, pp. 357–368, 2004.

[12] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2nd edition, 2003.

[13] I. J. Grabshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego, Calif, USA, 5th edition, 1994.

[14] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill, New York, NY, USA, 2nd edition, 1968.

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