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Research Article

A Comparison of Deformations and Geometric Study of Varieties of Associative Algebras

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The aim of this paper is to give an overview and to compare the different deformation theories of algebraic structures. In each case we describe the corresponding notions of degeneration and rigidity. We illustrate these notions by examples and give some general properties. The last part of this work shows how these notions help in the study of varieties of associative algebras. The first and popular deformation approach was introduced by M. Gerstenhaber for rings and algebras using formal power series. A noncommutative version was given by Pinczon and generalized by F. Nadaud. A more general approach called global deformation follows from a general theory by Schlessinger and was developed by A. Fialowski in order to deform infinite-dimensional nilpotent Lie algebras. In a nonstandard framework, M. Goze introduced the notion of perturbation for studying the rigidity of finite-dimensional complex Lie algebras. All these approaches share the common fact that we make an "extension" of the field. These theories may be applied to any multilinear structure. In this paper, we will be dealing with the category of associative algebras.

1. Introduction

Throughout this paper $\mathbb{K}$ will be an algebraically closed field, and $\mathcal{A}$ denotes an associative $\mathbb{K}$-algebra. Most examples will be finite dimensional, let V be the underlying n-dimensional vector space of $\mathcal{A}$ over $\mathbb{K}$ and $(e_1,\dots,e_n)$ be a basis of V. The bilinear map μ denotes the multiplication of $\mathcal{A}$ on V, and e_1 is the unit element. By linearity this can be done by specifying the n^3 structure constants $C_{ij}^k \in \mathbb{K}$ where $\mu(e_i,e_j) = \sum_{k=1}^n C_{ij}^k e_k$. The associativity condition limits the sets of structure constants, C_{ij}^k, to a subvariety of $\mathbb{K}^{n^3}$

which we denote by alg_n. It is generated by the polynomial relations

$$\sum_{l=1}^{n} C_{ij}^{l} C_{lk}^{s} - C_{il}^{s} C_{jk}^{l} = 0, \quad C_{1i}^{j} = C_{i1}^{j} = \delta_i^j, \quad 1 \leq i, j, k, s \leq n. \tag{1.1}$$

This variety is quadratic, nonregular, and in general nonreduced. The natural action of the group $\text{GL}(n, \mathbb{K})$ corresponds to the change of basis: two algebras μ_1 and μ_2 over V are isomorphic if there exists f in $\text{GL}(n, \mathbb{K})$ such that

$$\forall X, Y \in V, \quad \mu_2(X, Y) = (f \cdot \mu_1)(X, Y) = f^{-1}(\mu_1(f(X), f(Y))). \tag{1.2}$$

The orbit of an algebra $\mathcal{A}$ with multiplication μ_0, denoted by $\vartheta(\mu_0)$, is the set of all its isomorphic algebras. The deformation techniques are used to do the geometric study of these varieties. The deformation attempts to understand which algebra we can get from the original one by deforming. At the same time it gives more information about the structure of the algebra, for example, we can try to see which properties are stable under deformation.

The deformation of mathematical objects is one of the oldest techniques used by mathematicians. The different areas where the notion of deformation appears are geometry, complex manifolds (Kodaira and Spencer 1958, Kuranishi 1962), algebraic manifolds (Artin [1] and Schlessinger [2]), Lie algebras (Nijenhuis and Richardson [3]) and rings and associative algebras (Gerstenhaber [4]). The dual notion of deformation (in some sense) is the notion of degeneration which appears first in the physics literature (Segal 1951, Inonu and Wigner [5]). Degeneration is also called specialization or contraction.

The quantum mechanics and the theory of quantum groups had a large impact on the theory of deformation. The theory of deformation quantization was introduced by Bayen et al. [6–8] to describe the quantum mechanics as a deformation of classical mechanics. To a given Poisson manifold a star-product is associated, which is a one parameter family of associative algebras. On the other hand, the quantum groups are obtained by deforming the Hopf structure of an algebra, in particular universal enveloping algebras. In [9], we show the existence of associative deformation of a universal enveloping algebra using the linear Poisson structure of the Lie algebra.

In the following, we will first recall the different notions of deformation. The most frequently used one is the formal deformation introduced by Gerstenhaber for rings and algebras [4, 10, 11], it uses a formal series and links the theory of deformation to Hochschild cohomology. A noncommutative version, where the parameter no longer commutes with the element of algebra, was introduced by Pinczon [12] and generalized by Nadaud [13]. They describe the corresponding cohomology and show that the Weyl algebra, which is rigid for formal deformation, is nonrigid in the noncommutative case. In a nonstandard framework, Goze et al. introduced the notion of perturbation for studying the rigidity of Lie algebras [14, 15], it was also used to describe the 6-dimensional rigid associative algebras [16, 17]. The perturbation needs the concept of infinitely small elements, these elements are obtained here in an algebraic way. We construct an extension of the set of real or complex numbers containing these elements. A more general notion called global deformation was introduced by Fialowski, following Schlessinger, for Lie algebras [18]. All these approaches are compared in Section 2 and compared in Section 3. Section 4 is devoted to universal and versal deformations. In Section 5, we describe the corresponding notions of degeneration with some general properties and examples. Section 6 is devoted to the study of rigidity of algebra in each framework. The last section concerns the geometric study of the algebraic varieties alg_n using deformation tools.

In formal deformations the properties are described using Hochschild cohomology groups. The global deformation seems to be the good framework to solve the problem of universal or versal deformations, the deformations which generate the others. Nevertheless, the perturbation approach is sometimes more adapted to direct computation.

2. The world of deformations

2.1. Gerstenhaber's formal deformation. Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{K}$, V the underlying vector space, and μ_0 the multiplication.

Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and $V[[t]]$ be the extension of V by extending the coefficients domain from $\mathbb{K}$ to $\mathbb{K}[[t]]$. Then $V[[t]]$ is a $\mathbb{K}[[t]]$-module and $V[[t]] = V \bigotimes_{\mathbb{K}} \mathbb{K}[[t]]$ when V is finite dimensional. Note that V is a submodule of $V[[t]]$. We can obtain an extension of V with a structure of vector space by extending the coefficients domain from $\mathbb{K}$ to $\mathbb{K}((t))$, the field of formal Laurent series $\mathbb{K}[[t]]$.

Any bilinear map $f : V \times V \to V$ (in particular the multiplication in $\mathcal{A}$) can be extended to a bilinear map from $V[[t]] \times V[[t]]$ to $V[[t]]$.

Definition 2.1. Let μ_0 be the multiplication of the associative algebra $\mathcal{A}$. A deformation of μ_0 is a one parameter family μ_t in $\mathbb{K}[[t]] \otimes V$ over the formal power series ring $\mathbb{K}[[t]]$ of the form $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots$ where $\mu_i \in \operatorname{Hom}(V \times V, V)$ (bilinear maps) satisfying the (formally) condition of associativity:

$$\forall X, Y, Z \in V, \quad \mu_t(\mu_t(X, Y), Z) = \mu_t(X, \mu_t(Y, Z)). \tag{2.1}$$

We note that the deformation of $\mathcal{A}$ is a $\mathbb{K}$-algebra structure on $\mathcal{A}[[t]]$ such that $\mathcal{A}[[t]]/t\mathcal{A}[[t]]$ is isomorphic to $\mathcal{A}$.

The previous equation is equivalent to an infinite system of equations, and it is called the *deformation equation*. The resolution of the deformation equation links deformation theory to Hochschild cohomology. Let $C^d(\mathcal{A}, \mathcal{A})$ be the space of d-cochains, the space of multilinear maps from $V^{\times d}$ to V.

The coboundary operator δ^d, which we denote by δ if there is no ambiguity,

$$\delta^d : C^d(\mathcal{A}, \mathcal{A}) \longrightarrow C^{d+1}(\mathcal{A}, \mathcal{A}), \qquad \varphi \longrightarrow \delta^d\varphi \tag{2.2}$$

is defined for $(x_1, \dots, x_{d+1}) \in V^{\times(d+1)}$ by

$$\begin{aligned} \delta^d\varphi(x_1, \dots, x_{d+1}) = {} & \mu_0(x_1, \varphi(x_2, \dots, x_{d+1})) + \sum_{i=1}^{d}(-1)^i\varphi(x_1, \dots, \mu_0(x_i, x_{i+1}), \dots, x_d) \\ & + (-1)^{d+1}\mu_0(\varphi(x_1, \dots, x_d), x_{d+1}). \end{aligned} \tag{2.3}$$

The group of d-cocycles is $Z^d(\mathcal{A},\mathcal{A}) = \{\varphi \in C^d(\mathcal{A},\mathcal{A}) : \delta^d\varphi = 0\}$.

The group of d-coboundaries is

$$B^d(\mathcal{A},\mathcal{A}) = \{\varphi \in C^d(\mathcal{A},\mathcal{A}) : \varphi = \delta^{d-1}f,\ f \in C^{d-1}(\mathcal{A},\mathcal{A})\}. \tag{2.4}$$

The dth Hochschild cohomology group of the algebra $\mathcal{A}$ with coefficients in $\mathcal{A}$ is given by

$$H^d(\mathcal{A},\mathcal{A}) = \frac{Z^d(\mathcal{A},\mathcal{A})}{B^d(\mathcal{A},\mathcal{A})}. \tag{2.5}$$

We define two maps $\circ$ and $[\cdot,\cdot]_G$:

$$\circ, [\cdot,\cdot]_G : C^d(\mathcal{A},\mathcal{A}) \times C^e(\mathcal{A},\mathcal{A}) \longrightarrow C^{d+e-1}(\mathcal{A},\mathcal{A}) \tag{2.6}$$

by

$$\begin{gathered}(\varphi \circ \psi)(a_1,\ldots,a_{d+e-1}) = \sum_{i=0}^{d-1}(-1)^{i(e-1)}\varphi(a_1,\ldots,a_i,\psi(a_{i+1},\ldots,a_{i+e}),\ldots),\\ [\varphi,\psi]_G = \varphi\circ\psi - (-1)^{(e-1)(d-1)}\psi\circ\varphi.\end{gathered} \tag{2.7}$$

The space $(C(\mathcal{A},\mathcal{A}),\circ)$ is a pre-Lie algebra (see [4, 10, 11] for the definition) and $(C(\mathcal{A},\mathcal{A})[\cdot,\cdot]_G)$ is a graded Lie algebra (Lie superalgebra). The bracket $[\cdot,\cdot]_G$ is called Gerstenhaber's bracket. The square of $[\mu_0,\cdot]_G$ vanishes and defines the 2-coboundary operator. Up to a global sign, the multiplication μ_0 of $\mathcal{A}$ is associative if $[\mu_0,\mu_0]_G = 0$.

Now, we discuss the deformation equation in terms of cohomology. The deformation equation may be written

$$\sum_{i=0}^{k}\mu_i \circ \mu_{k-i} = 0, \quad k = 0,1,2,\ldots. \tag{$*$}$$

The first equation ($k = 0$) is the associativity condition for μ_0. The second equation shows that μ_1 must be a 2-cocycle for Hochschild cohomology ($\mu_1 \in Z^2(\mathcal{A},\mathcal{A})$).

More generally, suppose that μ_p is the first nonzero coefficient after μ_0 in the deformation μ_t. This μ_p is called the *infinitesimal* of μ_t and *is a* 2*-cocycle of the Hochschild cohomology of* $\mathcal{A}$ *with coefficients in itself.*

The cocycle μ_p is called integrable if it is the first term, after μ_0, of an associative deformation.

The integrability of μ_p implies an infinite sequence of relations which may be interpreted as the vanishing of the obstruction to the integration of μ_p.

For an arbitrary $k > 1$, the kth equation of the system $(*)$ may be written

$$\delta\mu_k = \sum_{i=1}^{k-1}\mu_i \circ \mu_{k-i}. \tag{2.8}$$

Suppose that the truncated deformation $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots + t^{m-1}\mu_{m-1}$ satisfies the deformation equation. The truncated deformation is extended to a deformation of order m, that is, $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots + t^{m-1}\mu_{m-1} + t^m\mu_m$, satisfying the deformation equation if

$$\delta\mu_m = \sum_{i=1}^{m-1} \mu_i \circ \mu_{m-i}. \tag{2.9}$$

The right-hand side of this equation is called the *obstruction* to find μ_m extending the deformation.

The obstruction is a Hochschild 3-cocycle. Then, if $H^3(\mathcal{A},\mathcal{A}) = 0$, *it follows that all obstructions vanish and every* $\mu_m \in Z^2(\mathcal{A},\mathcal{A})$ *is integrable.*

Given two associative deformations μ_t and μ_t' of μ_0, we say that they are equivalent if there is a formal isomorphism F_t which is a $\mathbb{K}[[t]]$-linear map that may be written in the form

$$F_t = id + tf_1 + t^2 f_2 + \cdots, \quad \text{where } f_i \in \operatorname{End}_{\mathbb{K}}(V), \tag{2.10}$$

such that $\mu_t = F_t \cdot \mu_t'$ defined by

$$\mu_t(X,Y) = F_t^{-1}(\mu_t'((F_t(X),F_t(Y)))) \quad \forall X,Y \in V. \tag{2.11}$$

A deformation μ_t of μ_0 is called *trivial* if and only if μ_t is equivalent to μ_0.

PROPOSITION 2.2. *Every nontrivial deformation* μ_t *of* μ_0 *is equivalent to* $\mu_t = \mu_0 + t^p\mu_p' + t^{p+1}\mu_{p+1}' + \cdots$ *where* $\mu_p' \in Z^2(\mathcal{A},\mathcal{A})$ *and* $\mu_p' \notin B^2(\mathcal{A},\mathcal{A})$.

Then we have this fundamental and well-known theorem.

THEOREM 2.3. *If* $H^2(\mathcal{A},\mathcal{A}) = 0$, *then all formal deformations of* $\mathcal{A}$ *are equivalent to a trivial deformation.*

Remark 2.4. The notion of formal deformation is extended to coalgebras and bialgebras in [19].

2.2. Noncommutative formal deformation. In the aforementioned formal deformation the parameter commutes with the elements of original algebra. Motivated by some nonclassical deformation appearing in the quantization of Nambu mechanics, Pinczon introduced a deformation called noncommutative deformation where the parameter no longer commutes with the original algebra. He also developed the associated cohomology [12].

Let $\mathcal{A}$ be a $\mathbb{K}$-vector space and let σ be an endomorphism of $\mathcal{A}$. We give $\mathcal{A}[[t]]$ a $\mathbb{K}[[t]]$-bimodule structure defined for every $a_p \in \mathcal{A}, \lambda_q \in \mathbb{K}$ by

$$\begin{aligned} \sum_{p\geq 0} a_p t^p \cdot \sum_{q\geq 0} \lambda_q t^q &= \sum_{p,q\geq 0} \lambda_q a_p t^{p+q}, \\ \sum_{q\geq 0} \lambda_q t^q \cdot \sum_{p\geq 0} a_p t^p &= \sum_{p,q\geq 0} \lambda_q \sigma^q(a_p) t^{p+q}. \end{aligned} \tag{2.12}$$

Definition 2.5. A σ-deformation of an algebra $\mathcal{A}$ is a $\mathbb{K}$-algebra structure on $\mathcal{A}[[t]]$ which is compatible with the previous $\mathbb{K}[[t]]$-bimodule structure and such that

$$\mathcal{A} \cong \mathcal{A}[[t]]/(\mathcal{A}[[t]]t). \tag{2.13}$$

The previous deformations were generalized by Nadaud in [13] where he considered deformations based on two commuting endomorphisms σ and τ. The $\mathbb{K}[[t]]$-bimodule structure on $\mathcal{A}[[t]]$ is defined for $a \in \mathcal{A}$ by the formulas $t \cdot a = \sigma(a)t$ and $a \cdot t = \tau(a)t$, ($a \cdot t$ being the right action of t on a).

The remarkable difference with commutative deformations is that the Weyl algebra of differential operators with polynomial coefficients over $\mathbb{R}$ is rigid for commutative deformations but has a nontrivial noncommutative deformation; it is given by the enveloping algebra of the Lie superalgebra $osp(1,2)$.

2.3. Global deformation. The approach follows from a general fact in Schlessinger's works [2] and was developed by Fialowski [20]. She applies it to construct deformations of Lie subalgebras of the Witt algebra. We summarize the notion of global deformations in the case of an associative algebra. Let $\mathbb{B}$ be a commutative unital algebra over a field $\mathbb{K}$ of characteristic 0 and augmentation morphism $\varepsilon : \mathcal{A} \to \mathbb{K}$ (a $\mathbb{K}$-algebra homomorphism, $\varepsilon(1_{\mathbb{B}}) = 1$). We set $m_\varepsilon = \operatorname{Ker}(\varepsilon)$; m_ε is a maximal ideal of $\mathbb{B}$. (A maximal ideal m of $\mathbb{B}$ such that $\mathcal{A}/m \cong \mathbb{K}$ defines naturally an augmentation.) We call $(\mathbb{B}, m)$ base of deformation.

Definition 2.6. A global deformation of base $(\mathbb{B}, m)$ of an algebra $\mathcal{A}$ with a multiplication μ is a structure of $\mathbb{B}$-algebra on the tensor product $\mathbb{B} \bigotimes_{\mathbb{K}} \mathcal{A}$ with a multiplication $\mu_{\mathbb{B}}$ such that $\varepsilon \otimes \mathrm{id} : \mathbb{B} \otimes \mathcal{A} \to \mathbb{K} \otimes \mathcal{A} = \mathcal{A}$ is an algebra homomorphism. That is, for all $a, b \in \mathbb{B}$ and for all $x, y \in \mathcal{A}$,

(1) $\mu_{\mathbb{B}}(a \otimes x, b \otimes y) = (ab \otimes \mathrm{id})\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y)$ ($\mathbb{B}$-linearity),
(2) the multiplication $\mu_{\mathbb{B}}$ is associative,
(3) $\varepsilon \otimes \mathrm{id}(\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y)) = 1 \otimes \mu(x, y)$.

Remark 2.7. Condition (1) shows that to describe a global deformation it is enough to know the products $\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y)$, where $x, y \in \mathcal{A}$. The conditions (1) and (2) show that the algebra is associative and the last condition insures the compatibility with the augmentation. We deduce

$$\mu_B(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \alpha_i \otimes z_i \quad \text{with } \alpha_i \in m,\ z_i \in \mathcal{A}. \tag{2.14}$$

2.3.1. Equivalence and push-out.

(a) A global deformation is called *trivial* if the structure of $\mathbb{B}$-algebra on $\mathbb{B} \bigotimes_{\mathbb{K}} \mathcal{A}$ satisfies $\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y)$.
(b) Two deformations of an algebra with the same base are called *equivalent* (or isomorphic) if there exists an algebra isomorphism between the two copies of $\mathbb{B} \bigotimes_{\mathbb{K}} \mathcal{A}$, compatible with $\varepsilon \otimes \mathrm{id}$.
(c) A global deformation with base $(\mathbb{B}, m)$ is called *local* if $\mathbb{B}$ is a local $\mathbb{K}$-algebra with a unique maximal ideal $m_{\mathbb{B}}$. If in addition $m_{\mathbb{B}}^2 = 0$, the deformation is called *infinitesimal*.

(d) Let $\mathbb{B}'$ be another commutative algebra over $\mathbb{K}$ with augmentation $\varepsilon' : \mathbb{B}' \to \mathbb{K}$ and $\Phi : \mathbb{B} \to \mathbb{B}'$ an algebra homomorphism such that $\Phi(1_{\mathbb{B}}) = 1_{\mathbb{B}'}$ and $\varepsilon' \circ \Phi = \varepsilon$. If a deformation $\mu_{\mathbb{B}}$ with a base $(\mathbb{B}, \mathrm{Ker}(\varepsilon))$ of $\mathscr{A}$ is given, we call push-out $\mu_{\mathbb{B}'} = \Phi_* \mu_{\mathbb{B}}$ a deformation of $\mathscr{A}$ with a base $(\mathbb{B}', \mathrm{Ker}(\varepsilon'))$ with the following algebra structure on $\mathbb{B}' \otimes \mathscr{A} = (\mathbb{B}' \bigotimes_{\mathbb{B}} \mathbb{B}) \otimes \mathscr{A} = \mathbb{B}' \bigotimes_{\mathbb{B}} (\mathbb{B} \otimes \mathscr{A})$:

$$\mu_{\mathbb{B}'}\left(a_1' \bigotimes_{\mathbb{B}} (a_1 \otimes x_1), a_2' \bigotimes_{\mathbb{B}} (a_2 \otimes x_2)\right) := a_1' a_2' \bigotimes_{\mathbb{B}} \mu_{\mathbb{B}}(a_1 \otimes x_1, a_2 \otimes x_2) \tag{2.15}$$

with $a_1', a_2' \in \mathbb{B}'$, $a_1, a_2 \in \mathbb{B}$, $x_1, x_2 \in \mathscr{A}$. The algebra $\mathbb{B}'$ is viewed as a $\mathbb{B}$-module with the structure $aa' = a'\Phi(a)$. Suppose that

$$\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \alpha_i \otimes z_i \tag{2.16}$$

with $\alpha_i \in m$, $z_i \in \mathscr{A}$, then

$$\mu_{\mathbb{B}'}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \Phi(\alpha_i) \otimes z_i. \tag{2.17}$$

2.3.2. Coalgebra and Hopf algebra global deformation. The global deformation may be extended to coalgebra structures, then to Hopf algebras. Let $\mathscr{C}$ be a coalgebra over $\mathbb{K}$, defined by the comultiplication $\Delta : \mathscr{C} \to \mathscr{C} \otimes C$. Let $\mathbb{B}$ be a commutative algebra over $\mathbb{K}$ and let ε be an augmentation $\varepsilon : \mathbb{B} \to \mathbb{K}$ with $m = \mathrm{Ker}(\varepsilon)$ a maximal ideal.

A global deformation with base $(\mathbb{B}, m)$ of coalgebra C with a comultiplication Δ is a structure of $\mathbb{B}$-coalgebra on the tensor product $\mathbb{B} \bigotimes_{\mathbb{K}} \mathscr{C}$ with the comultiplication $\Delta_{\mathbb{B}}$ such that $\varepsilon \otimes \mathrm{id} : \mathbb{B} \otimes \mathscr{C} \to \mathbb{K} \otimes \mathscr{C} = \mathscr{C}$ is a coalgebra homomorphism, that is, for all $a \in \mathbb{B}$ and for all $x \in \mathscr{C}$,

(1) $\Delta_{\mathbb{B}}(a \otimes x) = a\Delta_{\mathbb{B}}(1 \otimes x)$,

(2) the comultiplication $\Delta_{\mathbb{B}}$ is coassociative,

(3) $(\varepsilon \otimes \mathrm{id}) \otimes (\varepsilon \otimes \mathrm{id})(\Delta_{\mathbb{B}}(1 \otimes x)) = 1 \otimes \Delta(x)$.

The comultiplication $\Delta_{\mathbb{B}}$ may be written for all x as

$$\Delta_B(1 \otimes x) = 1 \otimes \Delta(x) + \sum_i \alpha_i \otimes z_i \otimes z_i' \quad \text{with } \alpha_i \in m,\ z_i, z_i' \in \mathscr{C}. \tag{2.18}$$

2.3.3. Equivalence and push-out for coalgebras. Two global deformations of a coalgebra with the same base are called *equivalent* (or isomorphic) if there exists a coalgebra isomorphism between the two copies of $\mathbb{B} \bigotimes_{\mathbb{K}} \mathscr{C}$, compatible with $\varepsilon \otimes \mathrm{id}$.

Let $\mathbb{B}'$ be another commutative algebra over $\mathbb{K}$ as in Section 2.3.1. If $\Delta_{\mathbb{B}}$ is a global deformation with a base $(\mathbb{B}, \mathrm{Ker}(\varepsilon))$ of a coalgebra $\mathscr{C}$, we call push-out $\Delta_{\mathbb{B}'} = \Phi_* \Delta_{\mathbb{B}}$ a global deformation with a base $(\mathbb{B}', \mathrm{Ker}(\varepsilon'))$ of $\mathscr{C}$ with the following coalgebra structure on $\mathbb{B}' \otimes \mathscr{C} = (\mathbb{B}' \bigotimes_{\mathbb{B}} \mathbb{B}) \otimes \mathscr{C} = \mathbb{B}' \bigotimes_{\mathbb{B}} (\mathbb{B} \otimes \mathscr{C})$:

$$\Delta_{\mathbb{B}'}(a' \otimes_{\mathbb{B}} a \otimes x := a' \otimes_{\mathbb{B}} \Delta_{\mathbb{B}}(a \otimes x)) \tag{2.19}$$

with $a' \in \mathbb{B}'$, $a \in \mathbb{B}$, $x \in \mathscr{C}$. The algebra $\mathbb{B}'$ is viewed as a $\mathbb{B}$-module with the structure $aa' = a'\Phi(a)$. Suppose that

$$\Delta_{\mathbb{B}}(1 \otimes x) = 1 \otimes \Delta(x) + \sum_i \alpha_i \otimes z_i \otimes z_i' \tag{2.20}$$

with $\alpha_i \in m$, $z_i \in \mathscr{C}$, then

$$\Delta_{\mathbb{B}'}(1 \otimes x) = 1 \otimes \Delta(x) + \sum_i \Phi(\alpha_i) \otimes z_i \otimes z_i'. \tag{2.21}$$

2.3.4. Hopf algebra global deformation. In a natural way, we can define Hopf algebra global deformation from the algebra and coalgebra global deformation.

2.3.5. Valued global deformation. In [21], Goze and Remm considered the case where the base algebra $\mathbb{B}$ is a commutative $\mathbb{K}$-algebra equipped with a valuation such that the residual field $\mathbb{B}/m$ is isomorphic to $\mathbb{K}$, where m is the maximal ideal (recall that $\mathbb{B}$ is a valuation ring of a field F if $\mathbb{B}$ is a local integral domain satisfying $x \in F\backslash\mathbb{B}$ which implies $x^{-1} \in m$). Such deformations are called valued global deformations.

2.4. Perturbation theory. The perturbation theory over the field of complex numbers is based on an enlargement of the field of real numbers with the same algebraic order properties as $\mathbb{R}$. The nonstandard extension $\mathbb{R}^\star$ of $\mathbb{R}$ induces the existence of infinitesimal element in $\mathbb{C}^\star$, hence in $(\mathbb{C}^n)^\star$. The “infinitely small number” and “unlimited number” have a long historical tradition (Euclid, Eudoxe, Archimedes,..., Cavalieri, Galilei,..., Leibniz, Newton). The infinitesimal methods were considered in heuristic and intuitive way until 1960 when Robinson gave a rigorous foundation to these methods [22]. He used methods of mathematical logic and constructed a nonstandard model for real numbers. However, there exist other frames for infinitesimal methods (see [23–25]). In order to study the local properties of complex Lie algebras, Goze introduced in 1980 the notion of perturbation of algebraic structures (see [14]) in Nelson’s framework [23]. The description given here is more algebraic, it is based on Robinson’s framework. First, we summarize a description of the field of hyperreals and then we define the perturbation notion over hypercomplex numbers.

2.4.1. Field of hyperreals and their properties. The construction of hyperreal numbers system needs four axioms. They induce a triple $(\mathbb{R}, \mathbb{R}^\star, \star)$ where $\mathbb{R}$ is the real field, $\mathbb{R}^\star$ is the hyperreal field, and $\star$ is a natural mapping.

Axiom 1. $\mathbb{R}$ is a complete ordered field.

Axiom 2. $\mathbb{R}^\star$ is a proper ordered field extension of $\mathbb{R}$.

Axiom 3. For each real function f of n variables, there is a corresponding hyperreal function $f^\star$ of n variables, called natural extension of f. The field operations of $\mathbb{R}^\star$ are the natural extensions of the field operations of $\mathbb{R}$.

Axiom 4. If two systems of formulae have exactly the same real solutions, they have exactly the same hyperreal solutions.

The following theorem shows that such extension exists.

THEOREM 2.8. *Let $\mathbb{R}$ be the ordered field of real numbers. There is an ordered field extension $\mathbb{R}^\star$ of $\mathbb{R}$ and a mapping $\star$ from real functions to hyperreal functions such that Axioms 1–4 hold.*

In the proof, the plan is to find an infinite set M of formulas $\varphi(x)$ which describe all properties of a positive infinitesimal x, and to built $\mathbb{R}^\star$ out of this set of formulas. See [26, pages 23-24] for a complete proof.

Definition 2.9. (1) $x \in \mathbb{R}^\star$ is called infinitely small or infinitesimal if $|x| < r$ for all $r \in \mathbb{R}^+$.

(2) $x \in \mathbb{R}^\star$ is called limited if there exists $r \in \mathbb{R}$ such that $|x| < r$.

(3) $x \in \mathbb{R}^\star$ is called unlimited or infinitely large if $|x| > r$ for all $r \in \mathbb{R}$.

(4) Two elements x and y of $\mathbb{R}^\star$ are called infinitely close ($x \simeq y$) if $x - y$ is infinitely small.

Definition 2.10. Given a hyperreal number $x \in \mathbb{R}^\star$, we set the following:

$$\text{halo}(x) = \{y \in \mathbb{R}^\star,\ x \simeq y\},$$
$$\text{galaxy}(x) = \{y \in \mathbb{R}^\star,\ x - y \text{ is limited}\}.$$

In particular, halo(0) is the set of infinitely small elements of $\mathbb{R}^\star$ and galaxy(0) is the set of limited hyperreal numbers which we denote by $\mathbb{R}^\star_L$.

PROPOSITION 2.11. (1) $\mathbb{R}^\star_L$ *and* halo(0) *are subrings of* $\mathbb{R}^\star$.

(2) halo(0) *is a maximal ideal in* $\mathbb{R}^\star_L$.

Proof. (1) The first property is easy to prove. (2) Let $\eta \simeq 0$ and $a \in \mathbb{R}$, then there exists $t \in \mathbb{R}$ such that $|a| < t$ and for all $r \in \mathbb{R}$, $|\eta| < r/t$, thus $|a\eta| < r$. Therefore, $a\eta \in \mathbb{R}^\star_L$ and halo(0) is an ideal in $\mathbb{R}^\star_L$. Let us show that the ideal is maximal. We note that $x \in \mathbb{R}^\star$ is unlimited, is equivalent to x^{-1} infinitely small. Assume that there exists an ideal I in $\mathbb{R}^\star_L$ containing halo(0) and let $a \in I \setminus \text{halo}(0)$, we have $a^{-1} \in \mathbb{R}^\star_L$, thus $1 = aa^{-1} \in I$, so $I \equiv \mathbb{R}^\star_L$. □

The following theorem shows that there is a ring homomorphism of $\mathbb{R}^\star_L$ onto the field of real numbers.

THEOREM 2.12. *Every x of $\mathbb{R}^\star_L$ admits a unique x_0 in $\mathbb{R}$ (called standard part of x and denoted by $st(x)$) such that $x \simeq x_0$.*

Proof. Let $x \in \mathbb{R}^\star_L$, assume that there exist two real numbers r and s such that $x \simeq r$ and $x \simeq s$, this implies that $r - s \simeq 0$. Since the unique infinitesimal real number is 0, then $r = s$. To show the existence, set $X = \{s \in \mathbb{R} : s < x\}$. Then $X \neq \emptyset$ and $X < r$, where r is a positive real number ($-r < x < r$). Let t be the smallest r, for all positive real r, we have $x \leq t + r$, then $x - t \leq r$ and $t - r < x$, it follows that $-(x - t) \leq r$. Then $x - t \simeq 0$, hence $x \simeq t$. □

We check easily that for $x, y \in \mathbb{R}^\star_L$, $st(x + y) = st(x) + st(y)$, $st(x - y) = st(x) - st(y)$, and $st(xy) = s(x)st(y)$. Also, for all $r \in \mathbb{R}$, $st(r) = r$.

Remark 2.13. All these concepts and properties may be extended to hyperreal vectors of $\mathbb{R}^{\star n}$. An element $v = (x_1,\dots,x_n)$, where $x_i \in \mathbb{R}^\star$, is called infinitely small if $|v| = \sqrt{\sum_{i=1}^n x_i^2}$ is an infinitely small hyperreal and called limited if $|v|$ is limited. Two vectors of $\mathbb{R}^{\star n}$ are infinitely close if their difference is infinitely small. The extension to complex numbers and vectors is similar ($\mathbb{C}^\star = \mathbb{R}^\star \times \mathbb{R}^\star$).

2.4.2. Perturbation of associative algebras. We introduce here the perturbation notion of an algebraic structure. Let $V = \mathbb{K}^n$ be a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $V^\star$ be a vector space over $\mathbb{K}^\star$ ($\mathbb{K}^\star = \mathbb{R}^\star$ or $\mathbb{C}^\star$). Let $\mathcal{A}$ be an associative algebra of alg_n with a multiplication μ_0 over V.

Definition 2.14. A morphism μ in $\mathrm{Hom}((V^\star)^{\times 2}, V^\star)$ is a perturbation of μ_0 if

$$\forall X_1, X_2 \in V : \mu(X_1,X_2) \simeq \mu_0(X_1,X_2) \tag{2.22}$$

and μ satisfies the associativity condition over V. We write $(\mu \simeq \mu_0)$.

Fixing a basis of V, the extension $V^\star$ of V is a vector space which may be taken with the same basis as V. Then μ is a perturbation of μ_0 and is equivalent to say that the difference between the structure constants, with respect to the same basis, of μ and μ_0 is an infinitesimal vector in $(V^\star)^{n^3}$.

Two perturbations μ and μ' of μ_0 are isomorphic if there is an invertible map f in $\mathrm{Hom}(V^\star, V^\star)$ such that $\mu' = f^{-1} \circ \mu \circ (f \otimes f)$.

The following decomposition of a perturbation follows from Goze's decomposition [14].

THEOREM 2.15. *Let $\mathcal{A}$ be an algebra in* alg_n *with a multiplication μ_0 and let μ be a perturbation of μ_0. The following decomposition of μ holds:*

$$\mu = \mu_0 + \varepsilon_1\varphi_1 + \varepsilon_1\varepsilon_2\varphi_2 + \cdots + \varepsilon_1 \cdot \cdots \cdot \varepsilon_k\varphi_k, \tag{2.23}$$

where

(1) $\varepsilon_1,\dots,\varepsilon_k$ *are nonzero infinitesimals in* $\mathbb{K}^\star$,

(2) $\varphi_1,\dots,\varphi_k$ *are independent bilinear maps in* $\mathrm{Hom}(V^{\times 2}, V)$.

Remark 2.16. (1) The integer k is called the length of the perturbation. It satisfies $k \le n^3$.

(2) The perturbation decomposition is generalized in the valued global deformation case [21] by taking the ε_i in maximal ideal of a valuation ring.

(3) The associativity of μ is equivalent to a finite system of equation called the perturbation equation. This equation is studied above by using Massey cohomology products.

2.4.3. Resolution of the perturbation equation. In the following we discuss the conditions on φ_1 such that it is a first term of a perturbation.

Let us consider a perturbation of length 2, $\mu = \mu_0 + \varepsilon_1\varphi_1 + \varepsilon_1\varepsilon_2\varphi_2$, the perturbation equation is equivalent to

$$\begin{gathered} \delta\varphi_1 = 0, \\ \varepsilon_1[\varphi_1,\varphi_1]_G + 2\varepsilon_1\varepsilon_2[\varphi_1,\varphi_2]_G + \varepsilon_1\varepsilon_2^2[\varphi_2,\varphi_2]_G + 2\varepsilon_2\delta\varphi_2 = 0, \end{gathered} \tag{2.24}$$

where $[\varphi_i,\varphi_j]_G$ is the trilinear map defined by Gerstenhaber's bracket (see Section 2.1).

It follows, as in the deformation equation, that φ_1 is a cocycle of $Z^2(\mathcal{A},\mathcal{A})$. The second equation has infinitesimal coefficients but the vectors

$$\{[\varphi_1,\varphi_1]_G,[\varphi_1,\varphi_2]_G,[\varphi_2,\varphi_2]_G,\delta\varphi_2\} \tag{2.25}$$

are in $\mathrm{Hom}(V^{\times 3},V)$ and form a system of rank 1.

Proposition 2.17. *Let μ be a perturbation of length k of the multiplication μ_0. The rank of the vectors $\{[\varphi_i,\varphi_j]_G,\delta\varphi_i\}$, $i=1,\dots,k$, and $i\le j\le k$ is equal to the rank of the vectors $\{[\varphi_i,\varphi_j]_G\}$, $i=1,\dots,k-1$, and $i\le j\le k-1$.*

Proof. We consider a nontrivial linear form ω containing in its kernel $\{[\varphi_i,\varphi_j]_G\}$, $i=1,\dots,k-1$, and $i\le j\le k-1$. We apply it to the perturbation equation then it follows that all the vectors $\{[\varphi_i,\varphi_j]_G,\delta\varphi_i\}$, $i=1,\dots,k$, and $i\le j\le k$ are in the kernel of ω. □

The following theorem, which uses the previous proposition, characterizes the cocycle which should be a first term of a perturbation, see [27] for the proof and [28] for the Massey products.

Theorem 2.18. *Let $\mathcal{A}$ be an algebra in alg_n with a multiplication μ_0. A vector φ_1 in $Z^2(\mathcal{A},\mathcal{A})$ is the first term of a perturbation μ of μ_0 (k being the length of μ) if and only if*

(1) *the Massey products $[\varphi_1^2],[\varphi_1^3],\dots,[\varphi_1^p]$ vanish until $p=k^2$,*
(2) *the representative's product in $B^3(\mathcal{A},\mathcal{A})$ forms a system of rank less than or equal to $k(k-1)/2$.*

2.5. Generalized coalgebra perturbations. The perturbation notion can be generalized to any algebraic structure on V. A morphism μ in $\mathrm{Hom}((V^\star)^{\times p},(V^\star)^{\times q})$ *is a perturbation of* μ_0 in $\mathrm{Hom}(V^{\times p},V^{\times q})(\mu\simeq\mu_0)$ *if*

$$\forall X_1,\dots,X_p\in V:\mu(X_1,\dots,X_p)\simeq\mu_0(X_1,\dots,X_p). \tag{2.26}$$

In particular, if $p=1$ and $q=2$, we obtain the concept of coalgebra perturbation.

3. Comparison of the deformations

We show that the formal deformations and the perturbation are global deformations with appropriate bases, and we show that over the field of complex numbers the perturbations contain all the convergent deformations.

Global deformation and formal deformation. The following proposition gives the link between formal deformation and global deformation.

Proposition 3.1. *Every formal deformation is a global deformation.*

Proof. Every formal deformation of an algebra $\mathcal{A}$, in Gerstenhaber's sense, is a global deformation with a basis $(\mathbb{B}, m)$ where $\mathbb{B} = \mathbb{K}[[t]]$ and $m = t\mathbb{K}[[t]]$. □

Global deformation and perturbation. Let $\mathbb{K}^\star$ be a proper extension of $\mathbb{K}$ described in Section 2.4, where $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$.

Proposition 3.2. *Every perturbation of an algebra over* $\mathbb{K}$ *is a global deformation with a base* $(\mathbb{K}_L^\star, \mathrm{halo}(0))$.

Proof. The set $\mathbb{B} := \mathbb{K}_L^\star$ is a local ring formed by the limited elements of $\mathbb{K}^\star$. We define the augmentation $\varepsilon : \mathbb{K}_L^\star \to \mathbb{K}$ which associates to any x its standard part $st(x)$. The kernel of ε corresponds to halo(0), the set of infinitesimal elements of $\mathbb{K}^\star$, which is a maximal ideal in $\mathbb{K}_L^\star$. □

The multiplication $\mu_{\mathbb{B}}$ of a global deformation with a base $(\mathbb{K}_L^\star, \mathrm{halo}(0))$ of an algebra with a multiplication μ may be written, for all $x, y \in \mathbb{K}^n$,

$$\mu_B(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \alpha_i \otimes z_i \quad \text{with } \alpha_i \in \mathrm{halo}(0) \subset \mathbb{K}^\star,\ z_i \in \mathbb{K}^n. \tag{3.1}$$

This is equivalent to say that $\mu_{\mathbb{B}}$ is a perturbation of μ.

Remark 3.3. The corresponding global deformations of perturbations are local but not infinitesimal because $\mathrm{halo}(0)^n \neq 0$.

Formal deformation and perturbation

Proposition 3.4. *Let* $\mathcal{A}$ *be a* $\mathbb{C}$*-algebra in* alg_n *with a multiplication* μ_0*. Let* μ_t *be a convergent deformation of* μ_0 *and let* α *be an infinitesimal in* $\mathbb{C}^\star$ *then* μ_α *is a perturbation of* μ_0.

Proof. Since the deformation is convergent, then μ_α corresponds to a point of $\mathrm{alg}_n \subset \mathbb{C}^{n^3}$ which is infinitely close to the corresponding point of μ_0. Then it determines a perturbation of μ_0. □

Remark 3.5. A perturbation should correspond to a formal deformation if one considers the more general power series rings $\mathbb{K}[[t_1, \ldots, t_r]]$.

4. Universal and versal deformations

Given an algebra, the problem is to find particular deformations which induce all the others in the space of all deformations or in a fixed category of deformations. This problem is too hard in general but the global deformation seems more adapted to construct universal or versal deformation. We say that the global deformation is universal if there is uniqueness of the homomorphism between base algebras, otherwise we say that the deformation is versal. This problem was considered in the case of Lie algebras for the categories of deformations over infinitesimal local algebras and complete local algebras (see [18, 29, 30]). They show that if we consider the infinitesimal deformations, that is, the deformations over local algebras $\mathbb{B}$ such that $m_{\mathbb{B}}^2 = 0$ where $m_{\mathbb{B}}$ is the maximal ideal, then there exists a universal deformation. If we consider the category of complete local rings, then there does not exist a universal deformation but only versal deformation.

Formal global deformation. Let $\mathbb{B}$ be a complete local algebra over $\mathbb{K}$, so $\mathbb{B}=\overleftarrow{\lim}_{n\to\infty}(\mathbb{B}/m^n)$ (inductive limit), where m is the maximal ideal of $\mathbb{B}$, and we assume that $\mathbb{B}/m \cong \mathbb{K}$.

Definition 4.1. A formal global deformation of $\mathcal{A}$ with base $(\mathbb{B}, m)$ is an algebra structure on the completed tensor product $\mathbb{B}\hat{\bigotimes}\mathcal{A} = \overleftarrow{\lim}_{n\to\infty}((\mathbb{B}/m^n)\bigotimes\mathcal{A})$ such that $\varepsilon\hat{\bigotimes}\mathrm{id} : \mathbb{B}\hat{\bigotimes}\mathcal{A} \to \mathbb{K}\bigotimes\mathcal{A} = \mathcal{A}$ is an algebra homomorphism.

Remark 4.2. (1) The formal global deformations of $\mathcal{A}$ with base $(\mathbb{K}[[t]], t\mathbb{K}[[t]])$ are the same as the formal parameter deformations of Gerstenhaber.

(2) The perturbations are complete global deformations because

$$\lim_{n\to\infty} \mathbb{K}^\star/(\mathrm{halo}(0))^n \tag{4.1}$$

is isomorphic to $\mathbb{K}^\star$ while $\mathbb{K}^\star$ is isomorphic to $\mathbb{K}$ as algebras.

We now assume that the algebra $\mathcal{A}$ satisfies $\dim(H^2(\mathcal{A},\mathcal{A})) < \infty$. We consider $\mathbb{B} = \mathbb{K} \oplus H^2(\mathcal{A},\mathcal{A})^{\mathrm{dual}}$. The following theorems due to Fialowski and Post [18, 29] show the existence of universal infinitesimal deformation under the previous assumptions.

Theorem 4.3. *There exists, in the category of infinitesimal global deformations, a universal infinitesimal deformation $\eta_{\mathcal{A}}$ with base $\mathbb{B}$ equipped with the multiplication $(\alpha_1,h_1)\cdot(\alpha_1,h_1) = (\alpha_1\alpha_2, \alpha_1h_2 + \alpha_2h_1)$.*

Let $\mathbb{P}$ be any finite-dimensional local algebra over $\mathbb{K}$. The theorem means that for any infinitesimal deformation of an algebra $\mathcal{A}$ defined by $\mu_{\mathbb{P}}$, there exists a unique homomorphism $\Phi : \mathbb{K} \oplus H^2(\mathcal{A},\mathcal{A})^{\mathrm{dual}} \to \mathbb{P}$ such that $\mu_{\mathbb{P}}$ is equivalent to the push-out $\Phi_*\eta_{\mathcal{A}}$.

Definition 4.4. A formal global deformation η of $\mathcal{A}$ parameterized by a complete local algebra $\mathbb{B}$ is called versal if for any deformation λ of $\mathcal{A}$, parameterized by a complete local algebra $(\mathbb{A}, m_{\mathbb{A}})$, there is a morphism $f : \mathbb{B} \to \mathbb{A}$ such that

(1) the push-out $f_*\eta$ is equivalent to λ,

(2) if $\mathbb{A}$ satisfies $m_{\mathbb{A}}^2 = 0$, then f is unique.

Theorem 4.5. *Let $\mathcal{A}$ be an algebra.*

(1) *There exists a versal formal global deformation of $\mathcal{A}$.*

(2) *The base of the versal formal deformation is formally embedded into $H^2(\mathcal{A},\mathcal{A})$ (it can be described in $H^2(\mathcal{A},\mathcal{A})$ by a finite system of formal equations).*

Examples. The Witt algebra is the infinite dimensional Lie algebra of polynomial vector fields spanned by the fields $e_i = z^{i+1}(d/dz)$ with $i \in \mathbb{Z}$. In [18], Fialowski constructed versal deformation of the Lie subalgebra L_1 of Witt algebra (L_1 is spanned by e_n, $n > 0$, while the bracket is given by $[e_n, e_m] = (m-n)e_{n+m}$).

The following three real deformations of the Lie algebra L_1 are nontrivial and pairwise nonisomorphic:

$$\begin{aligned} [e_i, e_j]_t^1 &= (j-i)(e_{i+j} + te_{i+j-1}), \\ [e_i, e_j]_t^2 &= \begin{cases} (j-i)e_{i+j} & \text{if } i, j > 1, \\ (j-i)e_{i+j} + tje_j & \text{if } i = 1, \end{cases} \\ [e_i, e_j]_t^3 &= \begin{cases} (j-i)e_{i+j} & \text{if } i, j \neq 2, \\ (j-i)e_{i+j} + tje_j & \text{if } i = 2. \end{cases} \end{aligned} \tag{4.2}$$

Fialowski and Post, in [29], study the L_2 case. A more general procedure using the Harrison cohomology of the commutative algebra $\mathbb{B}$ is described by Fialowski and Fuchs in [31].

Fialowski's global deformation of L_1 can be realized as a perturbation, the parameters t_1, t_2, and t_3 have to be different infinitesimals in $\mathbb{C}^\star$.

5. Degenerations

The notion of degeneration is fundamental in the geometric study of alg_n and helps, in general, to construct new algebras.

Definition 5.1. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two n-dimensional algebras. The algebra $\mathcal{A}_0$ is a degeneration of $\mathcal{A}_1$ if $\mathcal{A}_0$ belongs to $\overline{\vartheta(\mathcal{A}_1)}$, the Zariski closure of the orbit of $\mathcal{A}_1$.

In the following we define degenerations in different frameworks.

5.1. Global point of view.

A characterization of global degeneration was given by Grunewald and O'Halloran in [32].

THEOREM 5.2. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two n-dimensional associative algebras over $\mathbb{K}$ with the multiplications μ_0 and μ_1. The algebra $\mathcal{A}_0$ is a global degeneration of $\mathcal{A}_1$ if and only if there is a discrete valuation $\mathbb{K}$-algebra $\mathbb{B}$ with residue field $\mathbb{K}$ whose quotient field $\mathcal{K}$ is finitely generated over $\mathbb{K}$ of transcendence degree one (one parameter), and there is an n-dimensional algebra $\mu_{\mathbb{B}}$ over $\mathbb{B}$ such that $\mu_{\mathbb{B}} \otimes \mathcal{K} \cong \mu_1 \otimes \mathcal{K}$ and $\mu_{\mathbb{B}} \otimes \mathbb{K} \cong \mu_0$.*

5.2. Formal point of view.

Let t be a parameter in $\mathbb{K}$, let $\{f_t\}_{t\neq 0}$ be a continuous family of invertible linear maps on V over $\mathbb{K}$, and let $\mathcal{A}_1 = (V, \mu_1)$ be an algebra over $\mathbb{K}$. The limit (in case it exists) of a sequence $f_t \cdot \mathcal{A}_1$, $\mathcal{A}_0 = \lim_{t\to 0} f_t \cdot \mathcal{A}_1$, is a *formal degeneration* of $\mathcal{A}_1$ in the sense that $\mathcal{A}_0$ is in the Zariski closure of the set $\{f_t \cdot \mathcal{A}\}_{t\neq 0}$.

The multiplication μ_0 is given by

$$\mu_0 = \lim_{t\to 0} f_t \cdot \mu_1 = \lim_{t\to 0} f_t^{-1} \circ \mu_1 \circ f_t \times f_t. \tag{5.1}$$

(1) The multiplication $\mu_t = f_t^{-1} \circ \mu_1 \circ f_t \times f_t$ satisfies the associativity condition. Thus, when t tends to 0 the condition remains satisfied.

(2) The linear map f_t is invertible when $t \neq 0$ and may be singular when $t = 0$. Then, we may obtain a new algebra by degeneration.

(3) The definition of formal degeneration may be extended naturally to the infinite dimensional case.

(4) When $\mathbb{K}$ is the complex field, the multiplication given by the limit follows from a limit of the structure constants, using the metric topology. In fact, $f_t \cdot \mu$ corresponds to a change of basis when $t \neq 0$. When $t = 0$, the limit gives eventually a new point in $\text{alg}_n \subset \mathbb{K}^{n^3}$.

(5) If f_t is defined by a power series, the images over $V \times V$ of the multiplication of $f_t \cdot \mathcal{A}$ are in general in the Laurent power series ring $V[[t,t^{-1}]]$. But when the degeneration exists, it lies in the power series ring $V[[t]]$.

Proposition 5.3. *Every formal degeneration is a global degeneration.*

The proof follows from Theorem 5.2 and the last remark.

5.3. Contraction. The notion of degeneration over the hypercomplex field is called contraction. It is defined by the following.

Definition 5.4. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two algebras in alg_n with multiplications μ_0 and μ_1. The algebra $\mathcal{A}_0$ is a contraction of $\mathcal{A}_1$ if there exists a perturbation μ of μ_0 such that μ is isomorphic to μ_1.

The definition gives a characterization over the hypercomplex field that μ_0 is in the closure of the orbit of μ_1 (for the usual topology of $\mathbb{C}^{n^3}$).

5.4. Examples. (1) *The null algebra of* alg_n *is a degeneration of any algebra of* alg_n.
In fact, the null algebra is given in a basis $\{e_1,\dots,e_n\}$ (e_1 being the unit element) by the following nontrivial products $\mu_0(e_1,e_i) = \mu_0(e_i,e_1) = e_i$, $i = 1,\dots,n$.

Let μ_1 be a multiplication of any algebra of alg_n, we have $\mu_0 = \lim_{t\to 0} f_t \cdot \mu_1$ with f_t given by the diagonal matrix $(1,t,\dots,t)$.

For $i \neq 1$, and $j \neq 1$ we have

$$\begin{aligned} f_t \cdot \mu(e_i,e_j) &= f_t^{-1}(\mu(f_t(e_i),f_t(e_j))) = f_t^{-1}(\mu(te_i,te_j)) \\ &= t^2 C_{ij}^1 e_1 + t\sum_{k>1} C_{ij}^k e_k \longrightarrow 0 \quad (\text{when } t \longrightarrow 0). \end{aligned} \tag{5.2}$$

For $i = 1$, and $j \neq 1$ we have

$$f_t \cdot \mu(e_1,e_j) = f_t^{-1}(\mu(f_t(e_1),f_t(e_j))) = f_t^{-1}(\mu(e_1,te_j)) = f_t^{-1}(te_j) = e_j. \tag{5.3}$$

This shows that every algebra of alg_n degenerates formally to a null algebra.

By taking the parameter $t = \alpha$, α infinitesimal in the field of hypercomplex numbers, we get that the null algebra is a contraction of any complex associative algebra in alg_n. In fact, $f_\alpha \cdot \mu_1$ is a perturbation of μ_0 and is isomorphic to μ_1.

The same holds also in the global point of view: let $B = \mathbb{K}[t]_{\langle t\rangle}$ be the polynomial ring localized at the prime ideal $\langle t\rangle$ and let $\mu_B = t\mu_1$ on the elements different from the unit element and $\mu_B = \mu_1$ elsewhere. Then μ_1 is $\mathbb{K}(t)$-isomorphic to μ_B via f_t^{-1} and $\mu_B \otimes \mathbb{K} = \mu_0$.

(2) The classification of alg_2 yields two isomorphism classes.
Let $\{e_1, e_2\}$ be a basis of $\mathbb{K}^2$:

$$\begin{aligned}\mathcal{A}_1 : \mu_1(e_1,e_i) = \mu_1(e_i,e_1) = e_i, \quad i = 1,2;\ \mu_1(e_2,e_2) = e_2,\\ \mathcal{A}_0 : \mu_0(e_1,e_i) = \mu_0(e_i,e_1) = e_i, \quad i = 1,2;\ \mu_0(e_2,e_2) = 0.\end{aligned} \tag{5.4}$$

Consider the formal deformation μ_t of μ_0 defined by

$$\mu_t(e_1,e_i) = \mu_t(e_i,e_1) = e_i, \quad i = 1,2;\ \mu_t(e_2,e_2) = te_2. \tag{5.5}$$

Then μ_t is isomorphic to μ_1 through the change of basis given by the matrix $f_t = \left(\begin{smallmatrix}1 & 0\\ 0 & t\end{smallmatrix}\right)$, $\mu_t = f_t \cdot \mu_1$.

We have $\mu_0 = \lim_{t\to 0} f_t \cdot \mu_1$, thus μ_0 is a degeneration of μ_1. Since μ_1 is isomorphic to μ_t, it is a deformation of μ_0. We can obtain the same result over the field of complex numbers if we consider the parameter t infinitesimal in $\mathbb{C}^\star$.

5.4.1. Graded algebras. In this section, we give a relation between a finite-dimensional algebra and its associated graded algebra.

Theorem 5.5. *Let $\mathcal{A}$ be an algebra over $\mathbb{K}$ and $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n \subseteq \cdots$ an algebra filtration of $\mathcal{A}$, $\mathcal{A} = \bigcup_{n\geq 0}\mathcal{A}_n$. Then the graded algebra* $\text{gr}(\mathcal{A}) = \bigoplus_{n\geq 1}\mathcal{A}_n/\mathcal{A}_{n-1}$ *is a formal degeneration of $\mathcal{A}$.*

Proof. Let $\mathcal{A}[[t]]$ be a power series ring in one variable $t \in \mathbb{K}$ over $\mathcal{A}$, $\mathcal{A}[[t]] = \mathcal{A} \otimes \mathbb{K}[[t]] = \bigoplus_{n\geq 0}\mathcal{A} \otimes t^n$.

We denote by $\mathcal{A}_t$ the Rees algebra associated to the filtered algebra $\mathcal{A}$, $\mathcal{A}_t = \sum_{n\geq 0}\mathcal{A}_n \otimes t^n$. The Rees algebra $\mathcal{A}_t$ is contained in the algebra $\mathcal{A}[[t]]$.

For every $\lambda \in \mathbb{K}$, we set $\mathcal{A}_{(\lambda)} = \mathcal{A}_t/((t-\lambda)\cdot\mathcal{A}_t)$. For $\lambda = 0$, $\mathcal{A}_{(0)} = \mathcal{A}_t/(t\cdot\mathcal{A}_t)$. The algebra $\mathcal{A}_{(0)}$ corresponds to the graded algebra $\text{gr}(\mathcal{A})$ and $\mathcal{A}_{(1)}$ is isomorphic to $\mathcal{A}$. In fact, we suppose that the parameter t commutes with the elements of $\mathcal{A}$ then $t\cdot\mathcal{A}_t = \bigoplus_{n\geq 0}\mathcal{A}_n \otimes t^{n+1}$. It follows that $\mathcal{A}_{(0)} = \mathcal{A}_t/(t\cdot\mathcal{A}_t) = (\sum_n \mathcal{A}_n \otimes t^n)/(\bigoplus_n \mathcal{A}_n \otimes t^{n+1}) = \bigoplus_{n\geq 0}\mathcal{A}_n/\mathcal{A}_{n-1} = \text{gr}(\mathcal{A})$. By using the linear map from $\mathcal{A}_t$ to $\mathcal{A}$ where the image of $a_n \otimes t^n$ is a_n, we have $\mathcal{A}_{(1)} = \mathcal{A}_t/((t-1)\cdot\mathcal{A}_t) \cong \mathcal{A}$.

If $\lambda \neq 0$, the change of parameter $t = \lambda T$ shows that $\mathcal{A}_{(\lambda)}$ is isomorphic to $\mathcal{A}_{(1)}$. This ends the proof that $\mathcal{A}_{(0)} = \text{gr}(\mathcal{A})$ is a degeneration of $\mathcal{A}_{(1)} \cong \mathcal{A}$. □

5.5. Connection between degeneration and deformation. The following proposition gives the link between degeneration and deformation.

Proposition 5.6. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two algebras in* alg_n. *If $\mathcal{A}_0$ is a degeneration of $\mathcal{A}_1$, then $\mathcal{A}_1$ is a deformation of $\mathcal{A}_0$.*

In fact, let $\mathcal{A}_0 = \lim_{t\to 0} f_t \cdot \mathcal{A}_1$ be a formal degeneration of $\mathcal{A}$ then $\mathcal{A}_t = f_t \cdot \mathcal{A}_1$ is a formal deformation of $\mathcal{A}_0$.

In the contraction sense such a property follows directly from the definition.

In the global point of view, we get also that every degeneration can be realized by a global deformation. The base of the deformation is the completion of the discrete valuation $\mathbb{K}$-algebra (inductive limit of $\mu_n = \mu_B \otimes B/m_B^{n+1}$).

Remark 5.7. The converse is, in general, false. The following example shows that there is no duality in general between deformation and degeneration.

We consider, in alg_4, the family $\mathcal{A}_t = \mathbb{C}\{x,y\}/\langle x^2, y^2, yx - txy\rangle$ where $\mathbb{C}\{x,y\}$ stands for the free associative algebra with unity and generated by x and y.

Any two algebras $\mathcal{A}_t$ and $\mathcal{A}_s$ with $t\cdot s \neq 1$ are not isomorphic. They are isomorphic if and only if $t \cdot s = 1$.

Thus, $\mathcal{A}_t$ is a deformation of $\mathcal{A}_0$ but the family $\mathcal{A}_t$ is not isomorphic to one algebra and cannot be written $\mathcal{A}_t = f_t \cdot \mathcal{A}_1$.

We also have the following more geometric proposition.

Proposition 5.8. *If an algebra $\mathcal{A}_0$ is in the boundary of the orbit of $\mathcal{A}_1$, then this degeneration defines a nontrivial deformation of $\mathcal{A}_0$.*

With the contraction point of view, we can characterize the perturbation arising from degeneration by the following.

Proposition 5.9. *Let μ be a perturbation over hypercomplex numbers of an algebra of* alg_n *with a multiplication μ_1. Then μ arises from a contraction if there exists a multiplication μ_0 with structure constants in $\mathbb{C}$ belonging to the orbit of μ.*

In fact, if μ is a perturbation of μ_0 and there exists an algebra with multiplication μ_1 such that $\mu \in \vartheta(\mu_1)$ and $\mu \simeq \mu_0$, then, μ_0 is a contraction of μ_1. This shows that the orbit of μ passes through a point of alg_n over $\mathbb{C}$.

5.6. Infinitesimal degenerations. Let $f_t = v + tw$ be a family of endomorphisms where v is a singular linear map and w is a regular linear map. The aim of this section is to find necessary and sufficient conditions on v and w such that a degeneration of a given algebra $\mathcal{A} = (V,\mu)$ exists. We can set $w = \text{id}$ because $f_t = v + tw = (v\circ w^{-1} + t)\circ w$ which is isomorphic to $v \circ w^{-1} + t$. Then without loss of generality we can consider the family $f_t = \varphi + t\cdot \text{id}$ from V into V where φ is a singular map. The fitting lemma decomposes the vector space V by φ under the form $V_R \oplus V_N$ where V_R and V_N are φ-invariant defined in a canonical way such that φ is surjective on V_R and nilpotent on V_N. Let q be the smallest integer such that $\varphi^q(V_N) = 0$. The inverse of f_t may be written

$$f_t^{-1} = \begin{cases} \varphi^{-1}\left(t\varphi^{-1} + \text{id}\right)^{-1} & \text{on } V_R, \\ \dfrac{1}{t}\cdot \displaystyle\sum_{i=0}^{q-1}\left(-\frac{\varphi}{t}\right)^i & \text{on } V_N. \end{cases} \tag{5.6}$$

The action of $f_t = \varphi + t\cdot\text{id}$ on μ is defined by

$$f_t\cdot\mu(x,y) = f_t^{-1}\left(\mu(\varphi(x),\varphi(y)) + t\left(\mu(\varphi(x),y) + \mu(x,\varphi(y))\right) + t^2\mu(x,y)\right). \tag{5.7}$$

Since every element v of V decomposes in $v = v_R + v_N$, we set

$$\begin{aligned} A &= \mu(x,y) = A_R + A_N, \\ B &= \mu(\varphi(x),y) + \mu(x,\varphi(y)) = B_R + B_N, \\ C &= \mu(\varphi(x),\varphi(y)) = C_R + C_N. \end{aligned} \tag{5.8}$$

Then

$$f_t \cdot \mu(x,y) = \varphi^{-1}(t\varphi^{-1} + \mathrm{id})^{-1}(t^2 A_R + tB_R + C_R) + \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i (t^2 A_N + tB_N + C_N). \tag{5.9}$$

If the parameter t goes to 0, then $\varphi^{-1}(t\varphi^{-1} + \mathrm{id})^{-1}(t^2 A_R + tB_R + C_R)$ goes to $\varphi^{-1}(C_R)$. The term $(1/t) \cdot \sum_{i=0}^{q-1}(-\varphi/t)^i(t^2 A_N + tB_N + C_N)$ is equivalent to

$$tA_N + B_N - \varphi(A_N) + \left(\frac{\mathrm{id}}{t} - \frac{\varphi}{t^2} + \cdots + (-1)^{q-1}\frac{\varphi^{q-1}}{t^q}\right)(\varphi^2(A_N) - \varphi(B_N) + C_N). \tag{5.10}$$

Its limit exists if and only if

$$\varphi^2(A_N) - \varphi(B_N) + C_N = 0. \tag{5.11}$$

Then we have the following.

Proposition 5.10. *The degeneration of an algebra with a multiplication μ exists if and only if the condition*

$$\varphi^2 \circ \mu_N - \varphi \circ \mu_N \circ (\varphi \times \mathrm{id}) - \varphi \circ \mu_N \circ (\mathrm{id} \times \varphi) + \mu_N \circ (\varphi \times \varphi) = 0, \tag{5.12}$$

where $\mu_N(x,y) = (\mu(x,y))_N$, holds. And it is defined by

$$\mu_0 = \varphi^{-1} \circ \mu_R \circ (\varphi \times \varphi) + \mu_N \circ (\varphi \times \mathrm{id}) + \mu_N \circ (\mathrm{id} \times \varphi) - \varphi \circ \mu_N. \tag{5.13}$$

Remark 5.11. Using a more algebraic definition of a degeneration, Fialowski and O'Halloran [33] show that we have also a notion of universal degeneration and such degeneration exists for every finite-dimensional Lie algebra. The definition and the result holds naturally in the associative algebra case.

6. Rigidity

An algebra which has no nontrivial deformations is called rigid. In the finite-dimensional case, this notion is related geometrically to open orbits in alg_n. The Zariski closure of open orbit determines an irreducible component of alg_n.

Definition 6.1. An algebra $\mathcal{A}$ is called *algebraically rigid* if $H^2(\mathcal{A},\mathcal{A}) = 0$.

An algebra $\mathcal{A}$ is called *formally rigid* if every formal deformation of $\mathcal{A}$ is trivial.

An algebra $\mathcal{A}$ is called *geometrically rigid* if its orbit is Zariski open.

We have the following equivalence due to Gerstenhaber and Schack [34].

Proposition 6.2. *Let $\mathscr{A}$ be a finite-dimensional algebra over a field of characteristic* 0. *Then formal rigidity of $\mathscr{A}$ is equivalent to geometric rigidity.*

As seen in Theorem 2.3, $H^2(\mathscr{A},\mathscr{A}) = 0$ implies that every formal deformation is trivial, then algebraic rigidity implies formal rigidity. But the converse is false. An example of algebra which is formally rigid but not algebraically rigid, $H^2(\mathscr{A},\mathscr{A}) \neq 0$, was given by Gerstenhaber and Schack in positive characteristic and high dimension [34]. We do not know such examples in characteristic 0. However, there are many rigid Lie algebras, in characteristic 0, with a nontrivial second Chevalley-Eilenberg cohomology group.

Now, we give a sufficient condition for the formal rigidity of an algebra $\mathscr{A}$ using the following map:

$$Sq : H^2(\mathscr{A},\mathscr{A}) \longrightarrow H^3(\mathscr{A},\mathscr{A}), \qquad \mu_1 \longrightarrow Sq(\mu_1) = [\mu_1 \circ \mu_1]. \tag{6.1}$$

Let $\mu_1 \in Z^2(\mathscr{A},\mathscr{A})$, μ_1 is integrable if $Sq(\mu_1) = 0$. If we suppose that Sq is injective, then $Sq(\mu_1) = 0$ implies that the cohomology class $\mu_1 = 0$. Then every integrable infinitesimal is equivalent to the trivial cohomology class. Therefore, every formal deformation is trivial.

Proposition 6.3. *If the map Sq is injective, then A is formally rigid.*

We have the following definition for rigid complex algebra with the perturbation point of view.

Definition 6.4. A complex algebra of alg_n with multiplication μ_0 is *infinitesimally rigid* if every perturbation μ of μ_0 belongs to the orbit $\vartheta(\mu_0)$.

This definition characterizes the open sets over hypercomplex field (with metric topology). Since open orbit (with metric topology) is Zariski open [35]. Then we have the following.

Proposition 6.5. *For finite dimensional complex algebras, infinitesimal rigidity is equivalent to geometric rigidity, thus to formal rigidity.*

In the global point of view we set the following two concepts of rigidity.

Definition 6.6. Let $\mathbb{B}$ be a commutative algebra over a field $\mathbb{K}$ and let m be a maximal ideal of $\mathbb{B}$. An algebra $\mathscr{A}$ is called $(\mathbb{B},m)$-*rigid* if every global deformation parameterized by $(\mathbb{B},m)$ is isomorphic to $\mathscr{A}$ (in the push-out sense).

An algebra is called *globally rigid* if for every commutative algebra $\mathbb{B}$ and a maximal ideal m of $\mathbb{B}$, it is $(\mathbb{B},m)$-rigid.

The global rigidity implies the formal rigidity but the converse is false. Fialowski and Schlichenmaier show that over the complex field the Witt algebra, which is algebraically and formally rigid, is not globally rigid. They use families of Krichever-Novikov-type algebras [30].

Finally, we summarize the link between the different concepts of rigidity in the following theorem.

THEOREM 6.7. *Let $\mathcal{A}$ be a finite-dimensional algebra over a field $\mathbb{K}$ of characteristic zero. Then*

$$\begin{gathered}\text{algebraic rigidity} \Longrightarrow \text{formal rigidity}\\ \text{formal rigidity} \Longleftrightarrow \text{geometric rigidity} \Longleftrightarrow (\mathbb{K}[[t]], t\mathbb{K}[[t]])\text{-rigidity}\\ \text{global rigidity} \Longrightarrow \text{formal rigidity}.\end{gathered} \tag{6.2}$$

In particular, if $\mathbb{K} = \mathbb{C}$,

$$\text{infinitesimal rigidity} \Longleftrightarrow \text{formal rigidity}. \tag{6.3}$$

Recall that semisimple algebras are algebraically rigid. They are classified by Wedderburn's theorem. The classification of low-dimensional rigid algebras is known until $n < 7$, see [17, 36, 37].

7. The algebraic varieties alg_n

A point in alg_n is defined by n^3 parameters, which are the structure constants C_{ij}^k, satisfying a finite system of quadratic relations given by the associativity condition. The orbits are in 1-1-correspondence with the isomorphism classes of n-dimensional algebras.

The stabilizer subgroup of $\mathcal{A}$ ($\mathrm{stab}(\mathcal{A}) = \{f \in \mathrm{GL}_n(\mathbb{K}) : \mathcal{A} = f \cdot \mathcal{A}\}$) is $\mathrm{Aut}(\mathcal{A})$, the automorphism group of $\mathcal{A}$. The orbit $\vartheta(\mathcal{A})$ is identified with the homogeneous space $\mathrm{GL}_n(\mathbb{K})/\mathrm{Aut}(\mathcal{A})$. Then

$$\dim \vartheta(\mathcal{A}) = n^2 - \dim \mathrm{Aut}(\mathcal{A}). \tag{7.1}$$

The orbit $\vartheta(\mathcal{A})$ is provided, when $\mathbb{K} = \mathbb{C}$ (a complex field), with the structure of a differentiable manifold. In fact, $\vartheta(\mathcal{A})$ is the image through the action of the Lie group $\mathrm{GL}_n(\mathbb{K})$ of the point $\mathcal{A}$, considered as a point of $\mathrm{Hom}(V \otimes V, V)$. The Zariski tangent space to alg_n at the point $\mathcal{A}$ corresponds to $Z^2(\mathcal{A}, \mathcal{A})$ and the tangent space to the orbit corresponds to $B^2(\mathcal{A}, \mathcal{A})$.

The first approach to the study of varieties alg_n is to establish, for a fixed dimension, classifications of the algebras up to isomorphisms. Some incomplete classifications were known by mathematicians of the last centuries: R.S. Peirce (1870), E. Study (1890), G. Voghera (1908), and B.G. Scorza (1938). In [36], Gabriel has defined the *scheme* alg_n and gave the classification, up to isomorphisms, for $n \leq 4$ and Mazzola, in [37], has studied the case $n = 5$. The number of different isomorphism classes grows up very quickly, for example, there are 19 classes in alg_4 and 59 classes in alg_5.

The second approach is to describe the irreducible components of a given algebraic variety alg_n. This problem has already been proposed by Study and solved by Gabriel for $n \leq 4$ and Mazzola for $n = 5$. They used mainly the formal deformations and degenerations. The rigid algebras have a special interest, an open orbit of a given algebra is dense in the irreducible component in which it lies. Then, its Zariski closure determines an irreducible component of alg_n, that is, all algebras in this irreducible component are degenerations of the rigid algebra and there is no algebra which degenerates to the rigid algebra. Two nonisomorphic rigid algebras correspond to different irreducible components. So the number of rigid algebra classes gives a lower bound of the number of irreducible components of alg_n. Note that not all irreducible components are Zariski closure of open orbits.

Geometrically, $\mathcal{A}_0$ is a degeneration of $\mathcal{A}_1$ means that $\mathcal{A}_0$ and $\mathcal{A}_1$ belong to the same irreducible component in alg_n.

The following statement gives an invariant which is stable under perturbations [17], it was used to classify the 6-dimensional complex rigid associative algebras. Also it induces an algorithm to compute the irreducible components.

Theorem 7.1. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra in* alg_n *with a multiplication μ_0 and let X_0 be an idempotent of μ_0. Then, for every perturbation μ of μ_0, there exists an idempotent X such that $X \simeq X_0$.*

This has the following consequence: the number of linearly independent idempotents does not decrease by perturbation.

Then, we deduce a procedure that finds the irreducible components by solving some algebraic equations [38], this procedure finds all multiplication μ in alg_n with p idempotents, then the perturbations of each such μ's are studied. From this it can be decided whether μ belongs to a new irreducible component. By letting p run from n down to 1, one finds all irreducible components. A computer implementation enables to do the calculations.

Let us give the known results in low dimensions:

dimension	2	3	4	5	6
irreducible components	1	2	5	10	> 21
rigid algebras	1	2	4	9	21.

The asymptotic number of parameters in the system defining the algebraic variety alg_n is $4n^3/27 + O(n^{8/3})$; see [39]. Any change of basis can reduce this number by at most $n^2 (= \dim(\text{GL}(n,\mathbb{K})))$. The number of parameters for alg_2 is 2 and for alg_3 is 6. For large n the number of irreducible components alg_n satisfies $\exp(n) < \text{alg}_n < \exp(n^4)$, see [37].

Another way to study the irreducible components is the notion of compatible deformations introduced by Gerstenhaber and Giaquinto [40]. Two deformations $\mathcal{A}_t$ and $\mathcal{A}_s$ of the same algebra $\mathcal{A}$ are compatible if they can be joined by a continuous family of algebras. When $\mathcal{A}$ has finite dimension n, this means that $\mathcal{A}_t$ and $\mathcal{A}_s$ lie on a common irreducible component of alg_n. They proved the following theorem.

Theorem 7.2. *Let $\mathcal{A}$ be a finite dimensional algebra with multiplication μ_0 and let $\mathcal{A}_t$ and $\mathcal{A}_s$ be two deformations of $\mathcal{A}$ with multiplications $\alpha_t = \mu_0 + tF_1 + t^2F_2 + \cdots$ and $\beta_s = \mu_0 + sH_1 + s^2H_2 + \cdots$.*

If $\mathcal{A}_t$ and $\mathcal{A}_s$ are compatible, then the classes f and h of the cocycles F and H satisfy $[f,g]_G = 0$, that is, $[F,H]_G$ must be a coboundary.

The theorem gives a necessary condition for the compatibility of the deformations. It can be used to show that two deformations of $\mathcal{A}$ lie on different irreducible components of alg_n.

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