DIMENSION THEORY AND FUZZY TOPOLOGICAL SPACES

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J. M. Aarts introduced and studied a new dimension function, Hind, in 1975 and obtained several results on this function. In this paper, a new local inductive dimension function called local huge inductive dimension function denoted by loc Hind is introduced and studied. Furthermore, an effort is made to introduce and study dimension functions for fuzzy topological spaces. It has been possible to introduce and study the small inductive dimension function indf X and large inductive dimension function Indf X for a fuzzy topological space X.

1. Introduction

In Section 2, a new local dimension function called local huge inductive dimension function denoted by loc Hind is introduced and studied. Its relationship with other local dimension functions is established. A closed subset theorem and an open subset theorem are obtained for the local huge inductive dimension function. Further it is also proved that the local huge inductive dimension function coincides with the huge inductive dimension function for the class of weakly paracompact totally normal spaces.

The concept of a fuzzy subset was introduced and studied by Zadeh [9] and the concept of fuzzy topological spaces by Chang [2]. Many mathematicians have contributed to the development of fuzzy topological spaces.

In Section 3, two inductive types of dimension functions for fuzzy topological spaces have been introduced and studied. Several results have been obtained. It is observed that such dimensions are integers and not fractions.

2. A new local dimension function for topological spaces

The following definition is due to Aarts [1].

Definition 2.1 [1]. The huge inductive dimension function Hind is defined for every hereditarily normal space as follows. Hind $X = -1$ if and only if $X = \phi$. For each integer $n \geq 0$, Hind $X \leq n$ provided that for each pair of closed subsets F and G with

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$\operatorname{Hind}(F \cap G) \leq n-1$ there exists a pair of closed subsets K and L such that $F - G \subset K - L$, $G - F \subset L - K$, $K \cup L = X$, and $\operatorname{Hind}(K \cap L) \leq n-1$. $\operatorname{Hind} X = n$ if and only if $\operatorname{Hind} X \leq n$ is true and $\operatorname{Hind} X \leq n-1$ is not true. $\operatorname{Hind} X = \infty$ if and only if $\operatorname{Hind} X \leq n$ is not true for every n.

The following result is due to Aarts [1].

Proposition 2.2 [1]. *For each integer $n \geq 0$,* $\operatorname{Hind} X \leq n$ *if and only if for each pair of closed subsets F and G with* $\operatorname{Hind}(F \cap G) \leq n-1$ *there exists a closed set S such that $F - G$ and $G - F$ are separated by S and* $\operatorname{Hind} S \leq n-1$. *That is, $X - S = U \cup V$, where U, V are disjoint open sets in X, $F - G \subset U$, $G - F \subset V$, and* $\operatorname{Hind} S \leq n-1$.

The following concept of a barrier is due to Vaĭnšteĭn [6].

Definition 2.3 [6]. Let A, B be a pair of closed subsets of a space X. Then a closed subset C of X is said to be a barrier between A and B if $X - [C \cup (A \cap B)] = G \cup H$, where G, H are disjoint open sets in X such that $A - B \subset G$ and $B - A \subset H$.

The following result is proved.

Theorem 2.4. *Let X be a hereditarily normal space. If, for any two closed sets A, B in X with* $\operatorname{Hind}(A \cap B) \leq n-1$, *there is a barrier C between A and B in X such that* $\operatorname{Hind} C \leq n-1$, *then* $\operatorname{Hind} X \leq n$.

Proof. By hypothesis, there is a barrier C between A and B such that $\operatorname{Hind} C \leq n-1$. Since C is a barrier between A and B, $X - [C \cup (A \cap B)] = G \cup H$, where G, H are disjoint open sets in X such that $A - B \subset G$ and $B - A \subset H$. Clearly the closed set $C \cup (A \cap B)$ separates $A - B$ and $B - A$. Also $\operatorname{Hind} C \leq n-1$ and $\operatorname{Hind}(A \cap B) \leq n-1$. Therefore by the countable sum theorem for the huge inductive dimension [1, Theorem 1] it follows that $\operatorname{Hind}[C \cup (A \cap B)] \leq n-1$. Hence by Proposition 2.2 [1, Proposition 1] it follows that $\operatorname{Hind} X \leq n$. □

The concept of local dimension for the huge inductive dimension function is introduced in the following.

Definition 2.5. The local huge inductive dimension, loc Hind, is defined for every hereditarily normal space X as follows. $\operatorname{loc}\operatorname{Hind} X = -1$ if and only if $X = \phi$. $\operatorname{loc}\operatorname{Hind} X \leq n$ if and only if for each point $x \in X$ there exists an open set U containing x such that $\operatorname{Hind} \overline{U} \leq n$. $\operatorname{loc}\operatorname{Hind} X = n$, for $n = 0, 1, 2, \ldots, \infty$, are defined as usual.

Remark 2.6. If $\operatorname{loc}\operatorname{Hind} X \leq n$, $x \in X$, and U is an open set containing x, then there exists an open set V in X such that $x \in V \subset U$ and $\operatorname{Hind} \overline{V} \leq n$. Thus, for a hereditarily normal space X, $\operatorname{loc}\operatorname{Hind} X \leq n$ if and only if every open cover of X has an open refinement $\{U_\lambda : \lambda \in \Lambda\}$ such that $\operatorname{Hind} \overline{U}_\lambda \leq n$ for each $\lambda \in \Lambda$.

The relationships of loc Hind with some of the other dimension functions are obtained in the following.

THEOREM 2.7. *For a hereditarily normal space X, the following statements are true.*
(i) $\operatorname{loc}\operatorname{Hind} X \leq \operatorname{Hind} X$.
(ii) $\operatorname{loc}\dim X \leq \operatorname{loc}\operatorname{Ind} X \leq \operatorname{loc}\operatorname{Hind} X$.

Proof. (i) Let Hind X $\leq n$. Then every point $x \in X$ has the neighborhood X such that $\operatorname{Hind}\overline{X} = \operatorname{Hind} X \leq n$. Therefore $\operatorname{loc}\operatorname{Hind} X \leq n$.

(ii) Since X is hereditarily normal, it is normal, and hence $\operatorname{loc}\dim X \leq \operatorname{loc}\operatorname{Ind} X$. It is required to show that $\operatorname{loc}\operatorname{Ind} X \leq \operatorname{loc}\operatorname{Hind} X$. Let $\operatorname{loc}\operatorname{Hind} X \leq n$. Then each point $x \in X$ has a neighborhood U in X with $\operatorname{Hind}\overline{U} \leq n$. Now $\overline{U} \subset X$ and X is hereditarily normal. Therefore $\overline{U}$ is also hereditarily normal. Then from [1, Proposition 2] it follows that $\operatorname{Ind}\overline{U} \leq \operatorname{Hind}\overline{U}$. Therefore $\operatorname{Ind}\overline{U} \leq n$ and hence $\operatorname{loc}\operatorname{Hind} X \leq n$. Hence $\operatorname{loc}\operatorname{Ind} X \leq \operatorname{loc}\operatorname{Hind} X$. □

COROLLARY 2.8. *If X is hereditarily normal regular space, then* $\operatorname{ind} X \leq \operatorname{loc}\operatorname{Hind} X$.

Proof. The result follows since $\operatorname{ind} X \leq \operatorname{loc}\operatorname{Ind} X$ on the class of regular spaces [5]. □

The closed subset theorem for local huge inductive dimension function is obtained, which is contained in the following.

THEOREM 2.9. *If A is a closed subset of a hereditarily normal space X, then* $\operatorname{loc}\operatorname{Hind} A \leq \operatorname{loc}\operatorname{Hind} X$.

Proof. Let $\operatorname{loc}\operatorname{Hind} X \leq n$. Let $x \in A$. Now $x \in X$ and $\operatorname{loc}\operatorname{Hind} X \leq n$. Therefore there exists an open set U in X such that $x \in U$ and $\operatorname{Hind}\overline{U} \leq n$. Then $U \cap A$ is an open set in A containing x. Now $cl_A(U \cap A)$ is a closed subset of A and $\operatorname{Hind}\overline{U} \leq n$. Then by the closed subset theorem for Hind [1, Proposition 3] it follows that $\operatorname{Hind}[cl_A(U \cap A)] \leq n$. Therefore $\operatorname{loc}\operatorname{Hind} A \leq n$. Hence $\operatorname{loc}\operatorname{Hind} A \leq \operatorname{loc}\operatorname{Hind} X$. □

The open subset theorem for loc Hind is proved in the following.

THEOREM 2.10. *If X is a hereditarily normal regular space and Y is an open subset of X, then* $\operatorname{loc}\operatorname{Hind} Y \leq \operatorname{loc}\operatorname{Hind} X$.

Proof. Let $\operatorname{loc}\operatorname{Hind} X \leq n$. Let $y \in Y$. Then $y \in X$ and $\operatorname{loc}\operatorname{Hind} X \leq n$. Therefore there exists an open set U in X containing y such that $\operatorname{Hind}\overline{U} \leq n$. Now $U \cap Y$ is an open set in X containing y and X is regular. Therefore there exists an open set V such that $y \in V \subset \overline{V} \subset U \cap Y$. Then V is an open neighborhood of y in Y and $\overline{V}$ is the closure of V in Y. Since $\overline{V} \subset \overline{U}$, it follows that $\operatorname{Hind}\overline{V} \leq n$. Therefore $\operatorname{loc}\operatorname{Hind} X \leq n$. Hence $\operatorname{loc}\operatorname{Hind} Y \leq \operatorname{loc}\operatorname{Hind} X$. □

The next result is a sum theorem that is obtained for loc Hind.

THEOREM 2.11. *If a hereditarily normal space X is the union of two closed sets A, B and if $\operatorname{loc}\operatorname{Hind} A \leq n$ and $\operatorname{loc}\operatorname{Hind} B \leq n$, then $\operatorname{loc}\operatorname{Hind} X \leq n$.*

Proof. Let $x \in X$. If $x \in X - A$, then $x \in B$ and $\operatorname{loc}\operatorname{Hind} B \leq n$. Therefore there exists an open set $U \cap B$ in B containing x, where U is open in X such that $\operatorname{Hind}\overline{U \cap B} \leq n$. Let $W = U \cap (X - A)$. Then W is an open set in X containing x such that $\overline{W} \subset \overline{U \cap B}$ and $\operatorname{Hind}\overline{U \cap B} \leq n$. Therefore $\operatorname{Hind}\overline{W} \leq n$ and hence $\operatorname{loc}\operatorname{Hind} X \leq n$. Similarly if $x \in X - B$, then there exists an open set in X containing x, the closure of which has huge inductive dimension not exceeding n. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $\operatorname{loc}\operatorname{Hind} A \leq n$ and $\operatorname{loc}\operatorname{Hind} B \leq n$, there exist open sets $U \cap A$ and $V \cap B$ in A and B, respectively, each containing x such that $\operatorname{Hind}\overline{U \cap A} \leq n$ and $\operatorname{Hind}\overline{V \cap B} \leq n$. Let $W = [X - (A - U)] - (B - V)$. Then W is an open set containing x and $W \subset (U \cap A) \cup (V \cap B)$. Therefore $\overline{W} \subset (\overline{U \cap A}) \cup (\overline{V \cap B})$.

Therefore

$$\begin{aligned}\operatorname{Hind}\overline{W} &\leq \operatorname{Hind}\left[(\overline{U \cap A}) \cup (\overline{V \cap B})\right] \\ &\leq \sup\left\{\operatorname{Hind}(\overline{U \cap A}),\ \operatorname{Hind}\overline{(V \cap B)}\right\}\end{aligned} \tag{2.1}$$

by using the countable sum theorem for Hind [1]. Therefore $\operatorname{Hind} X \leq n$. Hence $\operatorname{loc}\operatorname{Hind} X \leq n$. □

The above result can be extended to a finite family and furthermore to a locally finite family which is contained in the following corollary.

Corollary 2.12. *If $\{A_\lambda : \lambda \in \Lambda\}$ is a locally finite closed covering of a hereditarily normal space X such that* $\operatorname{loc}\operatorname{Hind} A_\lambda \leq n$ *for each $\lambda \in \Lambda$, then* $\operatorname{loc}\operatorname{Hind} X \leq n$.

Proof. The straightforward proof is omitted. □

The following result shows that loc Hind coincides with Hind on the class of weakly paracompact totally normal spaces.

Theorem 2.13. *If X is a weakly paracompact totally normal space, then* $\operatorname{loc}\operatorname{Hind} X = \operatorname{Hind} X$.

Proof. Since X is totally normal, it is hereditarily normal, and hence from Theorem 2.7 it follows that $\operatorname{loc}\operatorname{Hind} X \leq \operatorname{Hind} X$.

On the other hand, since X is weakly paracompact totally normal, $\operatorname{loc}\operatorname{Ind} X = \operatorname{Ind} X$ from [5, Page 197]. But Ind = Hind on the class of totally normal spaces. Therefore $\operatorname{Hind} X = \operatorname{loc}\operatorname{Ind} X \leq \operatorname{loc}\operatorname{Hind} X$, since for hereditarily normal spaces $\operatorname{loc}\operatorname{Ind} \leq \operatorname{loc}\operatorname{Hind}$ from Theorem 2.7. Therefore $\operatorname{Hind} X \leq \operatorname{loc}\operatorname{Hind} X$. Hence $\operatorname{loc}\operatorname{Hind} X = \operatorname{Hind} X$. □

3. Inductive dimension functions for fuzzy topological spaces

The main purpose of this section is to introduce and study dimension functions on fuzzy topological spaces. It has been possible to introduce the small inductive dimension function, $\operatorname{indf} X$, and the large inductive dimension function, $\operatorname{Indf} X$, for a fuzzy topological space X. A subset theorem is obtained for $\operatorname{indf} X$. It is proved that if X is a fuzzy topological space such that $\operatorname{Indf} X = 0$ then X is a normal fuzzy topological space. A closed subset theorem for Indf is also obtained.

The following concept is due to Zadeh [9].

Definition 3.1 [9]. A fuzzy subset A in a set X is a function $A : X \to [0,1]$.

The elementary properties related to fuzzy sets are contained in [9]. The fuzzy topological spaces were introduced and studied by Chang [2].

Definition 3.2 [2]. Let X be a set and let T be family of fuzzy subsets of X. Then T is called a fuzzy topology on X if T satisfies the following conditions.

(i) $0, 1 \in T$.

(ii) If $\{G_\lambda : \lambda \in \Lambda\} \subset T$, then $\vee G_\lambda \in T$.

(iii) If $G, H \in T$, then $G \wedge H \in T$.

The pair (X, T) is called a fuzzy topological space (abbreviated as fts). The members of T are called open fuzzy sets. A fuzzy set B is called a closed fuzzy set if $1 - A$ is an open fuzzy set.

The concept of the boundary of a fuzzy subset was introduced and studied by Warren [7], which is contained in the following.

Definition 3.3 [7]. Let A be a fuzzy set in an fts X. The fuzzy boundary of A denoted by $\mathrm{bd}(A)$ is defined as the infimum of all the closed fuzzy sets D in X with the following property. $D(x) \geq \overline{A}(x)$ all $x \in X$ for which $(\overline{A} \wedge \overline{1 - A})(x) > 0$.

The following results of Warren [7] are used in the sequel.

Theorem 3.4 [7]. *Let A and B be fuzzy sets in an* fts X. *Then the following results hold good.*

(1) $\mathrm{bd}(A) = 0$ if and only if A is open, closed, and crisp.

(2) $\mathrm{bd}(A \wedge B) \leq \mathrm{bd}(A) \vee \mathrm{bd}(B)$.

The other elementary concepts, results, and developments on fuzzy topological spaces can be found in [2–4, 7, 8].

A new inductive dimension function for fuzzy topological spaces is introduced in the following.

Definition 3.5. Let X be a fuzzy topological space. The small inductive dimension of X, denoted by $\mathrm{indf}\, X$, is defined as follows. $\mathrm{indf}\, X = -1$ if $X = \phi$. For any nonnegative integer n, $\mathrm{indf}\, X \leq n$ if for each $x \in X$ and each open fuzzy set G such that $G(x) > 0$ there exists an open fuzzy set U in X such that $U(x) > 0$, $U \leq G$ and $\mathrm{indf}\, \mathrm{bd}(U) \leq n - 1$. $\mathrm{indf}\, X = n$ if $\mathrm{indf}\, X \leq n$ is true and $\mathrm{indf}\, X \leq n - 1$ is not true. $\mathrm{indf}\, X = \infty$ if there is no integer n such that $\mathrm{indf}\, X \leq n$.

Note that if X is a general topological space, then this concept reduces to that of ind.

A subset theorem for indf is proved in the following.

Theorem 3.6. *If A is a crisp subset of an* fts X, *then* $\mathrm{indf}\, A \leq \mathrm{indf}\, X$.

Proof. This is proved by induction on n. For $n = -1$, if $\mathrm{indf}\, X \leq -1$, then $\mathrm{indf}\, X = -1$, so that $X = \phi$. Since A is a crisp subset of X, it follows that $A = \phi$, and therefore $\mathrm{indf}\, A = -1$, that is, $\mathrm{indf}\, A \leq -1$. Thus if $\mathrm{indf}\, X \leq -1$, then $\mathrm{indf}\, A \leq -1$. Therefore the result holds for $n = -1$. Assume the result for $n - 1$. Then, to prove the result for n, that is, to prove if $\mathrm{indf}\, X \leq n$, then $\mathrm{indf}\, A \leq n$, let $\mathrm{indf}\, X \leq n$. Then to prove $\mathrm{indf}\, A \leq n$, let $x \in A$ and let G be an open fuzzy set in A, such that $G(x) > 0$. Since G is open in A by induced fuzzy topology on A [8], there exists an open fuzzy set H in X such that $G = A \wedge H$. Now $G(x) > 0$ implies $H(x) > 0$ and $A(x) > 0$. Since $\mathrm{indf}\, X \leq n$, H is an open fuzzy set in X such that $H(x) > 0$. By Definition 3.5 there exists an open fuzzy set V in X such that $V(x) > 0$, $V \leq H$, and $\mathrm{indf}\, \mathrm{bd}(V) \leq n - 1$. Let $U = A \wedge V$. Since V is an open fuzzy set in X, it follows that U is an open fuzzy set in A. Now $U(x) > 0$. We have $A(x) > 0$ and $V(x) > 0$. Therefore $A(x) \wedge V(x) > 0$, so that $(A \wedge V)(x) > 0$, and hence $U(x) > 0$. Also $U \leq G$. We have $V \leq H$. Therefore $A \wedge V \leq A \wedge H$, so that $U \leq G$. Further, $\operatorname{indf} \operatorname{bd}_A(U) = \operatorname{bd}_A(A \wedge V) \leq \operatorname{bd}_A(A) \vee \operatorname{bd}_A(V) = 0 \vee \operatorname{bd}_A(V) = \operatorname{bd}_A(V) \leq A \wedge \operatorname{bd}(V) \leq \operatorname{bd}(V)$. Thus $\operatorname{bd}_A(U) \leq \operatorname{bd}(V)$. Since $\operatorname{indf} \operatorname{bd}(V) \leq n-1$, by induction hypothesis it follows that $\operatorname{indf} \operatorname{bd}_A(U) \leq n-1$. Thus, for each $x \in A$ and each open fuzzy set G in A such that $G(x) > 0$, there exists an open fuzzy set U in A such that $U(x) > 0$, $U \leq G$, and $\operatorname{indf} \operatorname{bd}_A(U) \leq n-1$. Therefore by Definition 3.5 it follows that $\operatorname{indf} A \leq n$. Thus if $\operatorname{indf} X \leq n$, then $\operatorname{indf} A \leq n$. Therefore the result holds for n. Hence $\operatorname{indf} A \leq \operatorname{indf} X$. □

Another new inductive dimension function for fuzzy topological spaces is introduced in the following.

Definition 3.7. Let X be an fts. The large inductive dimension of X, denoted by $\operatorname{Indf} X$, is defined as follows. $\operatorname{Indf} X = -1$ if and only if $X = \phi$. $\operatorname{Indf} X \leq n$, for any nonnegative integer n, if for each closed fuzzy set E and each open fuzzy set G in X such that $E \leq G$, there exists an open fuzzy set U in X such that $E \leq U \leq G$ and $\operatorname{Indf} \operatorname{bd}(U) \leq n-1$. $\operatorname{Indf} X = n$ if $\operatorname{Indf} X \leq n$ is true and $\operatorname{Indf} X \leq n-1$ is not true. $\operatorname{Indf} X = \infty$ if $\operatorname{Indf} X \leq n$ is not true for every n.

Note that if X is a general topological space, then this concept reduces to that of Ind. A relationship between indf and Indf is obtained in the following.

Theorem 3.8. *If X is an* fts *with the property that each open fuzzy set in X is union of closed fuzzy sets in X, then* $\operatorname{indf} X \leq \operatorname{Indf} X$.

Proof. This is proved by induction on n. For $n = -1$, if $\operatorname{Indf} X \leq -1$, then $\operatorname{Indf} X = -1$, so that $X = \phi$. Therefore $\operatorname{indf} X = -1$, so that $\operatorname{indf} X \leq -1$. Thus if $\operatorname{Indf} X \leq -1$, then $\operatorname{indf} X \leq -1$. Therefore the result holds for $n = -1$. Assume that the result holds for $n = k-1$. That is, assume that if $\operatorname{Indf} X \leq k-1$, then $\operatorname{indf} X \leq k-1$. To prove that the result holds for $n = k$, suppose $\operatorname{Indf} X \leq k$. Then, to prove $\operatorname{indf} X \leq k$, let $x \in X$ and let G be an open fuzzy set in X such that $G(x) > 0$. Now G is an open fuzzy set. By hypothesis, G is union of closed fuzzy sets say $G = \vee E_\lambda$, where each E_λ is a closed fuzzy set. Since $G(x) > 0$, $(\vee E_\lambda)(x) > 0$, so that there exists a λ_0 such that $E_{\lambda_0}(x) > 0$. Also $E_{\lambda_0} \leq E_\lambda \leq G$. Now $E_{\lambda_0} \leq G$, where E_{λ_0} is a closed fuzzy set and G is an open fuzzy set. Since $\operatorname{Indf} X \leq k$ by Definition 3.7, there exists an open fuzzy set U in X such that $E_{\lambda_0} \leq U \leq G$ and $\operatorname{Indf} \operatorname{bd}(U) \leq k-1$. By induction hypothesis, it follows that $\operatorname{indf} \operatorname{bd}(U) \leq k-1$. Thus, for each $x \in X$ and each open fuzzy set G such that $G(x) > 0$, there exists an open fuzzy set U in X such that $U(x) > 0$, $U \leq G$, and $\operatorname{indf} \operatorname{bd}(U) \leq k-1$. Therefore by Definition 3.5 it follows that $\operatorname{indf} X \leq k$. Thus, if $\operatorname{Indf} X \leq k$, then $\operatorname{indf} X \leq k$. Therefore the result holds for $n = k$. Hence $\operatorname{indf} X \leq \operatorname{Indf} X$. □

We also have the following result.

Theorem 3.9. *If X is an* fts *such that* $\operatorname{Indf} X = 0$, *then X is a normal* fts.

Proof. Let a, b be closed fuzzy sets in X such that $a \leq 1-b$. Note that $1-b$ is an open fuzzy set. Since $\operatorname{Indf} X \leq 0$, by Definition 3.7, there exists an open fuzzy set c in X such that $a \le c \le 1 - b$ and $\operatorname{Indf}\operatorname{bd}(c) \le 0 - 1$. That is, $a \le c$, $c \le 1 - b$, and $\operatorname{Indf}\operatorname{bd}(c) \le -1$. Let $d = 1 - c$. Then $a \le c$, $b \le d$, $c = 1 - d$, and $\operatorname{bd}(c) = 0$. Since $\operatorname{bd}(c) = 0$, from Theorem 3.4, c is open, closed, and crisp. Thus, for each pair a, b of closed fuzzy sets in X with $a \le 1 - b$, there exist open fuzzy sets c, d in X, such that $a \le c$, $b \le d$, and $c \le 1 - d$. Therefore from [3, Theorem 5.2, page 36] it follows that X is a normal fts. □

A closed subset theorem for Indf is obtained, which is contained in the following.

Theorem 3.10. *If A is a closed crisp subspace of an* fts X, *then* $\operatorname{Indf} A \le \operatorname{Indf} X$.

Proof. This is proved by induction on n. For $n = -1$, if $\operatorname{Indf} X \le -1$, then $\operatorname{Indf} X = -1$, so that $X = \phi$. Therefore $A = \phi$ and so $\operatorname{Indf} A = -1$, so that $\operatorname{Indf} A \le -1$. Thus if $\operatorname{Indf} X \le -1$, then $\operatorname{Indf} A \le -1$. Therefore the result is true for $n = -1$. Assume that the result holds for $n = k - 1$. That is, assume that if $\operatorname{Indf} X \le k - 1$, then $\operatorname{Indf} A \le k - 1$. Then the result is to be proved for $n = k$, that is, to prove if $\operatorname{Indf} X \le k$, then $\operatorname{Indf} A \le k$. Suppose $\operatorname{Indf} X \le k$. To prove $\operatorname{Indf} A \le k$, let E be a closed fuzzy set in A and let G be an open fuzzy set in A such that $E \le G$. Since E is closed in A and A is closed in X, it follows that E is closed in X. Also G is an open fuzzy set in A. Therefore $G = A \wedge H$, where H is an open fuzzy set in X. Also since $E \le G$, we have $E \le A \wedge H \le H$, so that $E \le H$, where E is closed fuzzy set in X and H is open fuzzy set in X. Since $\operatorname{Indf} X \le k$, by definition, there exists an open fuzzy set V in X such that $E \le V \le H$ and $\operatorname{Indf}\operatorname{bd}(V) \le k - 1$. Therefore $E \wedge A \le V \wedge A \le H \wedge A$ which implies $E \le U \le G$, where $V \wedge A = U$ is a closed fuzzy set in A. Also $\operatorname{bd}_A(U)$ is a closed fuzzy set in A [7] and A is a closed fuzzy set in X. Therefore $\operatorname{bd}_A(U)$ is a closed fuzzy set in X. Further $\operatorname{bd}_A(U) \le \operatorname{bd}(V)$. Therefore $\operatorname{bd}_A(U)$ is closed in $\operatorname{bd}(V)$ and $\operatorname{Indf}\operatorname{bd}(V) \le k - 1$. Therefore by induction hypothesis $\operatorname{Indf}\operatorname{bd}_A(U) \le k - 1$.

Thus for each closed fuzzy set E in A and open fuzzy set G in A such that $E \le G$, there exists an open fuzzy set U in A such that $E \le U \le G$ and $\operatorname{Indf}\operatorname{bd}_A(U) \le k - 1$. Therefore, by definition, it follows that $\operatorname{Indf} A \le k$. Thus, if $\operatorname{Indf} X \le k$, then $\operatorname{Indf} A \le k$. Therefore the result holds for $n = k$. Thus the result holds for all values of n. Hence $\operatorname{Indf} A \le \operatorname{Indf} X$. □

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References

[1] J. M. Aarts, *A new dimension function*, Proceedings of the American Mathematical Society **50** (1975), no. 1, 419–425.
[2] C. L. Chang, *Fuzzy topological spaces*, Journal of Mathematical Analysis and Applications **24** (1968), no. 1, 182–190.
[3] S. R. Malghan and S. S. Benchalli, *On fuzzy topological spaces*, Glasnik Matematički. Serija III **16(36)** (1981), no. 2, 313–325.
[4] ———, *Open maps, closed maps and local compactness in fuzzy topological spaces*, Journal of Mathematical Analysis and Applications **99** (1984), no. 2, 338–349.

[5] A. R. Pears, *Dimension Theory of General Spaces*, Cambridge University Press, Cambridge, 1975.
[6] A. I. Vaĭnšteĭn, *A class of infinite-dimensional spaces*, Matematicheskiĭ Sbornik. Novaya Seriya **79 (121)** (1969), 433–443.
[7] R. H. Warren, *Boundary of a fuzzy set*, Indiana University Mathematics Journal **26** (1977), no. 2, 191–197.
[8] M. D. Weiss, *Fixed points, separation, and induced topologies for fuzzy sets*, Journal of Mathematical Analysis and Applications **50** (1975), no. 1, 142–150.
[9] L. A. Zadeh, *Fuzzy sets*, Information and Computation **8** (1965), 338–353.

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