STANDING WAVE SOLUTIONS OF SCHRÖDINGER SYSTEMS WITH DISCONTINUOUS NONLINEARITY IN ANISOTROPIC MEDIA

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We establish the existence of an entire solution for a class of stationary Schrödinger systems with subcritical discontinuous nonlinearities and lower bounded potentials that blow up at infinity. The proof is based on the critical point theory in the sense of Clarke and we apply Chang's version of the mountain pass lemma for locally Lipschitz functionals. Our result generalizes in a nonsmooth framework a result of Rabinowitz (1992) related to entire solutions of the Schrödinger equation.

1. Introduction and the main result

The Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behavior of a dynamic system. The linear form of Schrödinger's equation is

$$\Delta\psi + \frac{8\pi^2 m}{\hbar^2}\big(E(x) - V(x)\big)\psi = 0, \tag{1.1}$$

where ψ is the Schrödinger wave function, m is the mass, $\hbar$ denotes Planck's constant, E is the energy, and V stands for the potential energy. The structure of the nonlinear Schrödinger equation is much more complicated. This equation describes various phenomena arising in self-channelling of a high-power ultra-short laser in matter, in the theory of Heisenberg ferromagnets and magnons, in dissipative quantum mechanics, in condensed matter theory, in plasma physics (e.g., the Kurihara superfluid film equation). We refer to [7, 14] for a modern overview, including applications.

Consider the model problem

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \ (N \geq 2), \tag{1.2}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$. In the study of this equation, Oh [12] supposed that the potential V is bounded and possesses a nondegenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some a) introduced by Kato in [10]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov-Schmidt-type reduction, Oh [12] proved the existence of a standing wave solution of problem (1.2), that is, a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x). \tag{1.3}$$

Note that substituting the ansatz (1.3) into (1.2) leads to

$$-\frac{\hbar^2}{2}\Delta u + (V(x) - E)u = |u|^{p-1}u. \tag{1.4}$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2(V_\hbar(x) - E)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $V_\hbar(x) = V(\hbar x)$.

In a celebrated paper, Rabinowitz [13] continued the study of standing wave solutions of nonlinear Schrödinger equations. After making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

under suitable conditions on b and assuming that f is smooth, superlinear, and subcritical.

Inspired by Rabinowitz' paper, we consider the following class of coupled elliptic systems in $\mathbb{R}^N$ ($N \geq 3$):

$$\begin{aligned}
-\Delta u_1 + a(x)u_1 &= f(x,u_1,u_2) \quad \text{in } \mathbb{R}^N, \\
-\Delta u_2 + b(x)u_2 &= g(x,u_1,u_2) \quad \text{in } \mathbb{R}^N.
\end{aligned} \tag{1.7}$$

We point out that coupled nonlinear Schrödinger systems describe some physical phenomena such as the propagation in birefringent optical fibers or Kerr-like photorefractive media in optics. Another motivation to the study of coupled Schrödinger systems arises from the Hartree-Fock theory for the double condensate, that is, a binary mixture of Bose-Einstein condensates in two different hyperfine states (cf. [5]). System (1.7) is also important for industrial applications in fiber communications systems [8] and all-optical switching devices [9].

Throughout this paper, we assume that $a, b \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and there exist $\underline{a}, \underline{b} > 0$ such that $a(x) \geq \underline{a}$, $b(x) \geq \underline{b}$ a.e. in $\mathbb{R}^N$, and $\operatorname{esslim}_{|x| \to \infty} a(x) = \operatorname{esslim}_{|x| \to \infty} b(x) = +\infty$. Our aim in this paper is to study the existence of solutions to the above problem in the case when f, g are not continuous functions. Our goal is to show how variational methods can be used to find existence results for stationary nonsmooth Schrödinger systems.

Throughout this paper, we assume that $f(x,\cdot,\cdot), g(x,\cdot,\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. Denote

$$\underline{f}(x,t_1,t_2) = \lim_{\delta \to 0} \operatorname{essinf}\{f(x,s_1,s_2); \ |t_i - s_i| \leq \delta; \ i = 1,2\},$$

$$\overline{f}(x,t_1,t_2) = \lim_{\delta \to 0} \operatorname{esssup}\{f(x,s_1,s_2); \ |t_i - s_i| \leq \delta; \ i = 1,2\},$$

$$\underline{g}(x,t_1,t_2) = \lim_{\delta \to 0} \operatorname{essinf}\{g(x,s_1,s_2); \ |t_i - s_i| \leq \delta; \ i = 1,2\},$$

$$\overline{g}(x,t_1,t_2) = \lim_{\delta \to 0} \operatorname{esssup}\{g(x,s_1,s_2); \ |t_i - s_i| \leq \delta; \ i = 1,2\}. \tag{1.8}$$

Under these conditions, we reformulate problem (1.7) as follows:

$$-\Delta u_1 + a(x)u_1 \in \left[\underline{f}(x,u_1(x),u_2(x)), \overline{f}(x,u_1(x),u_2(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N,$$

$$-\Delta u_2 + b(x)u_2 \in \left[\underline{g}(x,u_1(x),u_2(x)), \overline{g}(x,u_1(x),u_2(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N. \tag{1.9}$$

Let $H^1 = H(\mathbb{R}^N, \mathbb{R}^2)$ denote the Sobolev space of all $U = (u_1, u_2) \in (L^2(\mathbb{R}^N))^2$ with weak derivatives $\partial u_1/\partial x_j$, $\partial u_2/\partial x_j$ $(j = 1,\ldots,N)$ also in $L^2(\mathbb{R}^N)$, endowed with the usual norm

$$\|U\|^2_{H_1} = \int_{\mathbb{R}^N} (|\nabla U|^2 + |U|^2)\,dx = \int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + u_1^2 + u_2^2\right) dx. \tag{1.10}$$

Given the functions $a,b : \mathbb{R}^N \to \mathbb{R}$ as above, define the subspace

$$E = \left\{ U = (u_1, u_2) \in H^1; \int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right) dx < +\infty \right\}. \tag{1.11}$$

Then the space E endowed with the norm

$$\|U\|^2_E = \int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right) dx \tag{1.12}$$

becomes a Hilbert space.

Since $a(x) \geq \underline{a} > 0$, $b(x) \geq \underline{b} > 0$, we have the continuous embeddings $H^1 \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ for all $2 \leq q \leq 2^* = 2N/(N-2)$.

We assume throughout the paper that $f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are nontrivial measurable functions satisfying the following hypotheses:

$$|f(x,t)| \leq C(|t| + |t|^p) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$$|g(x,t)| \leq C(|t| + |t|^p) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2, \tag{1.13}$$

where $p < 2^*$;

$$\lim_{\delta \to 0} \operatorname{esssup}\left\{\frac{|f(x,t)|}{|t|}; \ (x,t) \in \mathbb{R}^N \times (-\delta, +\delta)^2\right\} = 0,$$

$$\lim_{\delta \to 0} \operatorname{esssup}\left\{\frac{|g(x,t)|}{|t|}; \ (x,t) \in \mathbb{R}^N \times (-\delta, +\delta)^2\right\} = 0; \tag{1.14}$$

f and g are chosen so that the mapping $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, t_1, t_2) := \int_0^{t_1} f(x, \tau, t_2) d\tau + \int_0^{t_2} g(x, 0, \tau) d\tau$ satisfies

$$F(x, t_1, t_2) = \int_0^{t_2} g(x, t_1, \tau) d\tau + \int_0^{t_1} f(x, \tau, 0) d\tau, \quad F(x, t_1, t_2) = 0 \quad \text{iff } t_1 = t_2 = 0; \tag{1.15}$$

there exists $\mu > 2$ such that for any $x \in \mathbb{R}^N$,

$$0 \le \mu F(x, t_1, t_2) \le \begin{cases} t_1 \underline{f}(x, t_1, t_2) + t_2 \underline{g}(x, t_1, t_2), & t_1, t_2 \in [0, +\infty), \\ t_1 \underline{f}(x, t_1, t_2) + t_2 \overline{g}(x, t_1, t_2), & t_1 \in [0, +\infty),\ t_2 \in (-\infty, 0], \\ t_1 \overline{f}(x, t_1, t_2) + t_2 \overline{g}(x, t_1, t_2), & t_1, t_2 \in (-\infty, 0], \\ t_1 \overline{f}(x, t_1, t_2) + t_2 \underline{g}(x, t_1, t_2), & t_1 \in (-\infty, 0],\ t_2 \in [0, +\infty). \end{cases} \tag{1.16}$$

Definition 1.1. A function $U = (u_1, u_2) \in E$ is called solution to the problem (1.9) if there exists a function $W = (w_1, w_2) \in L^2(\mathbb{R}^N, \mathbb{R}^2)$ such that

(i)

$$\underline{f}(x, u_1(x), u_2(x)) \le w_1(x) \le \overline{f}(x, u_1(x), u_2(x)) \quad \text{a.e. } x \text{ in } \mathbb{R}^N,$$
$$\underline{g}(x, u_1(x), u_2(x)) \le w_2(x) \le \overline{g}(x, u_1(x), u_2(x)) \quad \text{a.e. } x \text{ in } \mathbb{R}^N; \tag{1.17}$$

(ii)

$$\int_{\mathbb{R}^N} (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x) u_1 v_1 + b(x) u_2 v_2) dx$$
$$= \int_{\mathbb{R}^N} (w_1 v_1 + w_2 v_2) dx, \quad \forall (v_1, v_2) \in E. \tag{1.18}$$

Our main result is the following.

THEOREM 1.2. *Assume that conditions (1.13)–(1.16) are fulfilled. Then problem (1.9) has at least a nontrivial solution in E.*

2. Auxiliary results

We first recall some basic notions from the Clarke gradient theory for locally Lipschitz functionals (see [3, 4] for more details). Let E be a real Banach space and assume that $I : E \to \mathbb{R}$ is a locally Lipschitz functional. Then the Clarke generalized gradient is defined by

$$\partial I(u) = \{\xi \in E^*;\ I^0(u, v) \ge \langle \xi, v \rangle,\ \forall v \in E\}, \tag{2.1}$$

where $I^0(u, v)$ stands for the directional derivative of I at u in the direction v, that is,

$$I^0(u, v) = \limsup_{\substack{w \to u \\ \lambda \searrow 0}} \frac{I(w + \lambda v) - I(w)}{\lambda}. \tag{2.2}$$

Let Ω be an arbitrary domain in $\mathbb{R}^N$. Set

$$E_\Omega = \left\{ U = (u_1, u_2) \in H^1(\Omega; \mathbb{R}^2); \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2 \right) dx < +\infty \right\}, \tag{2.3}$$

which is endowed with the norm

$$\|U\|_{E_\Omega}^2 = \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2 \right) dx. \tag{2.4}$$

Then E_Ω becomes a Hilbert space.

Lemma 2.1. *The functional* $\Psi_\Omega : E_\Omega \to \mathbb{R}$, $\Psi_\Omega(U) = \int_\Omega F(x, U) dx$ *is locally Lipschitz on* E_Ω.

Proof. We first observe that

$$\begin{aligned}
F(x, U) &= F(x, u_1, u_2) \\
&= \int_0^{u_1} f(x, \tau, u_2) d\tau + \int_0^{u_2} g(x, 0, \tau) d\tau \\
&= \int_0^{u_2} g(x, u_1, \tau) d\tau + \int_0^{u_1} f(x, \tau, 0) d\tau
\end{aligned} \tag{2.5}$$

is a Carathéodory functional which is locally Lipschitz with respect to the second variable. Indeed, by (1.13),

$$\begin{aligned}
\left| F(x, t_1, t) - F(x, s_1, t) \right| &= \left| \int_{s_1}^{t_1} f(x, \tau, t) d\tau \right| \\
&\leq \left| \int_{s_1}^{t_1} C(|\tau, t| + |\tau, t|^p) d\tau \right| \\
&\leq k(t_1, s_1, t) |t_1 - s_1|.
\end{aligned} \tag{2.6}$$

Similarly,

$$\left| F(x, t, t_2) - F(x, t, s_2) \right| \leq k(t_2, s_2, t) |t_2 - s_2|. \tag{2.7}$$

Therefore,

$$\begin{aligned}
\left| F(x, t_1, t_2) - F(x, s_1, s_2) \right| &\leq \left| F(x, t_1, t_2) - F(x, s_1, t_2) \right| + \left| F(x, t_1, s_2) - F(x, s_1, s_2) \right| \\
&\leq k(V) |(t_2, s_2) - (t_1, s_1)|,
\end{aligned} \tag{2.8}$$

where V is a neighborhood of (t_1, t_2), (s_1, s_2).

Set

$$\chi_1(x) = \max\{u_1(x), v_1(x)\}, \quad \chi_2(x) = \max\{u_2(x), v_2(x)\}, \quad \forall x \in \Omega. \tag{2.9}$$

It is obvious that if $U = (u_1, u_2)$, $V = (v_1, v_2)$ belong to E_Ω, then $(\chi_1, \chi_2) \in E_\Omega$. So, by Hölder's inequality and the continuous embedding $E_\Omega \subset L^p(\Omega; \mathbb{R}^2)$,

$$\left| \Psi_\Omega(U) - \Psi_\Omega(V) \right| \leq C \left(\|\chi_1, \chi_2\|_{E_\Omega} \right) \|U - V\|_{E_\Omega}, \tag{2.10}$$

which concludes the proof. □

The following result is a generalization of [11, Lemma 6].

LEMMA 2.2. *Let Ω be an arbitrary domain in $\mathbb{R}^N$ and let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function such that $f(x, \cdot) \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Then $\underline{f}$ and $\overline{f}$ are Borel functions.*

Proof. Since the requirement is local, we may suppose that f is bounded by M and it is nonnegative. Denote

$$f_{m,n}(x, t_1, t_2) = \left(\int_{t_1 - 1/n}^{t_1 + 1/n} \int_{t_2 - 1/n}^{t_2 + 1/n} |f(x, s_1, s_2)|^m ds_1 ds_2 \right)^{1/m}. \tag{2.11}$$

Since $\overline{f}(x, t_1, t_2) = \lim_{\delta \to 0} \text{esssup}\{f(x, s_1, s_2); |t_i - s_i| \leq \delta; i = 1, 2\}$, we deduce that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}^*$ such that for $|t_i - s_i| < 1/n$ $(i = 1, 2)$, we have $|\text{esssup} f(x, s_1, s_2) - \overline{f}(x, t_1, t_2)| < \varepsilon$ or, equivalently,

$$\overline{f}(x, t_1, t_2) - \varepsilon < \text{esssup} f(x, s_1, s_2) < \overline{f}(x, t_1, t_2) + \varepsilon. \tag{2.12}$$

By the second inequality in (2.12), we obtain

$$f(x, s_1, s_2) \leq \overline{f}(x, t_1, t_2) + \varepsilon \quad \text{a.e. } x \in \Omega \text{ for } |t_i - s_i| < \frac{1}{n} \quad (i = 1, 2) \tag{2.13}$$

which yields

$$f_{m,n}(x, t_1, t_2) \leq (\overline{f}(x, t_1, t_2) + \varepsilon) \left(\sqrt{\frac{4}{n^2}} \right)^{1/m}. \tag{2.14}$$

Let

$$A = \left\{ (s_1, s_2) \in \mathbb{R}^2; \ |t_i - s_i| < \frac{1}{n} \ (i = 1, 2); \ \overline{f}(x, t_1, t_2) - \varepsilon \leq f(x, s_1, s_2) \right\}. \tag{2.15}$$

By the first inequality in (2.12) and the definition of the essential supremum, we obtain that $|A| > 0$ and

$$f_{m,n} \leq \left(\int\!\!\int_A (f(x, s_1, s_2))^m ds_1 ds_2 \right)^{1/m} \geq (\overline{f}(x, s_1, s_2) - \varepsilon)|A|^{1/m}. \tag{2.16}$$

Since (2.14) and (2.16) imply that

$$\overline{f}(x, t_1, t_2) = \lim_{n \to \infty} \lim_{m \to \infty} f_{m,n}(x, t_1, t_2), \tag{2.17}$$

it suffices to prove that $f_{m,n}$ is Borel. Let

$$\mathcal{M} = \{f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R};\ |f| \le M \text{ and } f \text{ is a Borel function}\},$$

$$\mathcal{N} = \{f \in \mathcal{M};\ f_{m,n} \text{ is a Borel function}\}. \tag{2.18}$$

$\mathcal{M}$ is the smallest set of functions having the following properties (cf. [1, page 178]):

(i) $\{f \in C(\Omega \times \mathbb{R}^2; \mathbb{R});\ |f| \le M\} \subset \mathcal{M}$;

(ii) $f^{(k)} \in \mathcal{M}$ and $f^{(k)} \overset{k}{\to} f$ imply that $f \in \mathcal{M}$.

Since $\mathcal{N}$ contains obviously the continuous functions and (ii) is also true for $\mathcal{N}$, then by the Lebesgue dominated convergence theorem, we obtain that $\mathcal{M} = \mathcal{N}$. For $\underline{f}$, we note that $\underline{f} = -(-\overline{f})$ and the proof of Lemma 2.2 is complete. $\qquad\square$

Let us now assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain. By the continuous embedding $L^{p+1}(\Omega; \mathbb{R}^2) \hookrightarrow L^2(\Omega; \mathbb{R}^2)$, we may define the locally Lipchitz functional $\Psi_\Omega : L^{p+1}(\Omega; \mathbb{R}^2) \to \mathbb{R}$ by $\Psi_\Omega(U) = \int_\Omega F(x, U) dx$.

LEMMA 2.3. *Under the above assumptions and for any $U \in L^{p+1}(\Omega; \mathbb{R}^2)$,*

$$\partial\Psi_\Omega(U)(x) \subset [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))] \quad a.e.\ x \text{ in } \Omega, \tag{2.19}$$

in the sense that if $W = (w_1, w_2) \in \partial\Psi_\Omega(U) \subset L^{p+1}(\Omega; \mathbb{R}^2)$, then

$$\underline{f}(x, U(x)) \le w_1(x) \le \overline{f}(x, U(x)) \quad a.e.\ x \text{ in } \Omega, \tag{2.20}$$

$$\underline{g}(x, U(x)) \le w_2(x) \le \overline{g}(x, U(x)) \quad a.e.\ x \text{ in } \Omega. \tag{2.21}$$

Proof. By the definition of the Clarke gradient, we have

$$\int_\Omega (w_1 v_1 + w_2 v_2) dx \le \Psi_\Omega^0(U, V) \quad \forall V = (v_1, v_2) \in L^{p+1}(\Omega; \mathbb{R}^2). \tag{2.22}$$

Choose $V = (v, 0)$ such that $v \in L^{p+1}(\Omega)$, $v \ge 0$ a.e. in Ω. Thus, by Lemma 2.2,

$$
\begin{aligned}
\int_\Omega w_1 v &\le \limsup_{\substack{(h_1, h_2) \to U \\ \lambda \searrow 0}} \frac{\int_\Omega \left(\int_{h_1(x)}^{h_1(x) + \lambda v(x)} f(x, \tau, h_2(x)) d\tau \right) dx}{\lambda} \\
&\le \int_\Omega \left(\limsup_{\substack{(h_1, h_2) \to U \\ \lambda \searrow 0}} \frac{1}{\lambda} \int_{h_1(x)}^{h_1(x) + \lambda v(x)} f(x, \tau, h_2(x)) d\tau \right) dx \\
&\le \int_\Omega \overline{f}(x, u_1(x), u_2(x)) v(x) dx.
\end{aligned}
\tag{2.23}
$$

Analogously, we obtain

$$\int_\Omega \underline{f}(x, u_1(x), u_2(x)) v(x) dx \le \int_\Omega w_1 v \, dx \quad \forall v \ge 0 \text{ in } \Omega. \tag{2.24}$$

Arguing by contradiction, suppose that (2.20) is false. Then there exist $\varepsilon > 0$, a set $A \subset \Omega$ with $|A| > 0$, and w_1 as above such that

$$w_1(x) > \overline{f}(x, U(x)) + \varepsilon \quad \text{in } A. \tag{2.25}$$

Taking $v = \mathbf{1}_A$ in (2.23), we obtain

$$\int_\Omega w_1 v \, dx = \int_A w_1 \, dx \leq \int_A \overline{f}(x, U(x)) \, dx, \tag{2.26}$$

which contradicts (2.25). Proceeding in the same way, we obtain the corresponding result for g in (2.21). □

By Lemma 2.3, Chang [2, Lemma 2.1], and the embedding $E_\Omega \hookrightarrow L^{p+1}(\Omega, \mathbb{R}^2)$, we obtain also that for $\Psi_\Omega : E_\Omega \to \mathbb{R}$, $\Psi_\Omega(U) = \int_\Omega F(x, U) dx$, we have

$$\partial \Psi_\Omega(U)(x) \subset [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))] \quad \text{a.e. } x \in \Omega. \tag{2.27}$$

Let $V \in E_\Omega$. Then $\widetilde{V} \in E$, where $\widetilde{V} : \mathbb{R}^N \to \mathbb{R}^2$ is defined by

$$\widetilde{V} = \begin{cases} V(x) & x \text{ in } \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

For $W \in E^*$, we consider $W_\Omega \in E_\Omega^*$ such that $\langle W_\Omega, V \rangle = \langle W, \widetilde{V} \rangle$ for all V in E_Ω. Set $\Psi : E \to \mathbb{R}$, $\Psi(U) = \int_{\mathbb{R}^N} F(x, U)$.

LEMMA 2.4. *Let $W \in \partial \Psi(U)$, where $U \in E$. Then $W_\Omega \in \partial \Psi_\Omega(U)$, in the sense that $W_\Omega \in \partial \Psi_\Omega(U|_\Omega)$.*

Proof. By the definition of the Clarke gradient, we deduce that $\langle W, \widetilde{V} \rangle \leq \Psi^0(U, \widetilde{V})$ for all V in E_Ω,

$$\begin{aligned}
\Psi^0(U, \widetilde{V}) &= \limsup_{\substack{H \to U, H \in E \\ \lambda \to 0}} \frac{\Psi(H + \lambda \widetilde{V}) - \Psi(H)}{\lambda} \\
&= \limsup_{\substack{H \to U, H \in E \\ \lambda \to 0}} \frac{\int_{\mathbb{R}^N} (F(x, H + \lambda \widetilde{V}) - F(x, H)) \, dx}{\lambda} \\
&= \limsup_{\substack{H \to U, H \in E \\ \lambda \to 0}} \frac{\int_\Omega (F(x, H + \lambda \widetilde{V}) - F(x, H)) \, dx}{\lambda} \\
&= \limsup_{\substack{H \to U, H \in E_\Omega \\ \lambda \to 0}} \frac{\int_\Omega (F(x, H + \lambda \widetilde{V}) - F(x, H)) \, dx}{\lambda} \\
&= \Psi_\Omega^0(U, V).
\end{aligned} \tag{2.29}$$

Hence $\langle W_\Omega, V \rangle \leq \Psi_\Omega^0(U, V)$, which implies that $W_\Omega \in \partial \Psi_\Omega^0(U)$. □

By Lemmas 2.3 and 2.4, we obtain that for any $W \in \partial\Psi(U)$ (with $U \in E$), W_Ω satisfies (2.20) and (2.21). We also observe that for $\Omega_1, \Omega_2 \subset \mathbb{R}^N$, we have $W_{\Omega_1}|_{\Omega_1 \cap \Omega_2} = W_{\Omega_2}|_{\Omega_1 \cap \Omega_2}$.

Let $W_0 : \mathbb{R}^N \to \mathbb{R}$, where $W_0(x) = W_\Omega(x)$ if $x \in \Omega$. Then W_0 is well defined and

$$W_0(x) \in \left[\underline{f}(x, U(x)), \overline{f}(x, U(x))\right] \times \left[\underline{g}(x, U(x)), \overline{g}(x, U(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N, \quad (2.30)$$

and for all $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^2)$, $\langle W, \varphi \rangle = \int_{\mathbb{R}^N} W_0 \varphi$. By density of $C_c^\infty(\mathbb{R}^N, \mathbb{R}^2)$ in E, we deduce that $\langle W, V \rangle = \int_{\mathbb{R}^N} W_0 V \, dx$ for all V in E. Hence

$$W(x) = W_0(x) \in \left[\underline{f}(x, U(x)), \overline{f}(x, U(x))\right] \times \left[\underline{g}(x, U(x)), \overline{g}(x, U(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N.$$
$$(2.31)$$

3. Proof of Theorem 1.2

Define the energy functional $I : E \to \mathbb{R}$ by

$$I(U) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right) dx - \int_{\mathbb{R}^N} F(x, U) dx$$
$$= \frac{1}{2}\|U\|_E^2 - \Psi(U). \quad (3.1)$$

The existence of solutions to problem (1.9) will be justified by a nonsmooth variant of the mountain pass theorem (see [2]) applied to the functional I, even if the PS condition is not fulfilled. More precisely, we check the following geometric hypotheses:

$$I(0) = 0, \quad \exists V \in E, \quad \text{such that } I(V) \leq 0; \quad (3.2)$$

$$\exists \beta, \rho > 0 \quad \text{such that } I \geq \beta \text{ on } \{U \in E; \|U\|_E = \rho\}. \quad (3.3)$$

Verification of (3.2). It is obvious that $I(0) = 0$. For the second assertion, we need the following lemma.

LEMMA 3.1. *There exist two positive constants C_1 and C_2 such that*

$$f(x, s, 0) \geq C_1 s^{\mu-1} - C_2, \quad \text{for a.e. } x \in \mathbb{R}^N; \ s \in [0, +\infty). \quad (3.4)$$

Proof. We first observe that (1.16) implies that

$$0 \leq \mu F(x, s, 0) \leq \begin{cases} s\underline{f}(x, s, 0), & s \in [0, +\infty), \\ s\overline{f}(x, s, 0), & s \in (-\infty, 0], \end{cases} \quad (3.5)$$

which places us in the conditions of [11, Lemma 5].

Verification of (3.2) continued. Choose $v \in C_c^\infty(\mathbb{R}^N) - \{0\}$ so that $v \geq 0$ in $\mathbb{R}^N$. We have $\int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2 < \infty$, hence $t(v, 0) \in E$ for all $t \in \mathbb{R}$. Thus by Lemma 3.1, we obtain

$$
\begin{aligned}
I(t(v,0)) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2 dx - \int_{\mathbb{R}^N} \int_0^{tv} f(x,\tau,0)d\tau \\
&\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2 dx - \int_{\mathbb{R}^N} \int_0^{tv} (C_1\tau^{\mu-1} - C_2)d\tau \qquad (3.6) \\
&= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2 dx + C_2 t \int_{\mathbb{R}^N} v\, dx - C_1' t^\mu \int_{\mathbb{R}^N} v^\mu dx < 0
\end{aligned}
$$

for $t > 0$ large enough.

Verification of (3.3). We observe that (1.14), (1.15), and (1.16) imply that for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that

$$
\begin{aligned}
|f(x,s)| &\leq \varepsilon|s| + A_\varepsilon|s|^p \\
|g(x,s)| &\leq \varepsilon|s| + A_\varepsilon|s|^p
\end{aligned}
\qquad \text{for a.e. } (x,s) \in \mathbb{R}^N \times \mathbb{R}^2. \qquad (3.7)
$$

By (3.7) and Sobolev's embedding theorem, we have for any $U \in E$,

$$
\begin{aligned}
|\Psi(U)| &= |\Psi(u_1, u_2)| \\
&\leq \int_{\mathbb{R}^N} \int_0^{|u_1|} |f(x,\tau,u_2)|\, d\tau + \int_{\mathbb{R}^N} \int_0^{u_2} |g(x,0,\tau)|\, d\tau \\
&\leq \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2} |(u_1,u_2)|^2 + \frac{A_\varepsilon}{p+1} |u_1, u_2|^{p+1} \right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2} |u_2|^2 + \frac{A_\varepsilon}{p+1} |u_2|^{p+1} \right) dx \\
&\leq \varepsilon \|U\|_{L^2}^2 + \frac{2A_\varepsilon}{p+1} \|U\|_{L^{p+1}}^{p+1} \\
&\leq \varepsilon C_3 \|U\|_E^2 + C_4 \|U\|_E^{p+1},
\end{aligned}
\qquad (3.8)
$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus

$$
I(U) = \frac{1}{2} \|U\|_E^2 - \Psi(U) \geq \frac{1}{2} \|U\|_E^2 - \varepsilon C_3 \|U\|_E^2 - C_4 \|U\|_E^{p+1} \geq \beta > 0, \qquad (3.9)
$$

for $\|U\|_E = \rho$, with ρ, ε, and β sufficiently small positive constants.

Denote

$$
\begin{aligned}
\mathcal{P} &= \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) \neq 0, I(\gamma(1)) \leq 0\}, \\
c &= \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} I(\gamma(t)).
\end{aligned}
\qquad (3.10)
$$

Set

$$
\lambda_I(U) = \min_{\xi \in \partial I(U)} \|\xi\|_{E^*}. \qquad (3.11)
$$

Thus, by the nonsmooth version of the mountain pass lemma [2], there exists a sequence $\{U_M\} \subset E$ such that

$$I(U_m) \longrightarrow c, \qquad \lambda_I(U_m) \longrightarrow 0. \tag{3.12}$$

So, there exists a sequence $\{W_m\} \subset \partial\Psi(U_m)$, $W_m = (w_m^1, w_m^2)$, such that

$$(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + a(x)u_m^2 - w_m^2) \longrightarrow 0 \quad \text{in } E^*. \tag{3.13}$$

Note that by (1.16),

$$\Psi(U) \leq \frac{1}{\mu}\left(\int_{u_1 \geq 0} u_1(x)\underline{f}(x,U)dx + \int_{u_1 \leq 0} u_1(x)\overline{f}(x,U)dx \right. \\ \left. + \int_{u_2 \geq 0} u_1(x)\underline{g}(x,U)dx + \int_{u_2 \leq 0} u_2(x)\overline{g}(x,U)dx\right). \tag{3.14}$$

Therefore, by (2.31),

$$\Psi(U) \leq \frac{1}{\mu}\int_{\mathbb{R}^N} U(x)W(x)dx = \frac{1}{\mu}\int_{\mathbb{R}^N} (u_1w_1 + u_2w_2)dx, \tag{3.15}$$

for every $U \in E$ and $W \in \partial\Psi(U)$. Hence, if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E^* and E, we have

$$\begin{aligned}
I(U_m) &= \frac{\mu-2}{2\mu}\int_{\mathbb{R}^N}\left(|\nabla u_m^1|^2 + |\nabla u_m|^2 + a(x)|u_m|^1 + b(x)|u_m|^2\right)dx \\
&\quad + \frac{1}{\mu}\langle(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m\rangle \\
&\quad + \frac{1}{\mu}\langle W_m, U_m\rangle - \Psi(U_m) \\
&\geq \frac{\mu-2}{2\mu}\int_{\mathbb{R}^N}\left(|\nabla u_m^1|^2 + |\nabla u_m^2|^2 + a(x)|u_m^1|^2 + b(x)|u_m^2|^2\right)dx \\
&\quad + \frac{1}{\mu}\langle(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m\rangle \\
&\geq \frac{\mu-2}{2\mu}\|U_m\|_E^2 - o(1)\|U_m\|_E.
\end{aligned} \tag{3.16}$$

This relation in conjunction with (3.12) implies that the Palais-Smale sequence $\{U_m\}$ is bounded in E. Thus, it converges weakly (up to a subsequence) in E and strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$ to some U. Taking into account that $W_m \in \partial\Psi(U_m)$ and $U_m \rightharpoonup U$ in E, we deduce from (3.13) that there exists $W \in E^*$ such that $W_m \rightharpoonup W$ in E^* (up to a subsequence). Since the mapping $U \mapsto F(x,U)$ is compact from E to L^1, it follows that

$W \in \partial\Psi(U)$. Therefore,

$$W(x) \in \left[\underline{f}(x, U(x)), \overline{f}(x, U(x))\right] \times \left[\underline{g}(x, U(x)), \overline{g}(x, U(x))\right] \quad \text{a.e. } x \text{ in } \mathbb{R}^N,$$

$$
\begin{aligned}
&\left(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2\right) \\
&\quad = 0 \iff \int_{\mathbb{R}^N} \left(\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x)u_1 v_1 + b(x)u_2 v_2\right) dx \\
&\quad = \int_{\mathbb{R}^N} \left(w_1 v_1 + w_2 v_2\right) dx \quad \forall (v_1, v_2) \in E.
\end{aligned}
\tag{3.17}
$$

These last two relations show that U is a solution of the problem (1.9).

It remains to prove that $U \not\equiv 0$. If $\{W_m\}$ is as in (3.13), then by (1.16), (2.31), (3.12), and for large m,

$$
\begin{aligned}
\frac{c}{2} &\leq I(U_m) - \frac{1}{2}\left\langle\left(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2\right), U_m\right\rangle \\
&= \frac{1}{2}\langle W_m, U_m\rangle - \int_{\mathbb{R}^N} F(x, U_m)\, dx \\
&\leq \frac{1}{2}\left(\int_{u_1 \geq 0} u_1(x)\underline{f}(x, U)\, dx + \int_{u_1 \leq 0} u_1(x)\overline{f}(x, U)\, dx \right.\\
&\qquad\left. + \int_{u_2 \geq 0} u_1(x)\underline{g}(x, U)\, dx + \int_{u_2 \leq 0} u_2(x)\overline{g}(x, U)\, dx\right).
\end{aligned}
\tag{3.18}
$$

Now, taking into account the definition of $\overline{f}, \underline{f}, \overline{g}, \underline{g}$, we deduce that $\overline{f}, \underline{f}, \overline{g}, \underline{g}$ verify (3.2) too. So by (3.18), we obtain

$$\frac{c}{2} \leq \int_{\mathbb{R}^N} \left(\varepsilon |U_m|^2 + A_\varepsilon |u_m|^{p+1}\right) = \varepsilon \|U_m\|_{L^2}^2 + A_\varepsilon \|U_m\|_{L^{p+1}}^{p+1}. \tag{3.19}$$

So, $\{U_m\}$ does not converge strongly to 0 in $L^{p+1}(\mathbb{R}^N; \mathbb{R}^2)$. From now on, with the same arguments as in the proof of [6, Theorem 1], we deduce that $U \not\equiv 0$, which concludes our proof.   □

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