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Research Article
Fuzzy Entire Sequence Spaces

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We first investigate the notion of fuzzy entire sequence space with a suitable example. Also we deal with the properties of the space of fuzzy entire sequences. The concepts of subset and superset of the fuzzy entire sequence spaces are introduced and their properties are discussed.

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1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1] and subsequently several authors discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, fuzzy functional analysis, and fuzzy sequence spaces. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2] where every convergent sequence is bounded. Nanda [3] has studied the spaces of bounded and convergent sequences of fuzzy numbers and has shown that they are complete metric spaces. The space Γ of entire sequences was studied by Ganapathy Iyer [4]. In this paper, we introduce fuzzy topology into Γ. Then Γ turns out to be a fuzzy sequence spaces. Firstly we define the metric for the fuzzy sequence spaces and we try to introduce the notion of the fuzzy entire sequence spaces. Also we deal with topological properties of the space of fuzzy entire sequences. We first prove that the space of fuzzy entire sequence is a complete fuzzy metric space and also is a fuzzy linear metric space. The concepts of subset and superset of the fuzzy entire sequence space are introduced and their properties are discussed.

2. Preliminaries

At first, we recall some definitions and results about fuzzy numbers. A fuzzy number space is a fuzzy set on the real axis, that is, denote $F(R) = \{\widetilde{a} \mid \widetilde{a} : R^n \to [0,1]$, $\widetilde{a}$ has the following properties (a)–(d)$\}$:

(a) $\widetilde{a}$ is normal, that is, there exists an $x_0 \in R^n$ such that $\widetilde{a}(x_0) = 1$;

(b) $\widetilde{a}$ is convex, that is, $\widetilde{a}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{a}(x), \widetilde{a}(y)\}$ whenever $x, y \in R^n$ and $0 \leq \lambda \leq 1$;

(c) $\widetilde{a}(x)$ is upper semicontinuous;

(d) $[\widetilde{a}]^0 = \mathrm{cl}\{x \in R^n : \widetilde{a}(x) > 0\}$ is a compact set.

As obtained by Zadeh, $\widetilde{a}$ is convex if and only if each of its α-level sets $\widetilde{a}_\alpha$, where $\widetilde{a}_\alpha = \{x \in R^n : \widetilde{a}(x) \geq \alpha\}$ for each $\alpha \in (0,1]$, is a nonempty compact convex subset of R^n with compact support. The α-level set of an upper semicontinuous convex normal fuzzy number is a closed interval $[\widetilde{a}_\lambda^-, \widetilde{a}_\lambda^+]$, where the values $\widetilde{a}_\lambda^- = -\infty$ and $\widetilde{a}_\lambda^+ = +\infty$ are admissible. Since each $x \in R^n$ can be considered as a fuzzy number $\widetilde{a}$, defined by (2.1), the real number can be embedded in $F^*(R)$. A fuzzy number $\widetilde{a}$ is called nonnegative if $\widetilde{a}(x) = 0$ for all $x < 0$. The set of all nonnegative fuzzy numbers of $F^*(R)$ is denoted by $F(R)$. Let $k \in F(R)$ and $k = \bigcup_{\lambda \in [0,1]} \lambda[\underline{k}, \overline{k}]$,

$$\widetilde{a}(x) = \begin{cases} 1 & \text{for } x = k,\ k \in R^n, \\ 0 & \text{for } x \neq k,\ k \in R^n. \end{cases} \tag{2.1}$$

For any $\widetilde{a} \in F(R)$, $\widetilde{a}$ is called fuzzy number and $F(R)$ is called a fuzzy number space. For $\widetilde{a}, \widetilde{b} \in F(R)$, we define $\widetilde{a} \leq \widetilde{b}$ if and only if $[\widetilde{a}]_\lambda = [\widetilde{a}_\lambda^-, \widetilde{a}_\lambda^+] \leq [\widetilde{b}]_\lambda = [\widetilde{b}_\lambda^-, \widetilde{b}_\lambda^+]$ and $[\widetilde{a}]_\lambda \leq [\widetilde{b}]_\lambda$ if and only if $\widetilde{a}_\lambda^- \leq \widetilde{b}_\lambda^-$ and $\widetilde{a}_\lambda^+ \leq \widetilde{b}_\lambda^+$ for any $\lambda \in [0,1]$.

Theorem 2.1 (representation theorem). *For $\widetilde{a} \in F(R)$,*

(1) *$\widetilde{a}_\lambda^-$ is a bounded left continuous nondecreasing function on $(0,1]$;*

(2) *$\widetilde{a}_\lambda^+$ is a bounded left continuous nonincreasing function on $(0,1]$;*

(3) *$\widetilde{a}_\lambda^-$ and $\widetilde{a}_\lambda^+$ is right continuous at $\lambda = 0$;*

(4) *$\widetilde{a}_\lambda^- \leq \widetilde{a}_\lambda^+$.*

Moreover, if the pair of functions $a(\lambda)$ and $b(\lambda)$ satisfies (1)–(4), then there exists a unique $\widetilde{a} \in F(R)$ such that $\widetilde{a}_\lambda = [a(\lambda), b(\lambda)]$ for each $\lambda \in [0,1]$.

Define $F(R) \times F(R) \to R$ by $d(\widetilde{a}, \widetilde{b}) = \sup_{0 \leq a \leq 1} \delta_\infty(\widetilde{a}_\alpha, \widetilde{b}_\alpha)$ for $\widetilde{a}, \widetilde{b} \in F(R)$ and $\delta_\infty(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$. It is known that $(F(R), d)$ is a complete metric space.

Let $\widetilde{0}$ and $\widetilde{1}$ be defined by

$$\widetilde{0}(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0, \end{cases} \qquad \widetilde{1}(x) = \begin{cases} 1 & \text{for } x = 1, \\ 0 & \text{for } x \neq 1. \end{cases} \tag{2.2}$$

The absolute value $|\widetilde{a}|$ of $\widetilde{a} \in F(R)$ is defined by

$$|\widetilde{a}|(t) = \begin{cases} \max\{\widetilde{a}(t), \widetilde{a}(-t)\}, & t > 0, \\ 0, & t < 0. \end{cases} \tag{2.3}$$

Definition 2.2 [5]. A sequence $\widetilde{x} = (\widetilde{x}_k)$ of fuzzy numbers is said to be a Cauchy sequence to a fuzzy number $(\widetilde{x}_m)$, written as $\lim_{n\to k}\widetilde{x}_k = \widetilde{x}_m$, if for every $\varepsilon > 0$ there exists a positive integer N_0 such that $\widetilde{\rho}(\widetilde{x}_k,\widetilde{x}_m) < \varepsilon$ for $k,m > N_0$.

3. Fuzzy entire sequence $\Gamma(R)$

Now we introduce the new fuzzy sequence space and we show that this is a complete metric space.

For each fixed n, define the fuzzy metric

$$\widetilde{\rho}(\widetilde{x}_n,\widetilde{y}_n) = \bigcup_{\lambda\in[0,1]} \lambda\left\{ \left|(x_n)_1^- - (y_n)_1^-\right|^{1/n}, \sup_{\lambda\le\eta\le 1}\left\{ \left|(x_n)_\eta^- - (y_n)_\eta^-\right|^{1/n} \vee \left|(x_n)_\eta^+ - (y_n)_\eta^+\right|^{1/n}\right\}\right\}. \tag{3.1}$$

Let $\widetilde{x} = \{\widetilde{x}_n\}$ and $\widetilde{y} = \{\widetilde{y}_n\}$ be sequences of fuzzy real numbers. Define their distance by

$$\widetilde{\theta}(\widetilde{x},\widetilde{y}) = \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup_{(n)}\left|(x_n)_1^- - (y_n)_1^-\right|^{1/n}, \sup_{(n)}\left\{ \sup_{\lambda\le\eta\le 1}\left\{ \left|(x_n)_\eta^- - (y_n)_\eta^-\right|^{1/n} \vee \left|(x_n)_\eta^+ - (y_n)_\eta^+\right|^{1/n}\right\}\right\}\right\}. \tag{3.2}$$

Clearly, $(F(R),\widetilde{\rho})$ is a complete metric space [6].

Definition 3.1. $\widetilde{x} = \{\widetilde{x}_n\}$ is called a fuzzy entire sequence if $\widetilde{\rho}(\lim_{n\to\infty}\widetilde{x}_n) = \widetilde{0}$. In other words, given $\varepsilon > 0$ there exists a positive integer N such that $\widetilde{\rho}(\widetilde{x}_n,0) < \varepsilon$ for all $n \in N$. Let $\Gamma(R) = \{\text{all fuzzy entire sequences}\}$.

Example 3.2. Let $\widetilde{a}_n = 1/n!$ for $n = 1,2,\ldots$ be a sequence fuzzy numbers. Now we have $\widetilde{a} = (\widetilde{a}_n) = (1/n!)$, where $n \in F(R)$.

$$\widetilde{\rho}(\widetilde{a}_n,\widetilde{0}) = \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup\left|\left(\frac{1}{n!}\right)^- - 0^-\right|, \sup_n\left\{ \sup_{\lambda\le\eta\le 1}\left\{\left|\left(\frac{1}{n!}\right)_\eta^- - 0^-\right|^{1/n} \vee \left|\left(\frac{1}{n!}\right)_\eta^+ - 0^-\right|^{1/n}\right\}\right\}\right\} = \widetilde{0}. \tag{3.3}$$

This implies that $\widetilde{a} = (\widetilde{a}_n) = (1/n!) \in \Gamma(R)$. Hence, $\widetilde{a} = (\widetilde{a}_n)$ is a fuzzy entire sequence.

THEOREM 3.3. $(\Gamma(R),\widetilde{\theta})$ *is a fuzzy complete metric space.*

Proof. It can be seen that $\widetilde{\theta}$ is a metric for $\Gamma(R)$. Let $\{x^{(i)}\}$ be any fuzzy Cauchy sequence in $\Gamma(R)$. Then for every $\varepsilon > 0$, there exist a positive integer N such that

$$\begin{aligned}
&\widetilde{\rho}(\widetilde{x}_n^{(i)}, \widetilde{y}_n^{(j)}) \\
&\quad \leq \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup_n \left| x_{n_1}^{(i)^-} - x_{n_1}^{(j)^-} \right|^{1/n}, \sup_n \left\{ \sup_{\lambda\leq\eta\leq 1} \left\{ \left| x_{n_\eta}^{(i)^-} - x_{n_\eta}^{(j)^-} \right|^{1/n} \vee \left| x_{n_\eta}^{(i)^+} - x_{n_\eta}^{(j)^+} \right|^{1/n} \right\}\right\}\right\} \\
&\quad = \widetilde{\theta}(\widetilde{x}^{(i)}, \widetilde{x}^{(j)}) < \frac{\varepsilon}{5} \quad \text{for } i, j > N_n.
\end{aligned} \tag{3.4}$$

This implies that $\{x_n^{(i)}\}_{i=1}^{\infty}$ is a Cauchy sequence in $F(R)$ for each fixed n.

$(F(R), \widetilde{\rho})$ is a complete metric space; hence the Cauchy sequence $\{x_n^{(i)}\}_{i=1}^{\infty}$ converges to $\widetilde{x}_n$, that is $\rho(\widetilde{x}_n^{(i)}, \widetilde{x}_n) = 0$ as $i \to \infty$ for each fixed n

$$\begin{aligned}
&\Longrightarrow \lim_{i\to\infty} \bigcup_{\lambda\in[0,1]} \lambda\left\{ \left| x_{n_1}^{(i)^-} - x_{n_1}^- \right|^{1/n}, \sup_{\lambda\leq\eta\leq 1} \left\{ \left| x_{n_\eta}^{(i)^-} - x_{n_\eta}^- \right|^{1/n} \vee \left| x_{n_\eta}^{(i)^+} - x_{n_\eta}^+ \right|^{1/n} \right\}\right\} = 0 \quad \forall n \\
&\Longrightarrow \lim_{i\to\infty} \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup_{(n)} \left| x_{n_1}^{(i)^-} - x_{n_1}^- \right|^{1/n}, \sup_{(n)} \left\{ \sup_{\lambda\leq\eta\leq 1} \left\{ \left| x_{n_\eta}^{(i)^-} - x_{n_\eta}^- \right|^{1/n} \vee \left| x_{n_\eta}^{(i)^+} - x_{n_\eta}^+ \right|^{1/n} \right\}\right\}\right\} = 0 \quad \forall n \\
&\Longrightarrow \lim_{i\to\infty} \widetilde{\theta}(\widetilde{x}^{(i)}, \widetilde{x}) = 0, \quad \text{where } \widetilde{x} \in (\widetilde{x}_n).
\end{aligned} \tag{3.5}$$

Now we will show that $x \in \Gamma(R)$. In (3.4), letting $j \to \infty$, we get $\widetilde{\rho}(\widetilde{x}_n^{(i)}, \widetilde{x}_n) < \varepsilon/5$, since $\{x_n^{(i)}\}$ is a Cauchy sequence for each n.

This implies that $\widetilde{\rho}(\widetilde{x}_n^{(i)}, \widetilde{x}_k^{(i)}) < \varepsilon/5$ for $n, k \in N_i$.

Similarly, $\widetilde{\rho}(\widetilde{x}_n^{(j)}, \widetilde{x}_k^{(j)}) < \varepsilon/5$ for $n, k \in N_j$. For each fixed j, put $N = \max\{N_n, N_i, N_j\}$. Then for given $\varepsilon > 0$, there exist $x^{(i)}, x^{(j)} \in F(R)$ in connection with (3.4) such that

$$\begin{aligned}
&\widetilde{\rho}\left(\widetilde{x}_n^{(j)}, \widetilde{x}_k^{(j)}\right) < \frac{\varepsilon}{5}, \quad \widetilde{\rho}\left(\widetilde{x}_n^{(i)}, \widetilde{x}_n^{(j)}\right) < \frac{\varepsilon}{5} \quad \text{for } i, j, k, n > N \\
&\qquad \Longrightarrow \widetilde{\rho}\left(\widetilde{x}_k^{(i)}, \widetilde{x}_k^{(j)}\right) \leq \widetilde{\rho}\left(\widetilde{x}_n^{(i)}, \widetilde{x}_n^{(j)}\right) + \widetilde{\rho}\left(\widetilde{x}_n^{(i)}, \widetilde{x}_k^{(i)}\right) + \widetilde{\rho}\left(\widetilde{x}_n^{(j)}, \widetilde{x}_k^{(j)}\right) \leq \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \frac{3\varepsilon}{5} \\
&\qquad\qquad \text{for } i, j, k \geq N \\
&\qquad \Longrightarrow \{x_n^{(i)}\}_{i=1}^{\infty} \text{ is a fuzzy Cauchy sequence in } F(R) \\
&\qquad \Longrightarrow \text{by the completeness of } F(R), \widetilde{\rho}\left(\widetilde{x}_k^{(i)}, \widetilde{x}_k\right) \leq \frac{3\varepsilon}{5} \text{ for some } \widetilde{x}_k \in F(R) \\
&\qquad \Longrightarrow \widetilde{\rho}(\widetilde{x}_n, \widetilde{x}_k) \leq \widetilde{\rho}\left(\widetilde{x}_n, \widetilde{x}_n^{(i)}\right) + \widetilde{\rho}\left(\widetilde{x}_n^{(i)}, \widetilde{x}_k^{(i)}\right) + \widetilde{\rho}\left(\widetilde{x}_k^{(i)}, \widetilde{x}_k\right) \leq \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{3\varepsilon}{5} = \varepsilon \\
&\qquad\qquad \text{for each } n, k \geq N \\
&\qquad \Longrightarrow \{\widetilde{x}_n\} \text{ is a Cauchy sequence with respect to } \widetilde{\theta}.
\end{aligned} \tag{3.6}$$

This implies that $\widetilde{x} = \{\widetilde{x}_n\} \in \Gamma(R)$. Hence, $(\Gamma(R), \widetilde{\theta})$ is a fuzzy complete metric space. □

Theorem 3.4. *The space of the fuzzy entire sequences is a fuzzy linear space.*

Proof. Let $\tilde{\alpha},\tilde{\beta} \in \Gamma(R)$ and let $\tilde{\alpha} = (\tilde{a}_n)$ and $\tilde{\beta} = (\tilde{b}_n)$, where $\tilde{\rho}\lim_{n\to\infty} \tilde{a}_n = \tilde{0}$ and $\tilde{\rho}\lim_{n\to\infty} \tilde{b}_n = \tilde{0}$. To prove $a\tilde{\alpha} + b\tilde{\beta} \in \Gamma(R)$ and $a\tilde{\alpha} + b\tilde{\beta} = a(\tilde{a}_n) + b(\tilde{b}_n)$. It is enough to prove that $\tilde{\rho}\lim_{n\to\infty}(a\tilde{a}_n + b\tilde{b}_n) = \tilde{0}$. Since $\tilde{\rho}\lim_{n\to\infty} \tilde{a}_n = \tilde{0}$, given $\varepsilon > 0$, there exists a positive integer N_1 such that $\tilde{\rho}(\tilde{a}_n,\tilde{0}) < \varepsilon/|a|$ for all $n \geq N_1$. Since $\tilde{\rho}\lim_{n\to\infty} \tilde{b}_n = \tilde{0}$, given $\varepsilon > 0$, there exists a positive integer N_2 such that $\tilde{\rho}(\tilde{b}_n,\tilde{0}) < \varepsilon/|b|$ for all $n \geq N_2$.

Let $N = \max\{N_1,N_2\}$. Then

$$\begin{aligned}\tilde{\rho}\lim_{n\to\infty}(a\tilde{a}_n + b\tilde{b}_n) &= \tilde{\rho}(a\tilde{a}_n + b\tilde{b}_n,\tilde{0}) = \tilde{\rho}(a\tilde{a}_n,\tilde{0}) + \tilde{\rho}(b\tilde{b}_n,\tilde{0}) = a\tilde{\rho}(\tilde{a}_n,\tilde{0}) + b\tilde{\rho}(\tilde{b}_n,\tilde{0}) \\ &= \frac{a\varepsilon}{|a|} + \frac{b\varepsilon}{|b|} = \varepsilon + \varepsilon < \varepsilon \quad \forall n \geq N\ \tilde{\rho}(a\tilde{a}_n + b\tilde{b}_n,\tilde{0}) < \varepsilon.\end{aligned} \tag{3.7}$$

This implies that $\tilde{\rho}\lim_{n\to\infty}(a\tilde{a}_n + b\tilde{b}_n) = \tilde{0}$. Therefore, $a\tilde{\alpha} + b\tilde{\beta} \in \Gamma(R)$. Hence, $\Gamma(R)$ is a fuzzy linear space. □

Theorem 3.5. *The space of the fuzzy entire sequence is a fuzzy linear metric space.*

Proof. $\Gamma(R)$ is a fuzzy metric space if we define the metric $\tilde{\rho}$ by

$$\tilde{\rho}(\tilde{\alpha},\tilde{\beta}) = \bigcup_{\lambda\in[0,1]} \lambda\left\{ \left|(a_n)_1^- - (b_n)_1^-\right|^{1/n}, \sup_{\lambda\leq\eta\leq1}\left\{ \left|(a_n)_\eta^- - (b_n)_\eta^-\right|^{1/n} \vee \left|(a_n)_\eta^+ - (b_n)_\eta^+\right|^{1/n}\right\}\right\}, \tag{3.8}$$

where $\tilde{\alpha},\tilde{\beta} \in \Gamma(R)$ and $\tilde{\alpha} = (\tilde{a}_n)$ and $\tilde{\beta} = (\tilde{b}_n)$. To prove that $\Gamma(R)$ is a fuzzy linear metric space, it is enough to prove $\tilde{\alpha} + \tilde{\beta}$, $k\tilde{\alpha}$ where $\tilde{\alpha},\tilde{\beta} \in \Gamma(R)$ and $k \in R^+$. That $\tilde{\alpha} + \tilde{\beta}$ is fuzzy continuous follows from the property $|\tilde{\alpha}+\tilde{\beta}| \leq |\tilde{\alpha}| \oplus |\tilde{\beta}|$. To prove that $k\tilde{\alpha}$ is continuous; it is enough to prove that $\tilde{\alpha}_n \to \tilde{\alpha}$ in $\Gamma(R)$ and $k_n \to k \Rightarrow k_n\tilde{\alpha} \to k\tilde{\alpha}$ for each $\tilde{\alpha} \in \Gamma(R)$.
Case 1. Let $\tilde{\alpha}_n \to \tilde{\alpha}$ in $\Gamma(R)$. To prove that $k_n\tilde{\alpha} \to k\tilde{\alpha}$ since $\tilde{\alpha}_n \to \tilde{\alpha}$, given $\varepsilon > 0$, there exists N such that $\tilde{\rho}(\tilde{\alpha}_n,\tilde{\alpha}) < \varepsilon/|k|^n$ for all $n \geq N$. Now

$$\begin{aligned}&\tilde{\rho}(k\tilde{\alpha}_n,k\tilde{\alpha}) \\ &\quad\leq \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup_{(n)} \left|k\tilde{\alpha}_{n_1} - k\tilde{\alpha}\right|^{1/n}, \sup_{(n)}\left\{ \sup_{\lambda\leq\eta\leq1}\left\{ \left|k\tilde{\alpha}_{n_\eta}^- - k\tilde{\alpha}^-\right|^{1/n} \vee \left|k\tilde{\alpha}_{n_\eta}^+ - k\tilde{\alpha}^+\right|^{1/n}\right\}\right\}\right\}, \\ &\tilde{\rho}(k\tilde{\alpha}_n,k\tilde{\alpha}) \\ &\quad\leq \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup_{(n)} |k|^{1/n}\left|\tilde{\alpha}_{n_1} - \tilde{\alpha}\right|^{1/n}, \sup_{(n)}\left\{ \sup_{\lambda\leq\eta\leq1} |k|^{1/n}\left\{ \left|\tilde{\alpha}_{n_\eta}^- - \tilde{\alpha}^-\right|^{1/n} \vee |k|^{1/n}\left|\tilde{\alpha}_{n_\eta}^+ - \tilde{\alpha}^+\right|^{1/n}\right\}\right\}\right\}, \\ &\tilde{\rho}(k\tilde{\alpha}_n,k\tilde{\alpha}) \leq |k|^{1/n}\tilde{\rho}(\tilde{\alpha}_n,\tilde{\alpha}) \leq |k|^{1/n}\varepsilon/|k|^{1/n} < \varepsilon.\end{aligned} \tag{3.9}$$

Therefore, $k_n\tilde{\alpha} \to k\tilde{\alpha}$ as $n \to \infty$ that is $\tilde{\rho}\lim_{n\to\infty} \tilde{a}_n = \tilde{\alpha}$.

Case 2. Let $k_n \to k$. To prove $k_n\tilde{\alpha}$ for each $\tilde{\alpha} \in \Gamma(R)$, consider $k = \tilde{0}$. Let $\tilde{\alpha} = (\tilde{a}_n) \in \Gamma(R)$, where $\tilde{\rho}\lim_{n\to\infty} \tilde{a}_n = \tilde{0}$. Since $\tilde{\rho}\lim_{n\to\infty} \tilde{a}_n = \tilde{0}$, given $\varepsilon > 0$, there exists N_1 such that $\tilde{\rho}(\tilde{a}_n,\tilde{0}) < \varepsilon$ for all $n \geq N_1$. Since $k_n \to \tilde{0}$, we may suppose that $\tilde{\tilde{\rho}}(k_n,\tilde{0}) < 1$ for all $n \geq N_1$. From the fuzzy metric space [6], obviously we get $|k_n\tilde{\alpha}| \leq \varepsilon$ for $n \geq N_1$. That is, $k_n\tilde{\alpha} \to \tilde{0}$ as $n \to \infty$. This proves that $\Gamma(R)$ is a fuzzy linear metric space. □

Theorem 3.6. *A fuzzy entire sequence space is separable.*

Proof. Let $C = \{x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots\}$ be a countable subset of $\Gamma(R)$ and $x_i \in Q \subset \Gamma(R)$, where $Q = \{$all fuzzy rational numbers$\}$. Hence, $\Gamma(R)$ is separable. □

4. A subset of $\Gamma(R)$

In fact, χ is a subset of Γ. For each fixed n, define the fuzzy metric

$$\tilde{\mu}(\tilde{x}_n, \tilde{y}_n) = \bigcup_{\lambda\in[0,1]} \lambda\Big\{ |\angle n(x_n)_1^- - \angle n(y_n)_1^-|^{1/n}, \sup_{\lambda\leq\eta\leq 1}\Big\{ |\angle n(x_n)_\eta^- - \angle n(y_n)_\eta^-|^{1/n} \vee |\angle n(x_n)_\eta^+ - \angle n(y_n)_\eta^+|^{1/n} \Big\}\Big\}. \tag{4.1}$$

Let $\tilde{x} = \{\tilde{x}_n\}$ and $\tilde{y} = \{\tilde{y}_n\}$ be sequence of fuzzy real numbers. Define their distance by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \bigcup_{\lambda\in[0,1]} \lambda\left\{ \sup_{(n)} |\angle n(x_n)_1^- - \angle n(y_n)_1^-|^{1/n}, \sup_{(n)}\left\{ \sup_{\lambda\leq\eta\leq 1}\left\{ \begin{array}{l} |\angle n(x_n)_\eta^- - \angle n(y_n)_\eta^-|^{1/n} \\ \vee |\angle n(x_n)_\eta^+ - \angle n(y_n)_\eta^+|^{1/n} \end{array}\right\}\right\}\right\}. \tag{4.2}$$

Definition 4.1. $\tilde{x} = \{\tilde{x}_n\} \in \chi$ if $\tilde{\mu}(lim_{n\to\infty}\tilde{x}_n) = \tilde{0}$. In other words, given $\varepsilon > 0$ there exists a positive integer N such that $\tilde{\mu}(\tilde{x}_n, 0) < \varepsilon$ for all $n \in N$. The set of all fuzzy subsets of $\Gamma(R)$ is denoted by $\chi(R)$. Note that $\chi(R) \subset \Gamma(R)$.

Proposition 4.2. *$\chi(R)$ is a proper subset of $\Gamma(R)$.*

Proof. Let $|x_n| \leq \angle n|x_n|$. This implies that $|x_n|^{1/n} \leq (\angle n|x_n|)^{1/n}$. Let $(x_n) \in \chi(R) \Rightarrow (\angle n|x_n|)^{1/n} < \varepsilon \Rightarrow |x_n|^{1/n} < \varepsilon$ for all $n \geq n_0 \Rightarrow (x_n) \in \Gamma(R) \Rightarrow \chi(R) \subset \Gamma(R)$. □

Theorem 4.3. *$\chi(R)$ is a proper closed subspace of $\Gamma(R)$.*

Proof

Step 1. $(1/\angle n) \notin \chi(R)$ But $(1/\angle n) \in \Gamma(R)$. This implies that $\chi(R)$ is a proper subspace of $\Gamma(R)$.

Step 2. Suppose that $a^{(p)} \to a \in \chi(R) \Rightarrow |a_0^{(p)} - a_0|, [\angle n|a_n^{(p)} - a_n|]^{1/n} < \varepsilon$ for $p \geq p_0$,

$$\begin{aligned} \Longrightarrow \left[\angle n\,|\,a_n\,|\,\right]^{1/n} &\leq \left[\angle n\,|\,a_n^{(p)}\,|\,\right]^{1/n} + \left[\angle n\,|\,a_n^{(p)} - a_n\,|\,\right]^{1/n} \\ &< \left[\angle n\,|\,a_n^{(p)}\,|\,\right]^{1/n} + (\angle n)^{1/n}\varepsilon/n \quad \text{for } p \geq p_0 \\ \Longrightarrow \left[\angle n\,|\,a_n\,|\,\right]^{1/n} &< \varepsilon_1 + k\varepsilon, \text{ for sufficiently large } n. \end{aligned} \tag{4.3}$$

From Stirling's formula, where $\varepsilon \to 0$ as $p \to \infty$ and k is a positive constant

$$\Longrightarrow \tilde{\mu}\left(\left[\angle n\,|\,a_n\,|\,\right]^{1/n}, \tilde{0}\right) < \varepsilon \quad \forall n \geq n_0 \Longrightarrow a = (a_n) \in \chi(R). \tag{4.4}$$

□

Theorem 4.4. *A subset $\chi(R)$ of $\Gamma(R)$ is a fuzzy complete.*

Proof. Let $\{a_p : p \geq 1\}$ be a fuzzy Cauchy sequence in $\chi(R)$, where $|\angle n a_n^{(p)}|^{1/n} \to 0$ as $n \to \infty$, for each $p \geq 1$.

Let $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that

$$\begin{aligned} &\sup\left\{\,|\,a_0^{(p)} - a_0^{(m)}\,|, \left[\angle n\,|\,a_n^{(p)} - a_n^{(m)}\,|\,\right]^{1/n}, n \geq 1\right\} < \varepsilon \quad \text{for } m, p \geq N \\ &\quad \Longrightarrow |\,a_0^{(p)} - a_0^{(m)}\,| < \varepsilon, \left[\angle n\,|\,a_n^{(p)} - a_n^{(m)}\,|\,\right]^{1/n} < \varepsilon \quad \text{for } m, p \geq N \quad (n = 1, 2, \dots) \end{aligned} \tag{4.5}$$

$\Rightarrow \{\angle n|a_n^{(p)}|\}^{1/n}$ and so $\{a_n^{(p)}\}$ is a fuzzy Cauchy sequence in the complex plane for each $n \geq 1$.

$$\Longrightarrow a_n^{(p)} \longrightarrow a_n \quad \text{as } p \longrightarrow \infty \quad \forall n \geq 1 \Longrightarrow \angle n a_n^{(p)} \longrightarrow \angle n a_n \quad \text{as } p \longrightarrow \infty. \tag{4.6}$$

Now for $n \geq 1$, $p \geq 1$, $|\angle n a_n|^{1/n} \leq [\angle n|a_n - a_n^{(p)}|]^{1/n} + \{\angle n|a_n^{(p)}|\}^{1/n}$.

Let $\varepsilon > 0$, then there exists p_0 such that $[\angle n|a_n^{(p)} - a_n|]^{1/n} < \varepsilon$ and so this holds in particular for $p = p_0$.

Consequently,

$$|\,\angle n a_n\,|^{1/n} < \varepsilon + \left[\,|\,\angle n a_n^{(p_0)}\,|\,\right]^{1/n} \quad \text{for } n \geq 1. \qquad \text{Now } |\,\angle n a_n^{(p_0)}\,|^{1/n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.7}$$

Thus,

$$\widetilde{\mu}\Big(\lim_{n\to\infty} \angle n\,|\,a_n\,|^{1/n}\Big) \leq \varepsilon, \quad \Longrightarrow \lim_{n\to\infty} \widetilde{\mu}\big[\angle n\,|\,a_n\,|\,\big]^{1/n} = 0, \quad \Longrightarrow a = (a_n) \in \chi(R). \tag{4.8}$$

Also (4.5) and (4.6), $|a_0^{(p)} - a_0| < \varepsilon$; $[\angle n|a_n^{(p)} - a_n|]^{1/n} < \varepsilon$ for $p \geq N(n = 1,2,\dots) \Rightarrow |a_p - a| \to 0$ as $p \to \infty \Rightarrow \chi(R)$ is a fuzzy complete of $\Gamma(R)$. □

Corollary 4.5. *$(\chi(R), \widetilde{\theta})$ is a complete fuzzy metric space.*

Proof. From Proposition 4.2 and Theorem 3.3, it follows that $\chi(R)$ is a closed subspace of the complete fuzzy metric space of $\Gamma(R)$. Hence, $\chi(R)$ is complete fuzzy metric space. □

5. A superset of $\Gamma(R)$

In fact, Λ is a superset of Γ.

Definition 5.1. $\widetilde{x} = \{\widetilde{x}_n\} \in \Lambda$ is called a fuzzy analytic sequence Λ, if $\widetilde{\rho}(\widetilde{x}_n, \widetilde{0}) \leq M$, for all n and some constant $M > 0$. The set of all fuzzy analytic sequences is denoted by $\Lambda(R)$. Note that $\Gamma(R) \subset \Lambda(R)$.

Without proof, we state the following theorems.

Theorem 5.2. *$(\Lambda(R), \widetilde{\theta})$ is a complete fuzzy metric space.*

Proof. It is similar to the proof of Theorem 3.3. □

Theorem 5.3. *$\Gamma(R)$ is a closed subspace of $\Lambda(R)$.*

Proof. It is similar to the proof of Theorem 4.3. □

6. Conclusion

In this paper, we try to introduce the concept of fuzzy entire sequence spaces and we deal with some topological properties also. Many known sequence spaces can be fuzzified. There is considerable scope for further research in this area like in matrix transformation.

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